Explicit harmonic self-maps of complex projective spaces

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DIFFERENTIAL GEOMETRY **WORKSHOP 2023** – Alexandru Ioan Cuza University

Outline

(1) Harmonic maps and $\mathbb{CP}^n$

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(2) Reduction technique

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(3) Attacking the ODE

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(1) Harmonic maps and $\mathbb{CP}^n$

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(4) Energy

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(4) Energy

(5) Stability of solutions

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Remark:

▷ (Anti)holomorphic maps between Kähler manifolds are harmonic.

▷ Holomorphic harmonic maps between compact Kähler manifolds are weakly stable.

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Define for $t \in [0, \frac{\pi}{2}] (\simeq \mathbb{CP}^n / G)$ the curve

$$\gamma(t) = [\cos t : 0 : \ldots : 0 : \underbrace{\sin t}_{p+1} : 0 : \ldots : 0].$$

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$$g \cdot \gamma(t) \mapsto g \cdot \gamma(r(t))$$

where $r : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ is smooth and $r(0) = 0$, $r(\frac{\pi}{2}) = \frac{\pi}{2} + \pi\mathbb{Z}$.

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where $r : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ is smooth and $r(0) = 0$, $r(\frac{\pi}{2}) = \frac{\pi}{2} + \pi\mathbb{Z}$. The map ψ is well defined and smooth (Püttmann 2009).

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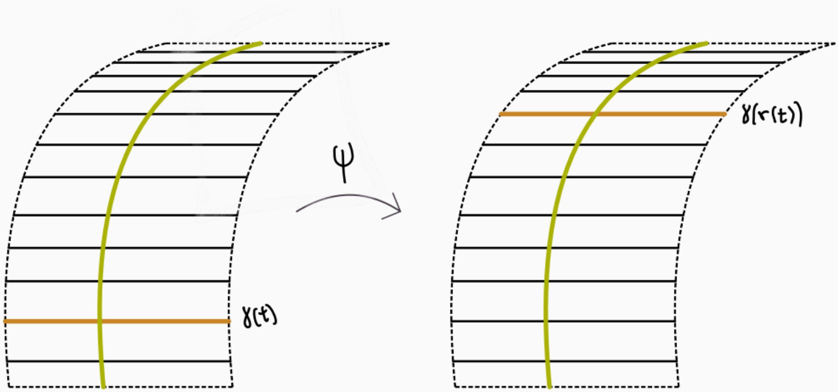

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Tension field
Take the biinvariant metric

$$Q(X, Y) = -\frac{1}{2}\text{trace}XY$$

for $X, Y \in \mathrm{Lie}(G)$ and split $\mathrm{Lie}(G) = \mathrm{Lie}(G_{\gamma(t)}) \oplus \mathfrak{n}$, $t \in (0, \frac{\pi}{2})$.

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Define for every $t \in (0, \frac{\pi}{2})$ the endomorphism $P_t : \mathfrak{n} \to \mathfrak{n}$ by

$$Q(P_t X, Y) = g_{FS}(X^*, Y^*)_{|\gamma(t)}$$

where $X^*_{|\gamma(t)} = \frac{d}{ds}\Big|_{s=0} \exp(sX) \cdot \gamma(t)$.

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where $X^*_{|\gamma(t)} = \frac{d}{ds}\Big|_{s=0} \exp(sX) \cdot \gamma(t)$. In our case,

$$P_t = \begin{pmatrix} \cos^2 t\, \mathbb{1}_{2p} & & \\ & \sin^2 t\, \mathbb{1}_{2(n-p-1)} & \\ & & \frac{\eta^2}{4}\sin^2 2t \end{pmatrix}.$$

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- $\tau^{\text{nor}}_{|\gamma(t)} = \left[\ddot{r}(t) + \frac{1}{2}\dot{r}(t)\,\text{tr}P_t^{-1}\,\dot{P}_t - \frac{1}{2}\text{tr}P_t^{-1}\,(\dot{P})_{r(t)}\right]\dot{\gamma}(r(t))$

