Beltrami representatives for homotopy classes of contact structures

Radu Slobodeanu [1]

[1] University of Bucharest, Faculty of Physics

Differential Geometry Workshop
University of Vienna, September 2022

Outline

1. fluids, (non-vanishing) Beltrami fields & contact structures.
2. Beltrami fields representatives on the 3-sphere.
3. Beltrami fields representatives on the 3-torus
4. ABC fields on the 3 torus & chaos.

References:
[**1**] Peralta-Salas D. and R. Slobodeanu, *Contact structures and Beltrami fields on the torus and the sphere*, arXiv:2004.10185, Indiana University Mathematics Journal, in press.
[**2**] Marciu M. and R. Slobodeanu, *ABC-like flows on the 3-torus*, Chaos, in press.
[**3**] Dombre T., et al, *Chaotic streamlines in the ABC flows*, Journal of Fluid Mechanics **167** (1986), 353-391.

Why precisely these compact spaces??

1. 3-sphere: $\mathbb{S}^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 | x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}$ endowed with constant curvature 1 induced metric.
2. 3-torus: $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ endowed with the flat metric.

Why?

1. conformally equivalent to the Euclidean 3-space (natural compactification of the physical space)
2. corresponds to periodic boundary conditions.

More generally, we prefer M = oriented closed Riemannian manifold of odd dimension n, since then, the eigenforms of curl associated to eigenvalues $\neq 0$ are smooth, the multiplicity of any nonzero eigenvalue is finite, and curl defined on $\Omega^{(n-1)/2}_{C^\infty}(M)$ is essentially self-adjoint in the Hilbert space $\Omega^{(n-1)/2}_{L^2}(M)$, cf. [Christian Bär, J. Math. Phys. 60, 031501 (2019)]

Part 1

1. fluids, (non-vanishing) Beltrami fields & contact structures.

Steady incompressible fluid flows & Beltramicity

Definition

(M, g) = Riemannian 3-manifold. $V \in \Gamma(TM)$ is a **steady Euler field/flow** if $\exists\, p \in C^1(M)$ s.t. $\operatorname{div}(V^\flat \otimes V^\flat + pg) = 0$, or, equivalently,

$$\begin{cases} \nabla_V V &= -\operatorname{grad} p \\ \operatorname{div} V &= 0 \end{cases}$$

- for steady incompressible fluids, the **Bernoulli function** $b = p + \frac{1}{2}|V|^2$ is conserved along the flow, $V(b) = 0$. In particular, if b non-constant, then the flow is laminar (aka integrable), and M is foliated by tori or cylinders
- solutions with b is constant are called **Beltrami fields**. They satisfy: $\operatorname{curl} V = fV$ and $\operatorname{div} V = 0$, for some $f \in C^\infty(M)$. In this case f is conserved, $V(f) = 0$.

$$\operatorname{curl} V \times V = \nabla_V V - \frac{1}{2} \operatorname{grad} |V|^2$$

Motivating Beltramicity

- if $f \equiv \lambda$ (constant), then Beltrami fields are simply **eigenfields of** curl **operator**.
- there is a dichotomy (*under some technical assumptions*):

a steady Euler flow is either laminar or Beltrami with $f \equiv$ const.

or, in other words: complex dynamics (chaos, as expected in Lagrangian turbulence) can appear in a fluid at equilibrium only through Beltrami fields.

- "Beltramization": experimentally observed phenomenon that the velocity field and its curl (i.e., the vorticity) tend to align in turbulent regions.
- Beltrami fields = another emergence of complexity in physics (*different from the chaotic behavior!*): Turing completeness of a system, related to the undecidability of its evolution. [Cardona et al, "Constructing Turing complete Euler flows in 3D", PNAS 2021]

Variational character - general case

$(M, g) =$ (compact) Riemannian 3-manifold.

- $\mathrm{SDiff}(M) =$ group of all diffeomorphisms of M preserving the volume form v_g
- $\Gamma_0(TM) =$ space of smooth divergence free vector fields
- **energy** $\mathcal{E} : \Gamma_0(TM) \to \mathbb{R}_+$, $\mathcal{E}(X) = \frac{1}{2}\int_M |X|^2 v_g$
- **variation**: $X_t = \mathrm{d}\psi_t(X)$, where $\psi_t \in \mathrm{SDiff}(M)$, $\psi_0 = Id_M$.
- **variation vector field**: $v = \left.\dfrac{\partial \psi_t}{\partial t}\right|_{t=0} \in \Gamma_0(TM)$.

