Biharmonic hypersurfaces with at most three distinct principal curvatures in space forms

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Harmonic and biharmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \; \overline{v}_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau(\varphi) &= \operatorname{trace} \nabla d\varphi \\ &= 0\end{aligned}$$

Critical points of E:
harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \; \overline{v}_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau_2(\varphi) =& -\Delta^{\varphi}\tau(\varphi) \\ & - \operatorname{trace} R^N(d\varphi(\cdot), \tau(\varphi))d\varphi(\cdot) \\ =& 0\end{aligned}$$

Critical points of E_2:
biharmonic maps

Any harmonic map is biharmonic. We will say that a map is proper biharmonic if it is biharmonic but not harmonic.

Conventions

Sign Conventions:

- $\Delta\sigma = -\operatorname{trace}\left(\nabla^2\sigma\right)$,
- $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

For a submanifold M^m in N^n we will use the following notations:

- B is the second fundamental form of M in N,
- $H = \frac{1}{m}\operatorname{trace} B$ is the mean curvature vector field.

If M^m is a hypersurface in N^{m+1} we also have:

- A is the shape operator of M in the direction of $\eta \in C(NM)$, $|\eta| = 1$,
- $f = \frac{1}{m}\operatorname{trace} A$ is the mean curvature function.

The biharmonic equation for submanifolds in space forms

Let $\varphi : M^m \to N^n(c)$ be a submanifold in a space form. Then

$$\tau(\varphi) = mH, \quad \tau_2(\varphi) = -m\Delta^\varphi H + m^2 cH,$$

thus φ is biharmonic if and only if $\Delta^\varphi H = mcH$.

1. The submanifold φ is biharmonic if and only if

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\cdot, A_H(\cdot)) - mcH = 0 \\ 2 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + \frac{m}{2} \operatorname{grad} |H|^2 = 0 \end{cases}.$$

2. If M^m is a hypersurface of $N^{m+1}(c)$, then M is biharmonic if and only if

$$\begin{cases} \Delta f + \left(|A|^2 - mc\right) f = 0 \\ 2A(\operatorname{grad} f) + mf \operatorname{grad} f = 0 \end{cases}.$$

Main examples of biharmonic submanifolds in $\mathbb{S}^n$ (Caddeo, Montaldo, Oniciuc-2001, 2002)

The composition property

$$\mathbb{S}^{n-1}(a) \xrightarrow{\ p.\ biharmonic\ } \mathbb{S}^n$$

$$\Updownarrow$$

$$a = \frac{1}{\sqrt{2}}$$

$$\begin{array}{ccc} M^m & \xrightarrow{\ minimal\ } & \mathbb{S}^{n-1}\left(\frac{1}{\sqrt{2}}\right) \\ & \searrow^{p.\ biharmonic} & \downarrow \varphi \\ & & \mathbb{S}^n \end{array}$$

Properties

- M has parallel mean curvature vector field, and $|H| = 1$.
- M is pseudo-umbilical in $\mathbb{S}^n$, i.e. $A_H = |H|^2 \operatorname{Id}$; $\nabla A_H = 0$.

Main examples of biharmonic submanifolds in $\mathbb{S}^n$ (Jiang-1986)

The product composition property

$$\mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2) \xrightarrow{p.\ biharmonic} \mathbb{S}^n$$

$$\Updownarrow$$

$$r_1 = r_2 = \frac{1}{\sqrt{2}} \quad and \quad n_1 \neq n_2$$

$$n_1 + n_2 = n - 1, \quad r_1^2 + r_2^2 = 1$$

$$M_1^{m_1} \times M_2^{m_2} \xrightarrow{minimal} \mathbb{S}^{n_1}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{n_2}\left(\frac{1}{\sqrt{2}}\right) \xrightarrow{\varphi} \mathbb{S}^n, \qquad M_1^{m_1} \times M_2^{m_2} \xrightarrow{p.\ biharmonic} \mathbb{S}^n$$

$$n_1 + n_2 = n - 1 \quad and \quad m_1 \neq m_2$$

Properties

- $M_1 \times M_2$ has parallel mean curvature vector field and $|H| \in (0,1)$.
- $M_1 \times M_2$ is not pseudo-umbilical in $\mathbb{S}^n$; $\nabla A_H = 0$.

