Classification of the biharmonic quadratic maps between spheres
Diferential Geometry Workshop

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September 7^{th}, 2023

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din IAŞI

- Biharmonic homogeneous polynomial maps between spheres
  Results in Mathematics (2023)
- The energy density of biharmonic quadratic maps between spheres
  https://arxiv.org/abs/2303.08905

1. Introduction
    - A short introduction to harmonic maps
    - Harmonic homogeneous polynomial maps between spheres
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Conventions

We use the following sign conventions for the rough Laplacian, that acts on the set $C\left(\phi^{-1}TN\right)$ of all sections of the pull-back bundle $\phi^{-1}TN$, and for the curvature tensor field

$$\Delta^{\phi}\sigma = -\mathrm{trace}_g \nabla^2\sigma, \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_Y Z - \nabla_{[X,Y]}Z.$$

Also, by $\mathbb{S}^m(r)$ we indicate the m-dimensional Euclidean sphere of radius r. When $r = 1$, we write $\mathbb{S}^m$ instead of $\mathbb{S}^m(1)$.

Harmonic maps

- Let $(M, g = (g_{ij}))$ and $(N, h = (h_{\alpha\beta}))$ be Riemannian manifolds.
- Let $\phi : M \to N$ be a smooth map.
- We define the *energy functional* $E : C^\infty(M, N) \to \mathbb{R}$ by

$$E(\phi) = \frac{1}{2} \int_M |\mathrm{d}\phi|^2 \ v_g = \frac{1}{2} \int_M g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} h_{\alpha\beta}(\phi) \ v_g .$$

- $E(\phi)$ is invariant under conformal transformations on M if $\dim M = 2$.

Harmonic maps

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- $E(\phi)$ is invariant under conformal transformations on M if $\dim M = 2$.
- *Harmonic maps* are the critical points of E. They are characterised by the equation

$$\tau(\phi) := \operatorname{trace}_g \nabla^\phi \mathrm{d}\phi = 0, \quad \tau(\phi) \in \mathcal{C}\left(\phi^{-1}TN\right),$$

  where ∇^ϕ represents the connection on $\phi^{-1}TN$.
- In terms of local coordinates we have

$$-\Delta_M \phi^\alpha + g^{ij}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\Gamma^\alpha_{\beta\gamma}(\phi) = 0.$$

Existence of harmonic maps

- Working with the L^2-gradient flow

$$\frac{\partial \phi_t}{\partial t} = \tau(\phi_t), \quad \phi(\cdot, 0) = \phi_0. \tag{2.1}$$

Theorem 2.1 (Eells - Sampson, 1964)

Let M and N be closed Riemannian manifolds and assume that the sectional curvature of N is non-positive. Then Equation (2.1) has a unique smooth solution $\phi_t \in C^\infty(M \times [0, \infty), N)$ for arbitrary $\phi_0 \in C^\infty(M, N)$ which for $t \to \infty$, converges to a harmonic map $\phi_\infty \in C^\infty(M, N)$ in $C^2(M, N)$.

Harmonic homogeneous polynomial maps between spheres

Proposition 2.1

Let $\varphi : M^m \to \mathbb{S}^n$ be an arbitrary smooth map and write $\Phi = \mathrm{i} \circ \varphi$, where $\mathrm{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the inclusion map. Then φ is harmonic if and only if

$$\Delta \Phi = \nu \Phi,$$

where ν is a smooth function. Moreover, in this case, $\nu = |\mathrm{d}\Phi|^2 = |\mathrm{d}\varphi|^2$.

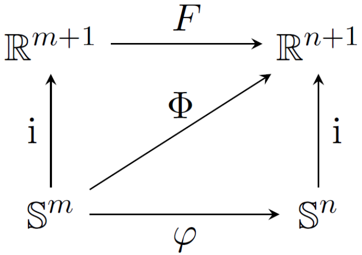
Corollary 2.1

Let $\varphi : M \to \mathbb{S}^n$ be a smooth map with constant energy density $e(\varphi) = (1/2)\,|\mathrm{d}\varphi|^2$. Then φ is harmonic if and only if Φ is an eigenmap with $\nu = 2e(\varphi)$.