First variation formula - any $X \in \Gamma_0(TM)$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(X_t)\right|_{t=0} = -\int_M \langle v, \nabla_X X\rangle v_g.$$

- A curl eigenfield X corresponding to the 1st positive eigenvalue λ_1 minimizes the energy $\mathcal{E}$ among all vectors fields obtained from X by push-forward through volume-preserving diffeo's.

Variational character- fully integrable case

Definition

Let $\overline{P} : N \to \mathbb{R}_+$ be a non-negative function on N. For every map $\varphi : M \to N$ the σ_2-*energy with potential* over a compact domain K is

$$\mathcal{E}_{\sigma_2,P}(\varphi, K) = \frac{1}{2}\int_K \{|\wedge^2 \mathrm{d}\varphi|^2 + 2\overline{P}\circ\varphi\}\nu_g, \qquad (1)$$

A map $\varphi : M \to N$ is called σ_2-*critical with potential* $\overline{P}$ if for every compact domain K in M and for any variation $\{\varphi_s\}_{s\in(-\epsilon,\epsilon)}$ supported in K, of $\varphi = \varphi_0$, we have $\frac{d}{ds}\Big|_{s=0}\mathcal{E}_{\sigma_2,P}(\varphi_s, K) = 0$.

Theorem [R.S. 2015]

If the C^2 mapping $\varphi : (M^3, g) \to (N^2, h)$ is σ_2-critical with potential $\overline{P}$, and ω is the area 2-form on N induced by h, then the vertical field $V = (*\varphi^*\omega)^\sharp$ satisfies the Euler equations for steady incompressible flows on M with Bernoulli function $P = \overline{P}\circ\varphi$. Conversely if V is a steady incompressible Euler solution on M, then it exists locally a σ_2-critical submersion with potential into some surface (N, h) with fibres tangent to V.

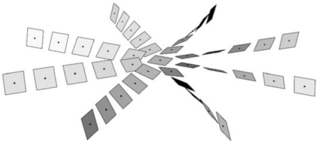
Reeb-Beltrami correspondence

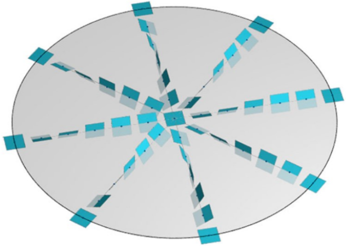
- contact form on (closed) 3-manifold M: a 1-form α s.t. $\alpha \wedge \mathrm{d}\alpha \neq 0$ (so $\alpha \wedge \mathrm{d}\alpha$ defines a volume form on M).
- (coorientable) contact structure: 2-plane field $\zeta \subset TM$ for which $\exists \alpha$ contact form s.t. $\zeta = \ker \alpha$
- Reeb vector field R: (uniquely) determined by: $\alpha(R) = 1$, $\mathrm{d}\alpha(R, \cdot) = 0$.
- Beltrami fields in terms of differential forms: $*\mathrm{d}\alpha = f\alpha$, $\delta\alpha = 0$. In particular, $\alpha \wedge \mathrm{d}\alpha = f|\alpha|^2 \mathrm{vol}_g$.
- If Beltrami is nonvanishing and $f > 0$, then $\alpha = V^\flat$ is a contact form and $R := V/\alpha(V)$ is the corresponding Reeb field. We say that $\zeta = \ker \alpha$ is the **contact structure engendered by the nonvanishing Beltrami field** V.

Contact geometry & hydrodynamics [Sullivan, Etnyre, Ghrist]

Any nonvanishing rotational Beltrami field is a reparametrization of a Reeb vector field for some contact form. Conversely, any reparametrization of a Reeb vector field of a contact structure is a nonvanishing rotational Beltrami field for some Riemannian metric.

