Spacelike hypersurfaces in the light cone of the Lorentz-Minkowski spacetime

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The results I am going to introduce in this talk have been obtained in collaboration with the following colleagues:

- ⋆ Verónica L. Cánovas, from Universidad de Murcia (Spain).
- ⋆ Marco Rigoli, from Università degli Studi di Milano (Italy).

- They where motivated by previous work by Palomo and Romero (On spacelike surfaces in four-dimensional Lorentz-Minkowski spacetime through a light cone, Proc. R. Soc. Edinb. A 143 (2013), 881–892)
- Our results can be found in the following paper: Codimension two spacelike submanifolds of the Lorentz-Minkowski spacetime into the light cone, Proc. R. Soc. Edinb. A 149 (2019), 1523–1553.
- They were also part of Veronica's PhD thesis.

The Lorentz-Minkowski spacetime

- Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz-Minkowski space, endowed with the Lorentzian metric

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n+2})^2, \quad x = (x_1, x_2 \ldots, x_{n+2})$$

- Consider on $\mathbb{L}^{n+2}$ the time-orientation induced by the globally defined unit timelike vector field

$$\boldsymbol{e}_1 = (1, 0, \ldots, 0).$$

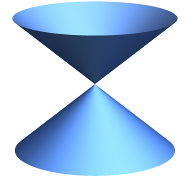
- Let Σ be a codimension-two spacelike submanifold immersed in $\mathbb{L}^{n+2}$. That is, Σ is an n-dimensional connected manifold admitting a smooth immersion $\psi : \Sigma \to \mathbb{L}^{n+2}$ such that the induced metric on Σ is Riemannian.
- In this talk, we are interested in the case where Σ is contained into the light cone of $\mathbb{L}^{n+2}$.

The light cone of the Lorentz-Minkowski space

Light cone of the Lorentz-Minkowski space

The light cone in $\mathbb{L}^{n+2}$ is the degenerate hyperquadric

$$\Lambda = \{x = (x_1, \ldots, x_{n+2}) \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x \neq \mathbf{0}\}.$$



- Geometrically, Λ corresponds to the subset of all points of the Lorentz-Minkowski space which can be reached from the origin **0** through a null geodesic starting at **0**.
- The future component of Λ is

$$\Lambda^+ = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0,\ x_1 > 0\}.$$

Spacelike hypersurfaces of the light cone

- Let $\psi : \Sigma^n \to \mathbb{L}^{n+2}$ be a codimension-two spacelike submanifold and assume that $\psi(\Sigma)$ is contained into the future connected component of the light cone. When this happens, we will refer to Σ as a spacelike hypersurface of the light cone Λ^+.
- In this case, there always exists a globally defined future-pointing normal null frame $\{\boldsymbol{k}_+, \boldsymbol{k}_-\}$ on Σ.
- Define the positive function $u : \Sigma \to (0, +\infty)$ by

$$u = -\langle \psi, \boldsymbol{e}_1 \rangle = \psi_1 > 0.$$

Future-pointing normal null frame for a hypersurface of the light cone

In these conditions

$$\boldsymbol{k}_+ = \psi \quad \text{and} \quad \boldsymbol{k}_- = -\frac{1 + \|\nabla u\|^2}{2u^2}\boldsymbol{k}_+ + \frac{1}{u}\boldsymbol{e}_1^\perp$$

gives two future-pointing null normal vector fields globally defined on Σ with $\langle \boldsymbol{k}_+, \boldsymbol{k}_+ \rangle = \langle \boldsymbol{k}_-, \boldsymbol{k}_- \rangle = 0$ and $\langle \boldsymbol{k}_+, \boldsymbol{k}_- \rangle = -1$, where

$$\boldsymbol{e}_1 = \boldsymbol{e}_1^\top(p) + \boldsymbol{e}_1^\perp(p), \quad p \in \Sigma.$$

The second fundamental form

Second fundamental form and mean curvature vector

Let $\mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^{\perp}(\Sigma)$ be the vector valued second fundamental form of the submanifold, that is the symmetric tensor

$$\mathrm{II}(X, Y) = -(\overline{\nabla}_X Y)^{\perp}.$$

The mean curvature vector field of Σ is $\boldsymbol{H} = \frac{1}{n}\mathrm{trace}(\mathrm{II}) \in \mathfrak{X}^{\perp}(\Sigma)$.