- *Eigenmaps.* We call a smooth map $\varphi : M \to \mathbb{S}^n$ an eigenmap if the components of $\Phi = \mathrm{i} \circ \varphi : M \to \mathbb{R}^{m+1}$ are all eigenfunctions of the Laplacian on M with the same eigenvalue. An eigenmap φ is a harmonic map with constant energy density.

- *Spherical harmonics.* Suppose that $\tilde{f} : \mathbb{R}^{m+1} \to \mathbb{R}$ is a harmonic homogeneous polynomial of degree $k \in \mathbb{N}$. Then, the restriction $f = \tilde{f}_{|\mathbb{S}^m}$ is an eigenfunction of the Laplacian Δ^{S^m} on the sphere with an eigenvalue $\lambda_k = k(k + m - 1)$. Such a function f is called a spherical harmonic of order k.

- Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ be a vector valued function such that each component is a homogeneous polynomial of degree k. We will assume that $F(\mathbb{S}^m) \subset \mathbb{S}^n$. Such a map F is called *form of degree* k. When $k = 2$, F is called a *quadratic form*. We will keep the same terminology for φ.



Proposition 2.2

Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ *be a harmonic form of degree* $k \in \mathbb{N}^*$. *Suppose that* F *restricts to the map* $\varphi : \mathbb{S}^m \to \mathbb{S}^n$. *Then* φ *is harmonic with constant energy density* $e(\varphi) = k(k+m-1)/2$, *i.e.* φ *is an eigenmap with* $\nu = k(k+m-1)$.

Proposition 2.2

Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ be a harmonic form of degree $k \in \mathbb{N}^*$. Suppose that F restricts to the map $\varphi : \mathbb{S}^m \to \mathbb{S}^n$. Then φ is harmonic with constant energy density $e(\varphi) = k(k+m-1)/2$, i.e. φ is an eigenmap with $\nu = k(k+m-1)$.

We determine all eigenmaps between spheres.

Proposition 2.3

Let $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ be a harmonic map with constant energy density $e(\varphi) = \alpha > 0$. Then there exists a unique $k \in \mathbb{N}^*$ such that $\alpha = k(m+k-1)/2$ and there exists a unique vector valued function $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ such that each component is either a harmonic homogeneous polynomial of degree k, or the null polynomial, and F restricts to φ.

The special case of quadratic forms ($k = 2$)

Proposition 2.4

Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ *be a quadratic form. Suppose that* F *restricts to* $\varphi : \mathbb{S}^m \to \mathbb{S}^n$*. Then, the following are equivalent*

i) $\tau(\varphi) = 0$,
ii) $\overset{o}{\Delta} F = 0$,
iii) $e(\varphi) = m + 1$.

Biharmonic maps

- Let $(M, g = (g_{ij}))$ and $(N, h = (h_{\alpha\beta}))$ be Riemannian manifolds.
- For $\phi : M \to N$ consider the *bienergy*

$$E_2(\phi) = \int_M |\tau(\phi)|^2 v_g.$$

- Critical points of E_2 are called (intrinsic) *biharmonic maps* and are characterized by the fourth order non-linear elliptic equation

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - R^N \left(\mathrm{d}\phi(e_i), \tau(\phi)\right) \mathrm{d}\phi(e_i) = 0,$$

where Δ^ϕ is the rough Laplacian on $\phi^{-1}TN$.

- Every harmonic map is biharmonic (in the compact case, a minimizer for E_2); a non-harmonic biharmonic map is called *proper biharmonic*.

Examples

- Any polynomial map of degree at most 3 between Euclidean spaces.
- The Almansi Property provides a method for constructing proper biharmonic maps by using harmonic ones. The Almansi property states that if $f : \mathbb{R}^m \to \mathbb{R}$ is harmonic, then the product function $r^2 f : \mathbb{R}^m \to \mathbb{R}$ is proper biharmonic, i.e.

$$\Delta f = 0 \Rightarrow \Delta^2 \left(r^2 f\right) = 0.$$

Here $r : \mathbb{R}^m \to \mathbb{R}$ denotes the distance function from the origin defined by

$$r\left(x^1, x^2, \ldots, x^m\right) = \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2 + \cdots + \left(x^m\right)^2}.$$

Biharmonic maps into spheres

Theorem 3.1 (Loubeau - Oniciuc, 2007)

Let M be a compact manifold and consider $\psi : M \to \mathbb{S}^n(r/\sqrt{2})$ a nonconstant map, where $\mathbb{S}^n(r/\sqrt{2})$ is a small hypersphere of radius $r/\sqrt{2}$ of $\mathbb{S}^{n+1}(r)$. The map $\varphi = \mathrm{i} \circ \psi : M \to \mathbb{S}^{n+1}(r)$, where i is the canonical inclusion, is proper biharmonic if and only if ψ is harmonic and the energy density $e(\psi)$ is constant.