Tight versus overtwisted contact structures - i

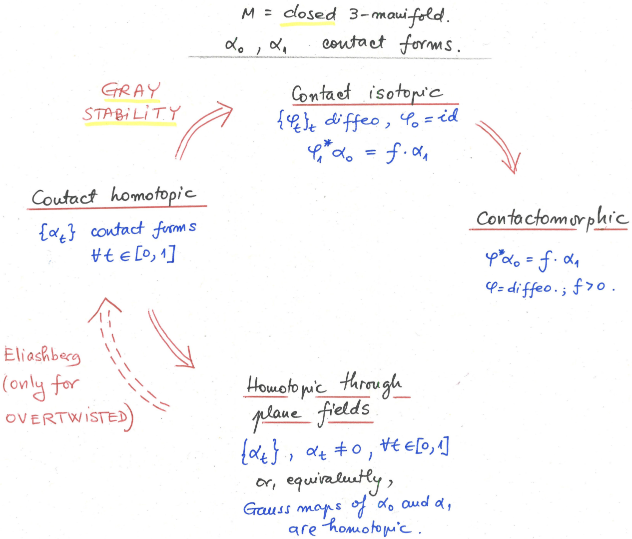




- a disk Δ embedded in M which, along its boundary, is tangent to ζ, and its interior is transverse to ζ everywhere except at one point, is called **overtwisted**.
- Contact structures are classified in overtwisted (if such a disk exists) and tight (if not).
- standard examples (in $\mathbb{R}^3$): $\alpha = dz + \rho^2 d\theta$ (tight), $\alpha = \cos\rho\, dz + \rho \sin\rho\, d\theta$ (OT), with OT disk $\Delta = \{z = 0, \rho \leq \pi\}$
- **Gauss map of a contact structure** with Reeb field R: $\varphi_R : M \to \mathbb{S}^2$, $\varphi_R(p) := \frac{1}{|R_p|} R_p$.

Tight versus overtwisted contact structures - ii

Equivalence classes of contact structures:



Tight versus overtwisted contact structures - iii

- *"Any contact structure can be "spoiled" and made overtwisted using a* ***Lutz twisting*** *(a surgery of the structure, but not of the manifold) along a closed transversal. It is possible to make Lutz twisting without changing the homotopy class of the contact structure as a plane field."* Yakov Eliashberg
- Eliashberg OT classification: contact isotopy classes of overtwisted contact structures on a closed 3-manifold are indexed by the Hopf invariant of their Gauss map. [Eliashberg, Invent. math. (1989)]
- OT Classification on $\mathbb{S}^3$. The homotopy classes of Gauss maps / plane-fields on $\mathbb{S}^3$ are identified with elements of $\pi_3\left(\mathbb{S}^2\right) = \mathbb{Z}$. The standard structure ζ_0 belongs to class 0. The class 0 contains exactly two nonequivalent (positive) contact structures: the standard and the overtwisted. All other classes k, $|k| = 1, 2, 3, \ldots$, contain only one contact structure, the overtwisted. [Eliashberg, Ann. Inst. Fourier (1992)]

Open realization problem: do these classes admit a (non-vanishing) Beltrami field representative?

Tight versus overtwisted contact structures - iii

Rigidity for tightness: All tight contact structures on $\mathbb{R}^3$ or $\mathbb{S}^3$ are isomorphic to the standard ones.

Criteria for tightness / OTness:

- (Eliashberg-Gromov) A symplectically fillable contact structure is tight
- (Giroux) Let ζ be an S^1-invariant contact structure on a principal circle bundle $\pi : P \to \Sigma$ over a closed oriented surface Σ, with bundle Euler number $e(P)$. Let $\Gamma = \pi\left(\Gamma_{S^1}\right)$ be a projection of the characteristic surface Γ_{S^1} onto Σ.
  1. ζ is universally tight if and only if one of the following holds:
     (i) For $\Sigma \neq \mathbb{S}^2$ none of the connected components of $\Sigma \backslash \Gamma$ is a disc.
     (ii) For $\Sigma = \mathbb{S}^2, e(P) < 0$ and $\Gamma = \emptyset$.
     (iii) For $\Sigma = \mathbb{S}^2, e(P) \geqslant 0$ and Γ is connected (non-empty).
  2. if $\Sigma \backslash \Gamma$ has a component diffeomorphic to a disk, the contact structure is tight only if Γ is connected.

Definition

For a vector field X preserving the contact distribution ζ (i.e. $\mathcal{L}_X \alpha = 0$), the **characteristic surface** is $\Gamma_X = \{p \in M : X_p \in \zeta_p\}$.

rigidity/flexibility in contact topology when a Riemannian metric is considered.

Definition: weakly compatible metrics

A Riemannian metric g on M is **weakly compatible** with a contact form α if there exists a function $f > 0$ such that

$$\star d\alpha = f\alpha,$$

where $\star$ is computed with the metric $g(\Longrightarrow$ the Reeb field R is g-orthogonal to the contact structure ξ). Moreover, if $|\alpha|_g = 1$ and $f =$ const, g is called **compatible** with α.