Conjectures (Balmuș, Montaldo, Oniciuc-2008)

Conjecture (C_1)

Let M^m be a proper biharmonic hypersurface in $\mathbb{S}^{m+1}$. Then M is either an open part of the small hypersphere $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Conjecture (C_2)

Any proper biharmonic submanifold in the Euclidean sphere has constant mean curvature.

- Obviously, (C_2) is weaker than (C_1). But, if one can prove (C_2), then it seems to be very difficult to prove (C_1).
- Assume that M^m is a non-minimal CMC hypersurface of $\mathbb{S}^{m+1}$. Then it is proper biharmonic if and only if

$$|A|^2 = m.$$

Therefore, the classification of CMC proper biharmonic hypersurfaces can be viewed as an interesting problem in the classical geometry of hypersurfaces in Euclidean spheres.
We also recall here that the minimal hypersurfaces M^m in $\mathbb{S}^{m+1}$ with $|A|^2 = m$ were already classified by Chern, do Carmo, Kobayashi-1970. Thus our problem can be viewed as a generalization from minimal to CMC.

Biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ with $\ell = 1$

We will denote by ℓ the number of distinct principal curvatures of M^m in $\mathbb{S}^{m+1}$.

If $\ell = 1$, then $A = f\,\mathrm{Id}$, i.e. M is an umbilical hypersurface of $\mathbb{S}^{m+1}$

Theorem (Caddeo, Montaldo, Oniciuc-2001)

If M^m is umbilical in $\mathbb{S}^{m+1}$, then it is proper biharmonic if and only if M is an open part of $\mathbb{S}^m(1/\sqrt{2})$.

Biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ with $\ell \leq 2$

Theorem (Balmuș, Montaldo, Oniciuc-2008)

Let M^m be a proper biharmonic hypersurface in $N^{m+1}(c)$ with at most two distinct principal curvatures at any point. Then M^m has constant mean curvature.

Theorem (Balmuș, Montaldo, Oniciuc-2008)

Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures at any point in $\mathbb{S}^{m+1}$. Then M is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ with $\ell \leq 3$

Theorem (Balmuș, Montaldo, Oniciuc-2010)

There exist no compact proper biharmonic hypersurfaces in the unit Euclidean sphere of constant mean curvature and with three distinct principal curvatures at any point.

Theorem (Balmuș, Montaldo, Oniciuc-2010)

A proper biharmonic hypersurface in $\mathbb{S}^4$ has constant mean curvature.

Theorem (Balmuș, Montaldo, Oniciuc-2010)

Let M^3 be a complete proper biharmonic hypersurface in $\mathbb{S}^4$. Then M^3 is either an open part of $\mathbb{S}^3(1/\sqrt{2})$, or an open part of $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.

Biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ with $\ell \leq 3$

Theorem (Fu-2015)

Let M^m be a proper biharmonic hypersurface in $N^{m+1}(c)$, $m \geq 4$, with at most three distinct principal curvatures at any point. Then M^m has constant mean curvature.

Other important results in the field of biharmonic hypersurfaces, obtained with other additional hypotheses (not involving the number of distinct principal curvatures) and employing various techniques, were proved by: Alias, Garcia-Martinez, Rigoli-2013; J.H. Chen-1993; Fu, Maeta, Ou-2021; Luo, Maeta-2017; Maeta-2017; Maeta, Ou-2020; Montaldo, Oniciuc, Ratto-2016; Ou-2010; Vieira-2022, etc.

Recall the following results:

Theorem (Balmuș, Montaldo, Oniciuc-2008)

Let M^m be a proper biharmonic hypersurface in $N^{m+1}(c)$ with at most two distinct principal curvatures at any point. Then M^m has constant mean curvature.

Theorem (Fu-2015)

Let M^m be a proper biharmonic hypersurface in $N^{m+1}(c)$, $m \geq 4$, with at most three distinct principal curvatures at any point. Then M^m has constant mean curvature.

The techniques used to prove these two results are similar and involve connection forms, Gauss and Codazzi equations. However, the Fu's proof is more computationally complex.
Studying the latter theorem, we found a gap in the proof of Fu. Because of that we cannot conclude and we had a doubt concerning the validity of the result.

We solved this problem in a positive way in
S. A., Y. Fu, C. Oniciuc, *On the biharmonic hypersurfaces with three distinct principal curvatures in space forms*, arXiv:2301.09354

More specifically, we introduced a new method involving algebraic tools and Mathematica programming. We managed to find all cases that the original proof missed and showed that all hypersurfaces of this type still have constant mean curvature, thus the original statement of Fu is correct.