Remark. We need compactness only for the direct implication.

Example

We can consider the map $\psi : \mathbb{S}^3(\sqrt{2}) \to \mathbb{S}^2(1/\sqrt{2})$ as being the classical Hopf map and then by composing with the inclusion of $\mathbb{S}^2(1/\sqrt{2})$ into $\mathbb{S}^3$, we obtain that

$$\varphi : \mathbb{S}^3(\sqrt{2}) = \left\{(z^1, z^2) \in \mathbb{C}^2 \, : \, \left|z^1\right|^2 + \left|z^2\right|^2 = 2\right\} \to \mathbb{S}^3$$

given by

$$\begin{aligned} \varphi(z^1, z^2) =& \frac{1}{2\sqrt{2}} \left(2z^1\overline{z^2}, \left|z^1\right|^2 - \left|z^2\right|^2, 2\right) \\ =& \frac{1}{2\sqrt{2}} \left(2z^1\overline{z^2}, \left|z^1\right|^2 - \left|z^2\right|^2, \left|z^1\right|^2 + \left|z^2\right|^2\right) \end{aligned} \tag{3.1}$$

is a proper biharmonic map (see [22]). As homothetic changes of the domain or codomain metrics preserves the harmoncity and biharmonicity, we can assume that φ maps $\mathbb{S}^3$ into $\mathbb{S}^3$. We also note that the components of φ are (restrictions of) homogeneous polynomials of degree 2.

Theorem 3.2 (Oniciuc - Ou, 2018)

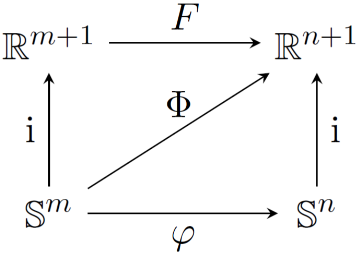
Let $\varphi : (M^m, g) \to \mathbb{S}^n$ *be a map and let* $\mathrm{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ *be the standard isometric embedding. Then,* φ *is a biharmonic map if and only if the vector function* $\Phi = \mathrm{i} \circ \varphi : (M^m, g) \to \mathbb{R}^{n+1}$ *solves the following PDE*

$$\begin{aligned} &\tau_2(\Phi) + 2|\mathrm{d}\Phi|^2 \tau(\Phi) \\ &+ \left(-\Delta|\mathrm{d}\Phi|^2 + 2\mathrm{div}\theta^\sharp - |\tau(\Phi)|^2 + 2|\mathrm{d}\Phi|^4\right)\Phi + 2\mathrm{d}\Phi\left(\mathrm{grad}|\mathrm{d}\Phi|^2\right) = 0. \end{aligned} \tag{3.2}$$

We denoted $\theta = \langle \mathrm{d}\Phi, \tau(\Phi) \rangle = \langle \mathrm{d}\varphi, \tau(\varphi) \rangle$.

Biharmonic homogeneous polynomial maps between spheres

Next, we give an application of Theorem 3.2 for a particular class of maps. Consider the diagram below



where $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ is a form of degree k. As usual, we assume that φ is not constant.

Theorem 3.3

The bitension field of the map φ is given by

$$
\begin{aligned}
\tau_2(\varphi) =& \overset{o}{\Delta}\overset{o}{\Delta} F + 2\left(mk + 2k^2 - 3k - m + 3 - \left|\overset{o}{\mathrm{d}} F\right|^2\right)\overset{o}{\Delta} F \\
&+ \left(-2\overset{o}{\Delta}\left(\left|\overset{o}{\mathrm{d}} F\right|^2\right) - 2\left|\overset{o}{\nabla}\overset{o}{\mathrm{d}} F\right|^2 + \left|\overset{o}{\Delta} F\right|^2 + 2\left|\overset{o}{\mathrm{d}} F\right|^4\right. \\
&\left. -2\left(2mk + 6k^2 - 6k - m + 3\right)\left|\overset{o}{\mathrm{d}} F\right|^2 + 4k^2(m + 2k - 1)\right)\Phi \\
&+ 2\overset{o}{\mathrm{d}} F\left(\overset{o}{\mathrm{grad}}\left(\left|\overset{o}{\mathrm{d}} F\right|^2\right)\right),
\end{aligned} \tag{3.3}
$$

where $\overset{o}{\mathrm{d}}$, $\overset{o}{\nabla}$, $\overset{o}{\Delta}$ *and* $\overset{o}{\mathrm{grad}}$ *denote operators that act on* $\mathbb{R}^{m+1}$.

