Elastic curves: a numerical approach

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Elastic curves (elastica)

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{n}}$ and $\mathbf{v}, \mathbf{w} \in S^{n-1}$ $(n \geq 2)$
Suppose there exists a C^∞ unit speed curve $c : [a, b] \to \mathbb{R}^n$ with the property

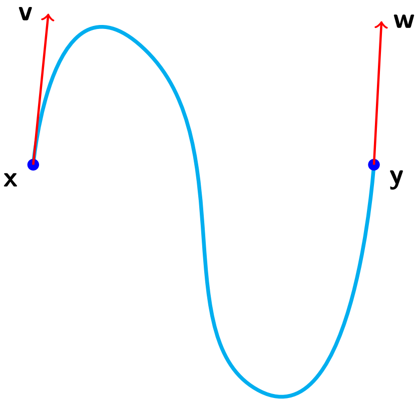
$$c(a) = \mathbf{x}, \quad c'(a) = \mathbf{v}, \quad c(b) = \mathbf{y}, \quad c'(b) = \mathbf{w}\,.$$

Such a curve has length $L(c) = b - a$ and is said to be *feasible* for $a, b, \mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w}$. An Euler-Bernouilli (fixed length) *elastica* is defined to be a critical point of the functional

$$K(c) = \frac{1}{2}\int_a^b ||c''(s)||^2 \mathrm{d}s\,,$$

as c varies over feasible curves.

Elastic curve - example



Euler-Lagrange equations

A feasible curve is an elastica when, for some C^∞ function $\mu : [a, b] \to \mathbb{R}$,

$$c''''(s) + (\mu(s)c'(s))' = 0 \quad \Leftrightarrow \quad \gamma''(s) + \mu(s)\gamma(s) = C \tag{1}$$

where $\gamma(s) = c'(s)$ and $C \in \mathbb{R}^n$ is constant.
Taking the inner product with $\gamma(s) \in S^{n-1}$:

$$\mu(s) = \langle C, \gamma(s)\rangle - \langle \gamma'', \gamma\rangle = \langle C, \gamma(s)\rangle + \kappa(s)^2$$

where the curvature $\kappa : [a, b] \to \mathbb{R}$ is defined by $\kappa(s) = ||\gamma''(s)||$.
Taking the inner product with $\gamma'(s)$:

$$\langle \gamma''(s), \gamma'(s)\rangle = \langle C, \gamma'(s)\rangle \Rightarrow \langle C, \gamma(s)\rangle = \frac{\kappa(s)^2 - a}{2} \Rightarrow \mu(s) = \frac{3\kappa(s)^2 - a}{2}$$

for a constant $a \in \mathbb{R}$.

Planar elastica

Suppose $n = 2$. Diff (1) and take the inner product with $\gamma'(s)$,

$$\begin{aligned} 0 &= \langle \gamma''', \gamma' \rangle + \mu\kappa^2 = \kappa\kappa'' + (\kappa')^2 - \langle \gamma'', \gamma'' \rangle + \mu\kappa^2 \\ &= \frac{\kappa(s)}{2} \left(2\kappa''(s) + \kappa^3(s) - a\kappa(s) \right) \end{aligned}$$

since $\gamma'' = -\kappa^2\gamma + \kappa'\nu$ where ν is the normal to γ (planar curve). Therefore

$$2\kappa''(s) = a\kappa(s) - \kappa(s)^3 \, .$$

Excluding circles (κ const.) solutions are given by

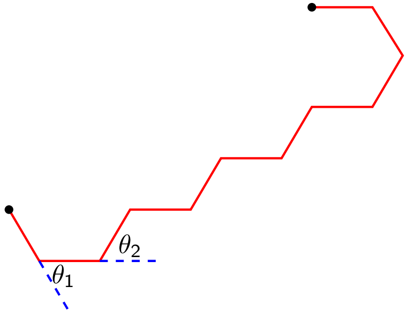
$$\kappa(s)^2 = \kappa_0 \left(1 - \frac{p^2}{w^2} \mathrm{sn}^2 \left(\frac{\kappa_0}{2w}(s - s_0), p \right) \right)$$