The set M_A of all point of M at which the number of distinct principal curvatures is locally constant is an open and dense set of M. On each non-empty connected component of M_A the multiplicities of the distinct principal curvatures are constant.
The main idea of the proof is to show that $\operatorname{grad} f = 0$ on every connected component of M_A and, using the density of M_A, it will imply that $\operatorname{grad} f = 0$ on M, i.e. f is constant.
Since the case when $\ell \leq 2$ was proved, we have to study only the case $\ell = 3$.
We consider a connected component of M_A such that each of its points has exactly three distinct principal curvatures, k_1, k_2 and k_3 with the multiplicities m_1, m_2 and m_3, respectively. Assume by way of contradiction that $\operatorname{grad} f \neq 0$ at each point of an open subset of this component.

Using Gauss and Codazzi equations, Fu arrives at two non-trivial polynomial equations along a specific integral curve γ. These equations take the forms

$$\sum_{i=0}^{9} a_{i,9-i} k_2^i f^{9-i} + c\left(\sum_{i=0}^{7} a_{i,7-i} k_2^i f^{7-i} + \sum_{i=0}^{5} a_{i,5-i} k_2^i f^{5-i} + \sum_{i=0}^{3} a_{i,3-i} k_2^i f^{3-i}\right) = 0$$

and

$$\sum_{i=0}^{12} b_{i,12-i} k_2^i f^{12-i} + c\left(\sum_{i=0}^{10} b_{i,10-i} k_2^i f^{10-i} + \sum_{i=0}^{8} b_{i,8-i} k_2^i f^{8-i} + \sum_{i=0}^{6} b_{i,6-i} k_2^i f^{6-i} + \sum_{i=0}^{4} b_{i,4-i} k_2^i f^{4-i}\right) = 0,$$

where $a_{i,j}$, $b_{i,j}$ represent constant coefficients depending on c, m and $r = m_1 + m_2$.

Using these two polynomial equations, Fu indicated a method to obtain a polynomial equation in the variable f with constant coefficients and claimed that this equation is non-trivial.
Using Mathematica we showed that this claim is not true at least for $m = 7$ and $r = 4$, that is for these values of m and r the above polynomial is the zero-polynomial.
Because the method pointed out by Fu yields high degree polynomials, we cannot determine explicit expressions for its coefficients. Even using Mathematica, this task is basically impossible.

Because of this, we changed the strategy and we determined two new polynomials of lower degrees. Then, we computed their resultant.

More precisely, we reduce the degrees of the previous polynomials found by Fu dividing the first one by f^3 and the second one by f^4. Denoting $z = k_2/f$, we obtain two new polynomials of lower degrees

$$\sum_{i=0}^{9} a_{i,9-i} z^i f^6 + c \left(\sum_{i=0}^{7} a_{i,7-i} z^i f^4 + \sum_{i=0}^{5} a_{i,5-i} z^i f^2 + \sum_{i=0}^{3} a_{i,3-i} z^i \right) = 0$$

and

$$\sum_{i=0}^{12} b_{i,12-i} z^i f^8 + c \left(\sum_{i=0}^{10} b_{i,10-i} z^i f^6 + \sum_{i=0}^{8} b_{i,8-i} z^i f^4 + \sum_{i=0}^{6} b_{i,6-i} z^i f^2 + \sum_{i=0}^{4} b_{i,4-i} z^i \right) = 0.$$

Using Mathematica we computed the resultant of these polynomials with respect to f in the general case and obtained a 40^{th}-degree polynomial in the variable z with constant coefficients depending on c, m and r. Next, we proved that it vanishes only when $m = 7$ and $r = 4$.

Now, in order to conclude, we need the following lemma.

Lemma

Along γ, the ratio $z = k_2/f$ cannot be constant.

Now, if $m \neq 7$ or $r \neq 4$, the previous resultant is a non-zero polynomial in the variable $z = k_2/f$ with constant coefficients, which implies that z is a constant, contradiction.

The case when $m = 7$ and $r = 4$ can be solved separately in a different way.

Now we conclude that any proper biharmonic hypersurface in a space form with at most three distinct principal curvatures is CMC.

Thank you!