Biharmonic quadratic maps between spheres

- Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ be a quadratic form. Then, F can be written in the form

$$F(\overline{x}) = \left(X^t A_1 X, X^t A_2 X, \ldots, X^t A_{n+1}\ X\right),$$

where $\overline{x} = \left(x^1, x^2, \ldots, x^{m+1}\right)$ coresponds to $X^t = \begin{bmatrix} x^1 & x^2 & \ldots & x^{m+1} \end{bmatrix}$, and A_1, ..., A_{n+1} are square symmetric matrices of order $m+1$. Assume that if $|\overline{x}|= 1$, then $|F(\overline{x})|= 1$.

- Since φ is not a constant map, therefore there exist $i_0 \in \{1, 2, \ldots, n+1\}$ such that A_{i_0} is not I_{m+1} multiplied by a non-zero real constant.

We obtain that on $\mathbb{R}^{m+1}$

$$\left|\overset{o}{\mathrm{d}}F(\overline{x})\right|^2 = 4X^t\left(A_1^2 + A_2^2 + \cdots + A_{n+1}^2\right)X = 4X^tSX,$$

$$\overset{o}{\Delta}F = -\left(2\mathrm{tr}A_1, 2\mathrm{tr}A_2, \ldots, 2\mathrm{tr}A_{n+1}\right),$$

$$\overset{o}{\Delta}\left(\left|\overset{o}{\mathrm{d}}F\right|^2\right) = -8\mathrm{tr}\left(A_1^2 + A_2^2 + \cdots + A_{n+1}^2\right) = -8\mathrm{tr}S, \tag{3.4}$$

$$\left|\overset{o}{\nabla}\overset{o}{\mathrm{d}}F\right|^2 = 4\left(|A_1|^2 + |A_2|^2 + \cdots + |A_{n+1}|^2\right),$$

$$\overset{o}{\mathrm{grad}}\left(\left|\overset{o}{\mathrm{d}}F\right|^2\right) = 8X^t\left(A_1^2 + A_2^2 + \cdots + A_{n+1}^2\right) = 8X^tS,$$

$$\overset{o}{\mathrm{d}}F\left(\overset{o}{\mathrm{grad}}\left(\left|\overset{o}{\mathrm{d}}F\right|^2\right)\right) = 16\left(X^tA_1SX, X^tA_2SX, \ldots, X^tA_{n+1}SX\right).$$

- We observe that, since the matrices A_1, ..., A_{n+1} are symmetric, then

$$|A_1|^2 + \cdots + |A_{n+1}|^2 = \mathrm{tr} S.$$

- We note that, the condition $S = \alpha I_{m+1}$, where the real constant α has to be greater than 1, is equivalent to $|\mathrm{d}\varphi|^2$, or $\left|\overset{o}{\mathrm{d}}F\right|^2$ restricted to $\mathbb{S}^m$, is constant.
- Since F is a quadratic map, it follows that $\overset{o}{\Delta}F$ is constant on $\mathbb{R}^{m+1}$.

Proposition 3.1

Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ be an arbitrary quadratic form. Then, with the above notations,

$$8\mathrm{tr} S + \left|\overset{o}{\Delta}F\right|^2 = 4(m+1)(m+3). \tag{3.5}$$

Theorem 3.4 (R.A., Oniciuc, Ou, 2023)