where sn denotes elliptic sine, w is either p or 1, and a is related to the parameters κ_0, p by

$$2a = \frac{{\kappa_0}^2}{w^2}(3w^2 - p^2 - 1) \, .$$

Discrete approximation

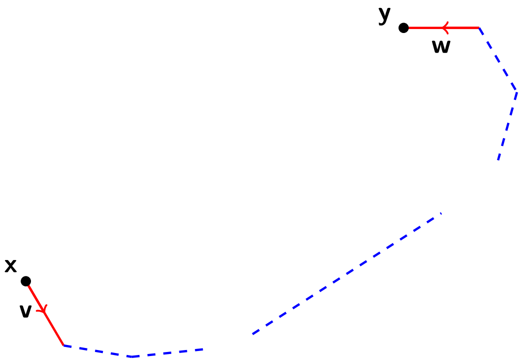
The discrete analogue of a unit speed curve is a polygonal curve with sides of equal (unit) length.



The curvature at a vertex is the modulus of the exterior angle: $|\theta|$, and the total square curvature is the sum $\sum_{i=1}^{n} {\theta_i}^2$.

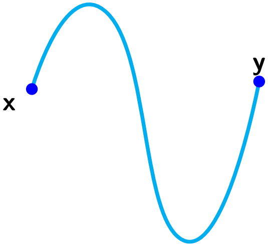
The discrete problem

Given two points **x** and **y** and unit directions at those two points **v** and **w**, find a polygonal curve of length n with sides of unit length beginning and ending in the given directions, which extremizes the discrete total square curvature amongst such curves.



Strategy to approximate smooth elastica

- Take smooth candidate curve satifying the required boundary and length conditions.
- Approximate this curve with a polygonal curve with sides of equal length.
- Perform a procedure (see below) on the polygonal curve to approach a discrete elastica.



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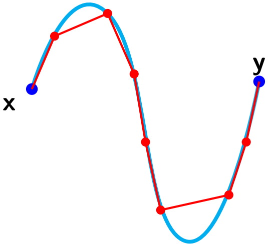
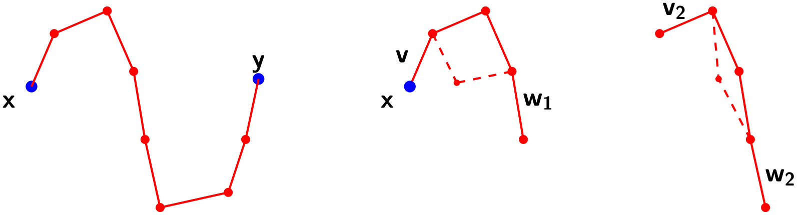


Optimizing the polygonal curve - leapfrog

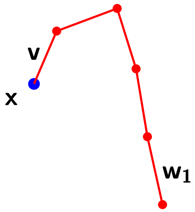
Start at **x** and choose a segment of the curve of fixed length, fixing the first and last edge, and improve it. Say, choose a segment of four edges.



In general there are just two choices: keep as is, or flip. In the example, a flip will increase the curvature, so keep as is. Then move on one edge and repeat. When one arrives at the far end, return doing the same procedure.

Leapfrog with 5 edges

Take a segment of five edges. Now have continuous flexibility for the middle three edges.



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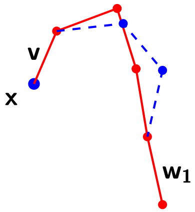
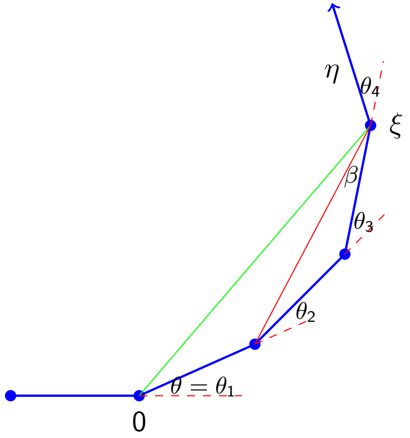


Parametrization of a configuration of five unit edge with boundary conditions



The parameter $\theta = \theta_1$ is variable and determines the configuration.