- As usual in relativity, we may decompose the second fundamental form into two scalar valued null second fundamental forms, the Weingarten (or shape) operators asociated to $\boldsymbol{k}_+$ and $\boldsymbol{k}_-$.
- That is, the symmetric operators $A_{\boldsymbol{k}_+}, A_{\boldsymbol{k}_-} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by

$$\langle A_{\boldsymbol{k}_+} X, Y \rangle = \langle \mathrm{II}(X, Y), \boldsymbol{k}_+ \rangle, \text{ and } \langle A_{\boldsymbol{k}_-} X, Y \rangle = \langle \mathrm{II}(X, Y), \boldsymbol{k}_- \rangle.$$

- Therefore, in terms of $\{\boldsymbol{k}_+, \boldsymbol{k}_-\}$ we have $\boldsymbol{H} = -\theta_- \boldsymbol{k}_+ - \theta_+ \boldsymbol{k}_-$, where

$$\theta_+ = \frac{1}{n}\mathrm{trace}(A_{\boldsymbol{k}_+}) \quad \text{and} \quad \theta_- = \frac{1}{n}\mathrm{trace}(A_{\boldsymbol{k}_-})$$

  define the null mean curvatures (or null expansion scalars) of Σ.

Mean curvature vector for a hypersurface of the light cone

Null shape operators for a submanifold into the light cone

The corresponding null second forms associated to the global null frame $\{\boldsymbol{k}_+, \boldsymbol{k}_-\}$ are given by

$$A_{\boldsymbol{k}_+} = I \qquad \text{and} \qquad A_{\boldsymbol{k}_-} = -\frac{1 + \|\nabla u\|^2}{2u^2} I + \frac{1}{u}\nabla^2 u,$$

where $\nabla^2 u$ is the Hessian operator of u.

- In particular, the null expansions are

$$\theta_+ = \frac{1}{n}\mathrm{tr}(A_{\boldsymbol{k}_+}) = 1 > 0$$

and

$$\theta_- = \frac{1}{n}\mathrm{tr}(A_{\boldsymbol{k}_-}) = \frac{2u\Delta u - n(1 + \|\nabla u\|^2)}{2nu^2},$$

where Δu is the Laplacian of u.

- Therefore, the mean curvature vector field of Σ is given by

$$\boldsymbol{H} = -\frac{2u\Delta u - n(1 + \|\nabla u\|^2)}{2nu^2}\boldsymbol{k}_+ - \boldsymbol{k}_-.$$

Totally umbilical hypersurfaces in Λ^+

- As a first main application of our approach, we derive a classification of the spacelike hypersurfaces of the light cone which are totally umbilical in $\mathbb{L}^{n+2}$.
- Recall that a submanifold Σ is said to be totally umbilical if it is umbilical with respect to all possible normal directions $\zeta \in \mathfrak{X}^\perp(\Sigma)$.
- Let $\psi : \Sigma \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a spacelike hypersurface and consider $\{\boldsymbol{k}_+, \boldsymbol{k}_-\}$ the associated normal null frame.
- We already know that $A_{\boldsymbol{k}_+} = I$, so that Σ is totally umbilical if and only if it is umbilical with respect to the normal direction $\boldsymbol{k}_-$.
- That is, if and only if there exists a smooth function λ such that $A_{\boldsymbol{k}_-} = \lambda I$, where $I =$ identity.
- A standard computation using Codazzi equation implies that λ is constant.
- Define $\boldsymbol{Q} = -\boldsymbol{k}_- + \lambda \boldsymbol{k}_+$. Then for every $X \in \mathfrak{X}(\Sigma)$

$$\overline{\nabla}_X \boldsymbol{Q} = -A_{\boldsymbol{k}_-} X + \lambda A_{\boldsymbol{k}_+} X = -\lambda X + \lambda X = 0,$$

which implies that $\boldsymbol{Q} \in \mathbb{L}^{n+2}$ is a constant vector, $\boldsymbol{Q} \neq 0$ with $\langle \boldsymbol{Q}, \boldsymbol{Q} \rangle = 2\lambda$.