Let $F:\mathbb{R}^{m+1}\to\mathbb{R}^{n+1}$ *be a quadratic form given by*

$$F(\overline{x})=\left(X^tA_1X,X^tA_2X,\ldots,X^tA_{n+1}\,X\right),$$

such that if $|\overline{x}|=1$ *then* $|F(\overline{x})|=1$*. We consider* $\varphi:\mathbb{S}^m\to\mathbb{S}^n$ *defined by* $\varphi(\overline{x})=F(\overline{x})$ *and* $\Phi=\mathrm{i}\circ\varphi:\mathbb{S}^m\to\mathbb{R}^{n+1}$*. Then, at a point* $\overline{x}\in\mathbb{S}^m$*, the bitension field of* φ *has the following expression*

$$\begin{aligned}\tau_2(\varphi)_{\overline{x}}=&-4\left(m+5-4X^tSX\right)\left(\mathrm{tr}A_1,\mathrm{tr}A_2,\ldots,\mathrm{tr}A_{n+1}\right)\\&+4\left((m+3)(m+5)-6(m+5)X^tSX+8\left(X^tSX\right)^2\right)\Phi(\overline{x})\\&+32\left(X^tA_1SX,X^tA_2SX,\ldots,X^tA_{n+1}SX\right).\end{aligned}\tag{3.6}$$

Proposition 3.2 (R.A., Oniciuc, Ou, 2023)

If the quadratic form φ has constant energy density, then φ is proper biharmonic if and only if we have

$$e(\varphi) = \frac{m+1}{2}. \tag{3.7}$$

Proof. Since the map φ is not harmonic and has constant energy density, it follows that $\overset{o}{\Delta} F \neq 0$ and $S = \alpha I_{m+1}$, for some $\alpha > 1$. Using Equation (3.6), we immediately obtain

$$\tau_2(\varphi)_{\overline{x}} = 8\left(\frac{m+5}{4} - \alpha\right)\overset{o}{\Delta} F + 32\left(\alpha - \frac{m+5}{4}\right)\left(\alpha - \frac{m+3}{2}\right)\Phi(\overline{x})$$

The conclusion follows.

Theorem 3.5 (R.A.,Oniciuc, Ou, 2023)

Up to orthogonal transformations of the domain and/or the codomain, the only proper biharmonic quadratic form $\varphi : \mathbb{S}^1 \to \mathbb{S}^n$, $n \geq 2$, is obtained from the restriction of the quadratic form $F : \mathbb{R}^2 \to \mathbb{R}^{n+1}$, given by

$$F(x, y) = \left(x^2, c^1 y^2 + 2\gamma^1 xy, \dots, c^n y^2 + 2\gamma^n xy\right),$$

such that

$$\left(c^1\right)^2 + \cdots + \left(c^n\right)^2 = 1, \quad c^1\gamma^1 + \cdots + c^n\gamma^n = 0$$

and

$$\left(\gamma^1\right)^2 + \cdots + \left(\gamma^n\right)^2 = \frac{1}{2}.$$

Moreover, the image of φ is the circle of radius $1/\sqrt{2}$ of $\mathbb{S}^n$.

Theorem 3.6 (R.A., Oniciuc, Ou, 2023)

There are no proper biharmonic quadratic forms from $\mathbb{S}^m$ *to* $\mathbb{S}^2$, $m \geq 2$.

Theorem 3.7 (R.A., Oniciuc, Ou, 2023)

Up to homothetic transformations of the domain and/or codomain, the only proper biharmonic quadratic form from $\mathbb{S}^m$ *to* $\mathbb{S}^3$, $m \geq 2$, *is the Hopf fibration* $\psi : \mathbb{S}^3 \to \mathbb{S}^2$ *followed by the inclusion, as described in example* (3.1).

Open Problem

All results obtained in the first paper[1] suggested the following

Open Problem. If $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ is a proper biharmonic quadratic form then, up to an isometry of $\mathbb{S}^n$, the first n components of φ are harmonic polynomials on $\mathbb{R}^{m+1}$ and form a map $\psi : \mathbb{S}^m \to \mathbb{S}^{n-1}(1/\sqrt{2})$.

Using the results presented above, we can give a positive answer to this problem.

[1] R. A., C. Oniciuc, Y.-L. Ou, *Biharmonic homogeneous polynomial maps between spheres*, Results Math. 78 (2023), no. 4, Paper No. 159.

Recall that if a quadratic form φ has constant energy density, then φ is proper biharmonic if and only if we have $e(\varphi) = (m+1)/2$.

[2] G. Toth, *Quadratic Eigenmaps between Spheres*, Geometriae Dedicata, 56 (1995), 35–52

Recall that if a quadratic form φ has constant energy density, then φ is proper biharmonic if and only if we have $e(\varphi) = (m+1)/2$.