Extremization problem

Problem: extremize (minimize) the total curvature: ${\theta_1}^2 + {\theta_2}^2 + {\theta_3}^2 + {\theta_4}^2$.
However, $\cos\theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4)$, so if we suppose we have chosen a large number of edges so the angles are small, instead maximize:

$$E := 2(\cos\theta + \cos\theta_2 + \cos\theta_3 + \cos\theta_4)$$

Constraints:
(i) $e^{\mathrm{i}\theta_1} + e^{\mathrm{i}(\theta_1+\theta_2)} + e^{\mathrm{i}(\theta_1+\theta_2+\theta_3)} = \xi \in \mathbf{C}$ fixed;
(ii) $e^{\mathrm{i}(\theta_1+\theta_2+\theta_3+\theta_4)} = \eta \in S^1 \subset \mathbf{C}$ fixed.

Geometry implies

$$E = -2 + 2\cos\theta + \frac{2}{|\xi - e^{\mathrm{i}\theta}|}\mathrm{Re}\left\{e^{-\mathrm{i}\beta}(\xi - e^{\mathrm{i}\theta})(\overline{\xi} - \overline{\eta})\right\} + 2\mathrm{Re}\left\{\xi\overline{\eta} - e^{\mathrm{i}\theta}\overline{\eta}\right\}.$$

where $\cos\beta = \frac{1}{2}|\xi - e^{\mathrm{i}\theta}|$.

Extremization problem cont.

Find the maximum of the function (for given ξ and η) :

$$F := \cos\theta + \frac{1}{|\xi - e^{i\theta}|}\mathrm{Re}\left\{e^{-i\beta}(\xi - e^{i\theta})(\overline{\xi} - \overline{\eta})\right\} - \mathrm{Re}\left\{e^{-i\theta}\eta\right\}$$

where $\cos\beta = \frac{1}{2}|\xi - e^{i\theta}|$.

$$\begin{aligned} F'(\theta) &= \mathrm{Im}\left\{e^{i\theta}(\overline{\xi} + \overline{\eta} - 2)\right\} \\ &\quad - \frac{1}{\sin 2\beta(1 + \cos 2\beta)}\Big\{\mathrm{Im}\left\{e^{i\theta}\overline{\xi}\right\}\mathrm{Im}\left\{(\xi - e^{i\theta})(\overline{\xi} - \overline{\eta})\right\} \\ &\qquad + (\sin^2 2\beta)\mathrm{Re}\left\{e^{i\theta}(\overline{\xi} - \overline{\eta})\right\}\Big\} \end{aligned}$$

Check: If the boundary conditions correspond to the configuration required for a regular polygon of n sides, then it is easily confirmed that the above expression has $F'(2\pi/n) = 0$.

Appropximate solution

For ease of notation, define six real constants:

$$A := |\xi - 1|, \quad B := \sqrt{4 - |\xi - 1|^2}, \quad C := \xi + \overline{\xi},$$
$$D := \eta + \overline{\eta}, \quad E := -\mathrm{i}(\xi - \overline{\xi}), \quad F := -\mathrm{i}(\eta - \overline{\eta})\,.$$

Lemma

(i)

$$\sin 2\beta \sim \frac{1}{2}\left(AB + \frac{E}{2}\left(\frac{A}{B} - \frac{B}{A}\right)\theta\right)\,.$$

(ii)

$$1 + \cos 2\beta \sim \frac{1}{2}(A^2 - E\theta)\,.$$

Approximate solution cont.