- If $\lambda \neq 0$, define $\tau = 1/\sqrt{2|\lambda|} > 0$ and let $\boldsymbol{a} = \tau \boldsymbol{Q} \in \mathbb{L}^{n+2}$.
- Then $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = c = \pm 1$ and $\langle \psi, \boldsymbol{a} \rangle = \tau$, which means that

$$\psi(\Sigma) \subset \Sigma(\boldsymbol{a}, \tau) = \{x \in \Lambda^+ : \langle x, \boldsymbol{a} \rangle = \tau\}.$$

.

- On the other hand, if $\lambda = 0$, let $\boldsymbol{a} = \boldsymbol{Q} = -\boldsymbol{k}_- \in \mathbb{L}^{n+2}$.
- Then $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = c = 0$ and $\langle \psi, \boldsymbol{a} \rangle = 1$, which means that

$$\psi(\Sigma) \subset \Sigma(\boldsymbol{a}, 1) = \{x \in \Lambda^+ : \langle x, \boldsymbol{a} \rangle = 1\}.$$

Theorem 1

The only spacelike hypersurfaces of the light cone Λ^+ which are totally umbilical in $\mathbb{L}^{n+2}$ are open pieces of the following submanifolds

$$\Sigma(\boldsymbol{a}, \tau) = \{x \in \Lambda^+ : \langle x, \boldsymbol{a} \rangle = \tau\}$$

with $\boldsymbol{a} \in \mathbb{L}^{n+2}$, $\boldsymbol{a} \neq 0$, $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = c \in \{-1, 0, 1\}$, with $\tau \in (0, +\infty)$ if $c = \pm 1$ and $\tau = 1$ if $c = 0$.

- It is not difficult to see that if $c = 0$ then $\Sigma(\boldsymbol{a}, 1)$ is isometric to the flat Euclidean space $\mathbb{R}^n$.
- On the other hand, when $c = -1$, $\Sigma(\boldsymbol{a}, \tau)$ is isometric to the Euclidean sphere $\mathbb{S}^n(\tau)$ with constant sectional curvature $1/\tau^2$.
- Finally, when $c = 1$, $\Sigma(\boldsymbol{a}, \tau)$ is isometric to the hyperbolic space $\mathbb{H}^n(\tau)$ with constant sectional curvature $-1/\tau^2$.
- In particular, the only compact spacelike hypersurfaces of Λ^+ which are totally umbilical in $\mathbb{L}^{n+2}$ are the submanifolds $\Sigma(\boldsymbol{a}, \tau)$ with $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = -1$.

Compactness of hypersurfaces into the light cone

- We have the following compactness criteria for a spacelike hypersurface of the light cone, under an appropriate bound on the growth of the function u.

Proposition 1

Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a spacelike hypersurface of the light cone Λ^+. Assume that Σ is complete and that the positive function $u = -\langle \psi, \boldsymbol{e}_1 \rangle$ satisfies

$$u(p) \leq C\, r(p) \log(r(p)), \quad r(p) \gg 1$$

where C is a positive constant and r denotes the Riemannian distance function from a fixed origin $o \in \Sigma$. Then Σ is compact and conformally diffeomorphic to the sphere $\mathbb{S}^n$. In particular, this holds if $\sup_\Sigma u < +\infty$ and, more generally, if $\limsup_{r\to+\infty} \frac{u}{r\log(r)} < +\infty$.

- For the proof of Proposition 1 we need the following technical lemma about completeness of conformal metrics, which is a slight generalization of Lemma 5.2 in [AMc][2]

Lemma 1

Let g be a complete metric on a Riemannian manifold Σ and let r denote the Riemannian distance function from a fixed origin $o \in \Sigma$. If a function w satisfies

$$w^{2/(n-2)}(p) \geq \frac{C}{r(p)\log(r(p))}, \quad r(p) \gg 1,$$

with C a positive constant, then the conformal metric $\tilde{g} = w^{4/(n-2)}g$ is also complete.

- In [AMc] the result is stated under the stronger hypothesis

$$w^{2/(n-2)}(p) \geq \frac{C}{r(p)}, \quad r(p) \gg 1,$$

but the proof is similar under our weaker hypothesis.