Reduction technique

Theorem
Consider the natural $\mathrm{SU}(p+1) \times \mathrm{SU}(n-p)$-action on $\mathbb{CP}^n$ with $0 \leq p < n$. The tension field of ψ vanishes if and only if r satisfies the boundary value problem

$$\begin{aligned}\ddot{r}(t) &+ [(2n-2p-1)\cot t - (2p+1)\tan t]\,\dot{r}(t) \\ &+ \left[\frac{p}{\cos^2 t} - \frac{(n-p-1)}{\sin^2 t}\right]\sin 2r(t) - \frac{\sin 4r(t)}{\sin^2 2t} = 0 \end{aligned} \qquad \text{(ODE)}$$

for smooth functions $r : (0, \frac{\pi}{2}) \to \mathbb{R}$ with

$$\lim_{t \to 0} r(t) = 0 \quad \text{and} \quad \lim_{t \to \frac{\pi}{2}} r(t) = k\frac{\pi}{2}, \qquad \text{(BC)}$$

where $k \in 2\mathbb{Z}+1$.

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Theorem
Let $\rho \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, the functions defined by

$$r_{\rho,\ell}(t) = \arctan(\rho \tan t) + \ell\pi, \quad \kappa_\ell(t) = \ell\frac{\pi}{2}$$

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2. The functions $r_{\rho,\ell}$ and $\kappa_{\rho,\ell}$ are solutions for the ordinary differential equation (ODE).
3. If $\rho \neq 0$, the function $r_{\rho,0}$ is the unique solution for the boundary value problem (ODE), (BC) satisfying $\dot{r}(t) \to \rho$ as $t \to 0^+$.

Attacking the ODE

Remark

1. If $\rho > 0$, ψ_ρ is a holomorphic harmonic map.
2. If $\rho < 0$, ψ_ρ is a non-holomorphic, non-antiholomorphic harmonic map.

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Proposition

For $\rho \neq 0$, the energy of the harmonic map ψ_ρ constructed above is given by

$$E(\psi_\rho) = n\mathrm{Vol}(\mathbb{CP}^n) = \frac{\pi^n}{(n-1)!}.$$

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Equivariant stability

Variations that are invariant under the cohomogeneity one action. We study equivariant stability by the following Sturm-Liouville problem:

$$\ddot{\xi}(t) + \frac{1}{2}\mathrm{trace}(P_t^{-1}\dot{P}_t)\,\dot{\xi}(t) - \frac{1}{2}\mathrm{trace}(P_t^{-1}\ddot{P}_{r(t)})\,\xi(t) + \lambda\xi(t) = 0$$

where $\xi \in C_0^\infty([0, \frac{\pi}{2}])$ (Branding and Siffert 2023).

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Theorem

For every $\rho \neq 0$, the map $\psi_\rho : \mathbb{CP}^n \to \mathbb{CP}^n$ is equivariantly weakly stable.

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for $\xi \in C_0^\infty(\mathbb{R})$.

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Theorem

The spectral problem (1) describing the equivariant stability of the maps ψ_1 and ψ_{-1}, is solved by

$$\xi_j(x) = \frac{1}{\cosh x}\,P_j^{(\frac{n+1}{2},\frac{n+1}{2})}(\tanh x), \quad \lambda_j = 4j(j+n+2)$$

for $j \in \mathbb{N}$, where $P_j^{(\frac{n+1}{2},\frac{n+1}{2})}$ are the so-called Jacobi polynomials.

Mulțumesc mult!

Thanks a lot!

Bibliography

[1] Püttmann, T. (2009). Cohomogeneity one manifolds and self-maps of nontrivial degree. Transformation Groups, 14(1), 225-247.

[2] Püttmann, T., and Siffert, A. (2019). Harmonic self-maps of cohomogeneity one manifolds. Mathematische Annalen, 375(1-2), 247-282.

[3] Gastel, A. (2004). On the harmonic Hopf construction. Proceedings of the American Math. Society, 132(2), 607-615.

[4] Branding, V., and Siffert, A. (2023). On the equivariant stability of harmonic self-maps of cohomogeneity one manifolds. Journal of Mathematical Analysis and Applications, 517(2), 126635.

[5] Balado-Alves, J. M. (2023). Explicit harmonic self-maps of complex projective spaces. arXiv preprint arXiv:2304.00851.