To first order, the solution is given by,

Lemma

$$\begin{aligned} F'(\theta) \quad \sim \quad & -A^3B(E+F)+2E(E-F)-E(ED-FC))-A^2B^2(C-D) \\ & +\theta\Bigg\{\frac{E(E+F)}{2}\left(3AB-\frac{A^3}{B}\right)+A^3B(C+D-4) \\ & \qquad -2E(C-D)-2C(E-F)+C(ED-FC)) \\ & \qquad\qquad -E(A^2-B^2)(C-D)-A^2B^2(E-F)\Bigg\} \end{aligned}$$

Questions: 1. Is there a simpler way to find an approximate solution?
2. Is there a geometric construction which determines an improved configuration?

Generalization to 2-dimensions

The Willmore energy of a closed surface S embedded in $\mathbb{R}^3$ is given by

$$\mathcal{W} = \int_S H^2 \mathrm{d}\sigma - \int_S K \mathrm{d}\sigma = \frac{1}{4}\int_S (k_1 - k_2)^2 \mathrm{d}\sigma = \int_S H^2 \mathrm{d}\sigma - 2\pi\chi(S)$$

Elastic energy: Wish to find critical points of the energy

$$E = \int_S H^2 \mathrm{d}\sigma$$

ranging over surfaces with fixed boundary $c = \partial S$ and fixed area A, which leave the bondary in a given direction, i.e. we define a vector field $\mathbf{v}$ along c, transverse to c and require this be tangent to the surface along c.

An approach using invariant frameworks:

A curious formula

For a unit speed curve in the plane $c(s)$ we have
$c''(s) = \pm \mathrm{i}\kappa(s)c'(s) \Rightarrow (c''(s))^2 = -\kappa(s)^2(c'(s))^2$.
Let $\varphi : (M^m, g) \to \mathbb{R}^{n+1}$ be a smooth isometric embedding and let
$\pi : \mathbb{R}^{n+1} \to \mathbf{C}$ be any orthog. projection to the the complex plane. Set
$\Phi = \pi \circ \varphi : (M^n, g) \to \mathbf{C}$ where H is the mean curvature of $\varphi(M^m)$. Then

Lemma

$$(\Delta_g \Phi)^2 = -H^2 (\nabla_g \Phi)^2$$

Proof.

Let $\mathbf{n}$ denote the unit normal to the hypersurface. On the one hand

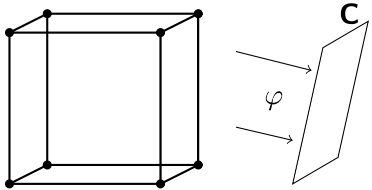
$$\Delta_g(\pi \circ \varphi) = \mathrm{d}\pi(\Delta_g \varphi) + \mathrm{Tr}\, \nabla \mathrm{d}\pi(\mathrm{d}\varphi, \mathrm{d}\varphi) = H \mathrm{d}\pi(\mathbf{n})$$

since π is totally geodesic. On the other hand if $\{e_i\}$ is an o.n. basis on M^m and $f_i = \mathrm{d}\varphi(e_i)$, then $\{f_i, \mathbf{n}\}$ is an o.n. basis in $\mathbb{R}^{n+1}$, and

$$\nabla_g(\pi \circ \varphi)^2 = \sum_i \mathrm{d}\pi(\mathrm{d}\varphi(e_i))^2 = \sum_i \mathrm{d}\pi(f_i)^2 = -\mathrm{d}\pi(\mathbf{n})^2$$

since π is a Riemannian submersion. □

Discrete version for frameworks



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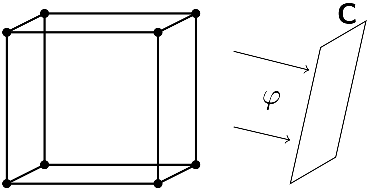

$\varphi : \{\text{vertices}\} \to \mathbf{C}$. Set

$\Delta\varphi := \frac{1}{d_x}\sum_{y\sim x}\left(\varphi(y) - \varphi(x)\right)$

$(\nabla\varphi)^2 :=$
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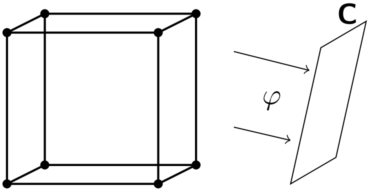