[2][AMc] P. Aviles and R.C. McOwen, Conformal deformations to constant negative scalar curvature on noncompact Riemannian manifolds. JDG 27 (1988), 225–239

Proof of Proposition 1

- Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a spacelike hypersurface of Λ^+.
- Then $\psi(p) = (u(p), \psi_2(p), \ldots, \psi_{n+2}(p))$ with

$$\sum_{i=2}^{n+2} \psi_i^2(p) = u^2(p) > 0.$$

- Define the function $\Psi : \Sigma^n \to \mathbb{S}^n$ by

$$\Psi(p) = \frac{1}{u(p)}(\psi_2(p), \ldots, \psi_{n+2}(p)).$$

- Then we can see that

$$\langle d\Psi_p(\boldsymbol{v}), d\Psi_p(\boldsymbol{w})\rangle_0 = \frac{1}{u^2(p)}\langle \boldsymbol{v}, \boldsymbol{w}\rangle$$

for every $p \in \Sigma$ and $\boldsymbol{v}, \boldsymbol{w} \in T_p\Sigma$, where $\langle , \rangle_0$ denotes the standard metric of the round sphere.

- In particular, $\Psi : (\Sigma, \langle,\rangle) \rightarrow (\mathbb{S}^n, \langle,\rangle_0)$ is conformal map and a local diffeomorphism.
- Assume now that Σ is complete (that is, $\langle,\rangle$ is a complete Riemannian metric on Σ) and $u \leq Cr\log(r)$, $r \gg 1$.
- Therefore, by Lemma 1 applied to the function $w = u^{-(n-2)/2}$, we know that the conformal metric $\widetilde{\langle,\rangle} = \frac{1}{u^2}\langle,\rangle$ is also complete on Σ.
- Then, the map

$$\Psi : (\Sigma^n, \widetilde{\langle,\rangle}) \rightarrow (\mathbb{S}^n, \langle,\rangle_0)$$

  is a local isometry between complete Riemannian manifolds.
- Hence, Ψ is a covering map, but $\mathbb{S}^n$ being simply connected this means that Ψ is in fact a global diffeomorphism.

Compact spacelike hypersurfaces in Λ^+

Example 1

- For each positive smooth function $f : \mathbb{S}^n \to (0, +\infty)$, consider the embedding $\psi_f : \mathbb{S}^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ given by

$$\psi_f(p) = (f(p), f(p)p).$$

- It is not difficult to see that for every $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{S}^n$

$$\langle d(\psi_f)_p(\boldsymbol{v}), d(\psi_f)_p(\boldsymbol{w})\rangle = f^2(p)\langle \boldsymbol{v}, \boldsymbol{w}\rangle_0.$$

- That is $\psi_f^*(\langle,\rangle) = f^2\langle,\rangle_0$, which means that ψ_f defines a spacelike immersion of $\mathbb{S}^n$ into Λ^+ with induced metric conformal to $\langle,\rangle_0$.
- We already know that $\theta_{\boldsymbol{k}_+} = 1$.
- Moreover, we can also compute $\theta_{\boldsymbol{k}_-}$ to see that

$$\theta_{\boldsymbol{k}_-} = \frac{2f\Delta_0 f + (n-4)\|\nabla^0 f\|_0^2 - nf^2}{2nf^4},$$

where $\|\cdot\|_0^2$, ∇^0 and Δ_0 denote the norm, the gradient and the Laplacian on $\mathbb{S}^n$ with respect to the standard metric $\langle,\rangle_0$.

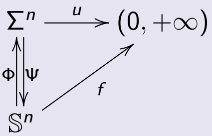
As a consequence of Proposition 1, we observe that every compact spacelike hypersurface in Λ^+ is, up to a conformal diffeomorphism, as in Example 1.

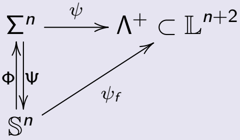
Theorem 2

Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a compact spacelike hypersurface of the light cone Λ^+. There exists a conformal diffeomorphism

$$\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0) \quad \text{such that} \quad \langle , \rangle = u^2 \Psi^*(\langle , \rangle_0),$$

with $u = -\langle \psi, \boldsymbol{e}_0 \rangle = \psi_1 > 0$, and $\psi = \psi_f \circ \Psi$ where $f = u \circ \Psi^{-1}$.