Theorem 4.1 (R.A., Oniciuc, 2023)

Let $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ be a quadratic form. Then φ is proper biharmonic if and only if $e(\varphi) = (m+1)/2$.

[2] G. Toth, *Quadratic Eigenmaps between Spheres*, Geometriae Dedicata, 56 (1995), 35–52

Recall that if a quadratic form φ has constant energy density, then φ is proper biharmonic if and only if we have $e(\varphi) = (m+1)/2$.

Theorem 4.1 (R.A., Oniciuc, 2023)

Let $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ be a quadratic form. Then φ is proper biharmonic if and only if $e(\varphi) = (m+1)/2$.

Proof. By using the standard coordinates, any quadratic form $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ can be written as

$$F(\overline{x}) = \sum_{i=1}^{m+1} \overline{a}_i \left(x^i\right)^2 + \sum_{1 \le i < j \le m+1} \overline{a}_{ij} x^i x^j,$$

where $\overline{a}_i \in \mathbb{R}^{n+1}$, for $i = 1, \ldots, m+1$, and $\overline{a}_{ij} \in \mathbb{R}^{n+1}$, for $1 \le i < j \le m+1$ satisfy 5 conditions (see [2]).

[2] G. Toth, *Quadratic Eigenmaps between Spheres*, Geometriae Dedicata, 56 (1995), 35–52

Proof

We transform the non-homogeneous polynomial map $\tau_2(\varphi)$ from Equation (3.6) into a homogeneous polynomial map of degree 6 because it is well known that if a homogeneous polynomial vanishes on the sphere $\mathbb{S}^m$, then it vanishes on $\mathbb{R}^{m+1}$. Thus, we obtain

$$
\begin{aligned}
&-4\left|\overline{x}\right|^4\left((m+5)|\overline{x}|^2-4X^tSX\right)\left(\mathrm{tr}A_1,\mathrm{tr}A_2,\ldots,\mathrm{tr}A_{n+1}\right) \\
&+4\left((m+3)(m+5)|\overline{x}|^4-6(m+5)|\overline{x}|^2X^tSX+8\left(X^tSX\right)^2\right)F(\overline{x}) \\
&+32|\overline{x}|^4\left(X^tA_1SX,X^tA_2SX,\ldots,X^tA_{n+1}SX\right)=\overline{0},\quad \text{on } \mathbb{R}^{m+1}
\end{aligned} \tag{4.1}
$$

The matrix S defines a quadratic map. We perform an orthogonal change of the domain variables x^1, x^2, ..., x^{m+1} which brings S in diagonal form, $S=(s_i)_{1\leq i\leq m+1}$. We analyse the coefficient list for each component of the above homogeneous polynomial equation.

Proof

For any $i \in \{1, 2, \ldots, n+1\}$ we notice that the coefficient of $(x^k)^6$, which has to vanish, gives

$$4\left(5 + m - 4s_k\right)\left(a^i_k\left(3 + m - 2s_k\right) - \mathrm{tr}A_i\right) = 0, \quad \forall k \in \{1, 2, \ldots, m+1\}, \tag{4.2}$$

Thus, for any k arbitrarily fixed, we have either

$$s_k = \frac{m+5}{4},$$

or

$$a^i_k\left(3 + m - 2s_k\right) - \mathrm{tr}A_i = 0, \quad \forall i \in \{1, 2, \ldots, n+1\}.$$

Corollary 4.1

Let $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ be a proper biharmonic quadratic form. Then $\left|\overset{o}{\Delta} F\right|^2 = 2(m+1)^2$.

The answer to the open problem

Theorem 4.2 (R.A., Oniciuc, 2023)

If $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ *is a proper biharmonic quadratic form then, up to an isometry of* $\mathbb{S}^n$*, the first n components of* φ *are harmonic polynomials on* $\mathbb{R}^{m+1}$ *and form a map* $\psi : \mathbb{S}^m \to \mathbb{S}^{n-1}(1/\sqrt{2})$*.*

Applications

Using the result of Calabi concerning the uniqueness of compact minimal 2-dimensional round spheres in $\mathbb{S}^n$, i.e. the uniqueness of the Boruvka spheres (see [3] and also [4] and [5]), we obtain

Theorem 4.3

Let $\varphi : \mathbb{S}^2 \to \mathbb{S}^n$ be a full quadratic map. Assume that φ is homothetic. Then φ is proper biharmonic if and only if $n = 5$, $\varphi\left(\mathbb{S}^2\right) \subset \mathbb{S}^4\left(1/\sqrt{2}\right)$, and up to homothetic changes of domain and codomain, $\psi : \mathbb{S}^2 \to \mathbb{S}^4\left(1/\sqrt{2}\right)$ is the Veronese map.