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At a given vertex x, consider the equation

$$\frac{\gamma(x)}{d_x}\left\{\sum_{y\sim x}(\varphi(y)-\varphi(x))\right\}^2 = \sum_{y\sim x}\left(\varphi(x)-\varphi(y)\right)^2 \qquad (2)$$

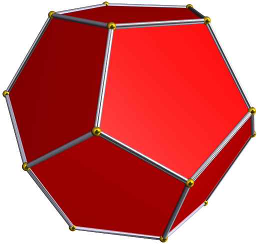
$$\iff \quad \gamma(\Delta\varphi)^2 = (\nabla\varphi)^2$$

where d_x is the degree of vertex x and $\gamma : \{\text{vertices}\} \to \mathbb{R}$ is a real function on the vertices. For cube $\gamma \equiv 0$ and this is essentially the Gauss Theorem of Axonometry (1876).

Regular polytopes

The regular polytopes in Euclidean space satisfy (2) with some examples of γ given in the following table:

polyhedron	γ
tetrahedron	$3/4$
cube	0
octahedron	$1/2$
icosahedron	$\frac{2-\sqrt{5}}{3-\sqrt{5}} < 0$
dodecahedron	$\frac{3(1-\sqrt{5})}{2(3-\sqrt{5})} < 0$
600-cell	$\frac{5(1-2\sqrt{5})}{3} < 0$



The 600-cell is a convex 4-dimensional regular polytope made up of 600 tetrahedral 3-polytopes. It has 120 vertices and 720 edges.

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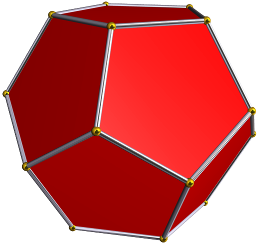

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We will interpret γ as a curvature – see below.

Invariant frameworks

A *framework* $\mathcal{F}$ in $\mathbb{R}^N$ is a finite collection of points $\{\vec{x}_1, \ldots, \vec{x}_n\}$ connected in various ways by edges (bars) which are straight line segments.

Let $\varphi : \mathbb{R}^N \to \mathbf{C}$ be an orthogonal projection. Then say that the framework is *invariant* if φ satisfies (2) for some fixed function $\gamma : V \to \mathbb{R}$, independently of any similitude of the framework.

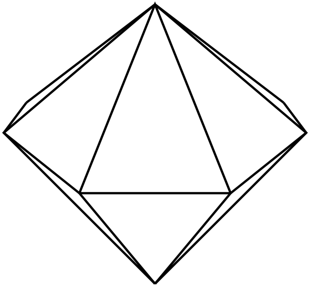
Two requirements: that γ be real and that the equation be satisfied independent of the orthogonal projection. For a curve in the plane, there is only the reality requirement which is satisfied if and only if the edges have equal length.

The property of invariance means that geometry is intrinsic to the structure and not dependent on how it is embedded in space.

Under the assumption of invariance, for a given graph, we may be able to recover its "shape" from its combinatorial structure and the function $\gamma : V \to \mathbb{R}$.

Framework invariance

Only certain frameworks fit the invariance requirement: for a double pyramid on a regular polygon it has a unique height for which it is invariant.



The height decreases to zero with the number n of vertices of the regular polygon so it converges to a disc as $n \to \infty$.

Program for elastic surfaces

- Given an initial surface of area A satisfying required boundary conditions, approximate it with an invariant framework (triangular, rectangular, ...). Is this possible?
- Adjust the framework locally to decrease the total squared mean curvature. Is this possible while maintaining area and invariance?
- Take a pragmatic approach: for simplicity assume the surface is a topological disc - decompose it with a suitable rectangular tiling; don't worry about invariance, but use (2) anyway to calculate mean curvature; systematically improve the mean curvature while keeping the area constant, as one might attempt to deform chain mail, for example by spiralling into the centre and spiralling out again, repeating the procedure.

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Thank you for your attention!

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