In particular, the immersion ψ is an embedding.

Proof of Theorem 2

- For the proof of Theorem 2, simply consider u and Ψ as in the proof of Proposition 1, and recall that in this situation

$$\Psi : (\Sigma^n, \langle , \rangle) \to (\mathbb{S}^n, \langle , \rangle_0)$$

is a conformal diffeomorphism with

$$\Psi^*(\langle , \rangle_0) = \frac{1}{u^2}\langle , \rangle.$$

- Let $\Phi : \mathbb{S}^n \to \Sigma^n$ be the inverse of Ψ.
- Then taking $f = u \circ \Phi$ one has $f \circ \Psi = u$ and $\psi = \psi_f \circ \Psi$, since

$$\begin{aligned} \psi_f \circ \Psi(p) &= (f(\Psi(p)), f(\Psi(p))\Psi(p)) \\ &= (u(p), \psi_2(p), \ldots, \psi_{n+2}(p)) \\ &= \psi(p). \end{aligned}$$

Trapped submanifolds in the Lorentz-Minkowski space

A codimension-two spacelike submanifold Σ in the Lorentz-Minkowski space is said to be

- Future (past) trapped if $\boldsymbol{H}$ is timelike and future-pointing (past-pointing) on Σ.
- Future (past) marginally trapped if $\boldsymbol{H}$ is null and future-pointing (past-pointing) on Σ.
- Future (past) weakly trapped if $\boldsymbol{H}$ is causal and future-pointing (past-pointing) on Σ.
- The extreme case $\boldsymbol{H} = 0$ corresponds to a minimal submanifold.

Recall that, in terms of a null frame $\{\boldsymbol{k}_+, \boldsymbol{k}_-\}$ on Σ, we have

$$\boldsymbol{H} = -\theta_- \boldsymbol{k}_+ - \theta_+ \boldsymbol{k}_-.$$

In particular

$$\langle \boldsymbol{H}, \boldsymbol{H} \rangle = -2\theta_+ \theta_-.$$

Therefore, since $\langle \boldsymbol{H}, \boldsymbol{H} \rangle = -2\theta_+\theta_-$, we have that

- Σ is a trapped submanifold if and only if
  - i) either both $\theta_+ < 0$ and $\theta_- < 0$ (future trapped),
  - ii) or both $\theta_+ > 0$ and $\theta_- > 0$ (past trapped).
- Σ is a marginally trapped submanifold if and only if
  - i) either $\theta_+ = 0$ and $\theta_- \neq 0$ (marginally future trapped if $\theta_- < 0$ and marginally past trapped if $\theta_- > 0$),
  - ii) or $\theta_+ \neq 0$ and $\theta_- = 0$ (marginally future trapped if $\theta_+ < 0$ and marginally past trapped if $\theta_+ > 0$).
- Σ is a weakly trapped submanifold if and only if
  - i) either both $\theta_+ \leq 0$ and $\theta_- \leq 0$ with $\theta_+^2 + \theta_-^2 > 0$ (weakly futur trapped),
  - ii) or both $\theta_+ \geq 0$ and $\theta_- \geq 0$ with $\theta_+^2 + \theta_-^2 > 0$ (weakly past trapped).
- This was the original formulation of trapped surfaces given by Penrose (1965) in terms of the signs or the vanishing of the null expansions.

Trapped submanifolds into Λ^+

Recall that for a spacelike hypersurface Σ of the light cone Λ^+ we already know that $\theta_+ = \frac{1}{n}\mathrm{tr}(A_{\boldsymbol{k}_+}) = 1 > 0$ and

$$\theta_- = \frac{1}{n}\mathrm{tr}(A_{\boldsymbol{k}_-}) = \frac{2u\Delta u - n(1+\|\nabla u\|^2)}{2nu^2}.$$

Corollary 1

Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a spacelike hypersurface of the light cone of the Lorentz-Minkowski space.