[3] E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geometry 1 (1967), 111–125

[4] R.L. Bryant, *Minimal surfaces of constant curvature in S^n*, Trans. Amer. Math. Soc. 290 (1985), no. 1, 259–271

[5] K. Kenmotsu, *Minimal surfaces with constant curvature in 4-dimensional space forms*, Proc. Amer. Math. Soc. 89 (1983), no. 1, 133–138

Theorem 4.4 (Toth, 1987)

Full quadratic harmonic maps of $\mathbb{S}^3$ into $\mathbb{S}^n$ exist only if $2 \leq n \leq 8$ and $n \neq 3$. Moreover, if $\varphi : \mathbb{S}^3 \to \mathbb{S}^n$ is such a map, then there exist $U \in O(4)$, $V \in O(n+1)$ and a symmetric positive definite matrix $B \in \mathbb{S}^2\left(\mathbb{R}^{n+1}\right)$ such that

$$V \circ \varphi \circ U = B \circ \varphi_n,$$

where $\varphi_n : \mathbb{S}^3 \to \mathbb{S}^n$ is defined by

$$\varphi_n(\overline{x}) = \begin{cases} \left(\left(x^1\right)^2 + \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2, 2\left(x^1x^3 - x^2x^4\right), 2\left(x^1x^4 + x^2x^3\right)\right), & n = 2 \\ \left(\left(x^1\right)^2 + \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2, 2x^1x^3, 2x^1x^4, 2x^2x^3, 2x^2x^4\right), & n = 4 \\ \left(\left(x^1\right)^2 - \left(x^2\right)^2, \left(x^3\right)^2 - \left(x^4\right)^2, 2x^1x^2, \sqrt{2}\left(x^1x^3 + x^2x^4\right), \right. & \\ \left. \sqrt{2}\left(x^2x^3 - x^1x^4\right), 2x^3x^4\right), & n = 5 \\ \left(\frac{1}{\sqrt{2}}\left(\left(x^1\right)^2 + \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right), \frac{1}{\sqrt{2}}\left(\left(x^1\right)^2 - \left(x^2\right)^2\right), \right. & \\ \frac{1}{\sqrt{2}}\left(\left(x^3\right)^2 - \left(x^4\right)^2\right), \sqrt{2}x^1x^2, \sqrt{3}\left(x^1x^3 + x^2x^4\right), & \\ \left. \sqrt{3}\left(x^2x^3 - x^1x^4\right), \sqrt{2}x^3x^4\right), & n = 6 \\ \left(\left(x^1\right)^2 - \left(x^2\right)^2, \left(x^3\right)^2 - \left(x^4\right)^2, 2x^1x^2, \sqrt{2}x^1x^3, \sqrt{2}x^1x^4, \right. & \\ \left. \sqrt{2}x^2x^3, \sqrt{2}x^2x^4, 2x^3x^4\right), & n = 7 \\ \varphi_{\lambda_2}\left(x^1, x^2, x^3, x^4\right), (\varphi_{\lambda_2} = \text{ a standard minimal immersion}) & n = 8 \end{cases}$$

Proposition 4.1

Full quadratic proper biharmonic maps of $\mathbb{S}^3$ *into* $\mathbb{S}^n$ *exist only if* $3 \leq n \leq 9$ *and* $n \neq 4$. *Moreover, if* $\varphi : \mathbb{S}^3 \to \mathbb{S}^n$ *is such a map, then there exist* $U \in O(4)$, $V \in O(n+1)$ *and a symmetric positive definite matrix* $B \in \mathbb{S}^2\left(\mathbb{R}^{n+1}\right)$ *such that*

$$V \circ \varphi \circ U = B \circ \left(\frac{1}{\sqrt{2}} \varphi_n, \frac{1}{\sqrt{2}} \right).$$

Also,

Proposition 4.2

There is no full quadratic proper biharmonic maps of $\mathbb{S}^3$ *into* $\mathbb{S}^4$.

Thank You!

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