- Σ is (necessarily past) marginally trapped if and only if $u = -\langle \psi, \boldsymbol{e}_0 \rangle$ satisfies the differential equation

$$2u\Delta u - n(1+\|\nabla u\|^2) = 0 \quad \text{on } \Sigma.$$

- Σ is (necessarily past) weakly trapped if and only if $u = -\langle \psi, \boldsymbol{e}_0 \rangle$ satisfies the differential inequality

$$2u\Delta u - n(1+\|\nabla u\|^2) \geq 0 \quad \text{on } \Sigma.$$

- On the other hand, it follows from the Gauss equation that the Ricci and the scalar curvatures of Σ are given by

$$\mathrm{Ric}(X,Y)=(n-1)\langle \boldsymbol{H},\boldsymbol{H}\rangle\langle X,Y\rangle+\frac{n-2}{nu}(\Delta u\langle X,Y\rangle-n\mathrm{Hess}\,u(X,Y)),$$

and

$$\mathrm{Scal}=n(n-1)\langle \boldsymbol{H},\boldsymbol{H}\rangle=-(n-1)\frac{2u\Delta u-n(1+\|\nabla u\|^2)}{u^2}.$$

Corollary 2

Let $\psi:\Sigma^n\to\Lambda^+\subset\mathbb{L}^{n+2}$ be a spacelike hypersurface of the light cone of the Lorentz-Minkowski space. Let u be the positive function $u=-\langle\psi,\boldsymbol{e}_1\rangle$. The following are equivalent:

i) Σ is marginally (resp. weakly) trapped.

ii) u satisfies the differential equation (resp. inequality) on Σ

$$2u\Delta u-n(1+\|\nabla u\|^2)=0 \quad (\text{resp.} \geq 0).$$

iii) Σ has zero scalar curvature, $\mathrm{Scal}=0$ (resp. $\mathrm{Scal}\leq 0$).

Examples of trapped hypersurfaces in the light cone Λ^+

Example 2 (B.Y. Chen and J. Van der Veken, HJM 36 (2010), 421–449)

- Let $\psi : \mathbb{R}^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be the map given by

$$\psi(p) = \left(\frac{\|p\|^2 + 1}{2}, \frac{\|p\|^2 - 1}{2}, p \right), \quad u(p) = \frac{\|p\|^2 + 1}{2}.$$

- Is is not difficult to see that for every $\mathbf{v}, \mathbf{w} \in T_p\mathbb{R}^n$,

$$\langle d\psi_p(\mathbf{v}), d\psi_p(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{R}^n}.$$

- That is $\psi^*(\langle , \rangle) = \langle , \rangle_{\mathbb{R}^n}$, which means that ψ is an isometric immersion of $(\mathbb{R}^n, \langle , \rangle_{\mathbb{R}^n})$ into $\Lambda^+ \subset \mathbb{L}^{n+2}$.
- In particular, $\nabla u(p) = \nabla^{\mathbb{R}^n} u(p) = p$ and $\Delta u(p) = \Delta_{\mathbb{R}^n} u(p) = n$, and u satisfies

$$2u\Delta u - n(1 + \|\nabla u\|^2) = n(\|p\|^2 + 1) - n(1 + \|p\|^2) = 0$$

which means ψ is a marginally trapped immersion of $\mathbb{R}^n$ into Λ^+.

Example 3 (B.Y. Chen and J. Van der Veken, HJM 36 (2010), 421–449)

- Let $\phi : (0, +\infty) \times \mathbb{H}^{n-1} \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be the map given by

$$\psi(t, p) = (p, \cos(t), \sin(t)), \quad u(p) = p_1.$$

- Is is not difficult to see that $\phi^*(\langle , \rangle) = dt^2 + \langle , \rangle_{\mathbb{H}^{n-1}}$, which means that ϕ gives an isometric immersion of the Riemannian product manifold $(0, +\infty) \times \mathbb{H}^{n-1}$ into $\Lambda^+ \subset \mathbb{L}^{n+2}$.
- In particular, and after some computations, we have

$$\|\nabla u\|^2 = -1 + u^2 \quad \text{and} \quad \Delta u = (n-1)u,$$

which implies that

$$2u\Delta u - n(1 + \|\nabla u\|^2) = (n-2)u^2 \geq 0.$$

- Therefore, Σ is a weakly trapped submanifold, and it is marginally trapped if, and only if $n = 2$.

Non-existence of weakly trapped hypersurfaces in Λ^+

Proposition 2

There exists no compact hypersurface in Λ^+ which is weakly trapped in $\mathbb{L}^{n+2}$.

- The proof of Proposition 2 follows from that fact that

$$\langle \boldsymbol{H}, \boldsymbol{H} \rangle = -\frac{2u\Delta u - n(1 + \|\nabla u\|^2)}{nu^2}.$$

- Assume that Σ is compact and let $p_0 \in \Sigma$ a point where u attains its maximum.
- At the point p_0 we have $\|\nabla u(p_0)\| = 0$ and $\Delta u(p_0) \leq 0$, so that

$$\langle \boldsymbol{H}, \boldsymbol{H} \rangle(p_0) = \frac{n - 2u(p_0)\Delta u(p_0)}{nu^2(p_0)} \geq \frac{1}{u^2(p_0)} > 0$$

what implies that $\boldsymbol{H}(p_0)$ is spacelike and Σ cannot be weakly trapped.

Non-existence of weakly trapped hypersurfaces in Λ^+

As a consequence, using our compactness result for spacelike hypersurfaces of the light cone given in Proposition 1, we have

Corollary 3

There exists no complete hypersurface in Λ^+ which is weakly trapped in $\mathbb{L}^{n+2}$ for which the positive function $u = -\langle \psi, \boldsymbol{e}_1 \rangle$ satisfies

$$u \leq Cr \log r, \qquad r >> 1.$$

In particular, there is no complete hypersurface in Λ^+ which is weakly trapped in $\mathbb{L}^{n+2}$ for which the positive function u is bounded from above.

More generally, with the aid of the weak maximum principle we can extend this non-existence result to the case of stochastically complete hypersurfaces as follows

Theorem 3

There exists no stochastically complete hypersurface in Λ^+ which is weakly trapped in $\mathbb{L}^{n+2}$ for which the positive function u is bounded from above.

Stochastic completeness and the weak maximum principle

- The *weak* maximum principle is said to hold on Σ if, for any $u \in \mathcal{C}^2(\Sigma)$ with $\sup_\Sigma u = u^* < +\infty$ there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in Σ with

$$\text{(i)} \quad u(p_k) > u^* - \frac{1}{k}, \quad \text{and} \quad \text{(ii)} \quad \Delta u(p_k) < \frac{1}{k}.$$

- Pigola, Rigoli and Setti (2003) proved that the weak maximum principle holds on a (non-necessarily complete) Riemannian manifold Σ if and only if Σ is stochastically complete.
- We recall that Σ is said to be stochastically complete if its Brownian motion is stochastically complete, i.e, the probability of a particle to be found in the state space is constantly equal to 1.
- This is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on Σ is the constant $u = 0$.
- In particular, every parabolic manifold is stochastically complete. Hence, the weak max principle holds on every parabolic manifold.

Proof of Theorem 3

- Let $\psi : \Sigma^n \to \Lambda^+ \subset \mathbb{L}^{n+2}$ be a stochastically complete hypersurface in Λ^+ which is weakly trapped in $\mathbb{L}^{n+2}$.
- Consider $u = -\langle \psi, \boldsymbol{e}_1 \rangle$ as usual, which satisfies

$$2u\Delta u - n(1 + \|\nabla u\|^2) \geq 0. \tag{1}$$

- Supose that $u^* = \sup_\Sigma u < +\infty$. Since Σ is stochastically complete, by the weak maximum principle there exists a sequence $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma$ with

$$\Delta u(p_k) < \frac{1}{k} \quad \text{for every} \quad k \in \mathbb{N}$$

- Putting this into (1) we obtain

$$n \leq n(1 + \|\nabla u(p_k)\|^2) \leq 2u(p_k)\Delta u(p_k) < 2\frac{u(p_k)}{k},$$

  and making $k \to +\infty$ we get

$$n \leq 0$$

  which is not possible.

That's all !!

Thank you very much for your attention