Remark: if V is a nonvanishing curl eigenfield on (M, g), then g is weakly compatible with the contact form α engendered by V.

Compatible metrics are severely restricted, as shown by the following pinching theorem:

Theorem (Etnyre, Komendarczyk & Massot, 2012)

Let (M, α) be a contact 3-manifold. If there exists a compatible metric g with pinched sectional curvature

$$0 < \frac{4}{9}K_0 < \sec(g) < K_0,$$

then α is tight and M is covered by $\mathbb{S}^3$.

Open problem: does the contact sphere theorem hold for weakly compatible metrics? In particular, can an overtwisted contact structure be engendered by a nonvanishing curl eigenfield on the round $\mathbb{S}^3$?

Part 2

1.

2. Beltrami fields representatives on the 3-sphere.

the spectrum of the curl operator on $\mathbb{S}^3$ is given by

$$\{\lambda = \pm(k+2), k \in \mathbb{N}\}.$$

Theorem 1 (R.S. and D. Peralta-Salas)

Any nonvanishing curl eigenfield on $\mathbb{S}^3$ has even eigenvalue $\lambda = 2m$, $m \in \{\pm 1, \pm 2, \cdots\}$. Moreover, for each $|m| \geq 2$ there exists a nonvanishing curl eigenfield V_m whose associated contact structure is overtwisted. The homotopy classes of the corresponding contact plane fields have Hopf index

$$\text{Hopf index} = \frac{1}{2}(\text{sign}(m)(-1)^{m+1} - 1)$$

Corollary. The round metric on $\mathbb{S}^3$ is weakly compatible with an OT contact structure.

sphere case proof - i

- we work in Hopf coordinates (s, ϕ_1, ϕ_2), $s \in [0, \pi/2]$, $\phi_{1,2} \in [0, 2\pi)$:

$$z_1 = \cos s\, e^{i\phi_1}, \quad z_2 = \sin s\, e^{i\phi_2}$$

- the **Hopf** and **anti-Hopf fields** are given by:

$$R = \partial_{\phi_1} + \partial_{\phi_2}, \quad R' = \partial_{\phi_1} - \partial_{\phi_2}$$

  and we have: $\operatorname{curl} R = 2R$ and $\operatorname{curl} R' = -2R'$.
- $\{s = 0\} \cup \{s = \pi/2\}$ corresponds to the Hopf link in $\mathbb{S}^3$.
- the standard round metric is: $ds^2 + \cos^2 s\, d\phi_1^2 + \sin^2 s\, d\phi_2^2$
- in terms of the (positively oriented) **standard orthonormal global frame** $\{R, X_1, X_2\}$, a Beltrami field V reads:

$$V = fR + f_1 X_1 + f_2 X_2$$

sphere case proof - ii

Key Proposition

f, f_1, f_2 are eigenfunctions of the Laplacian on $\mathbb{S}^3$ with eigenvalue $-\lambda(\lambda - 2)$.

$\Longrightarrow$ If λ is odd, then V has zeros. Indeed, f, f_1, f_2 are restrictions on $\mathbb{S}^3$ of homogeneous harmonic polynomials in $\mathbb{R}^4$ of odd degree. Borsuk-Ulam theorem then implies that the map $(f, f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^3$ has a non-empty zero set.

sphere case proof - iii

KKPS construction (Khesin, Kuksin & P-S, 2014)

Let $F, G : \mathbb{R} \to \mathbb{R}$ be smooth functions. Then the vector field

$$V = F\left(\cos^2 s\right) R + G\left(\cos^2 s\right) R'$$

is a steady Euler flow on the round sphere.

Step 1: For a suitable choice of F, G, V is a Beltrami field. Indeed, taking $\lambda = 2m, m \geqslant 2$,

$$F \equiv F_m = \frac{1}{m} P^{(1,1)}_{m-1}(1 - 2z), \quad G \equiv G_m = \frac{1}{m+1} P^{(1,1)}_{m-2}(1 - 2z)$$

where $z = \cos^2 s$ and $\left\{ P^{(1,1)}_* \right\}$ is the family of orthogonal Jacobi polynomials of degree $*$. $\Longrightarrow$ Since the zeros of the Jacobi polynomials interlace, the Beltrami fields $V \equiv V_m$ are nonvanishing.

sphere case proof - iv

Properties of V_m

1. The Hopf link is a set of periodic orbits of V_m.
2. V_m is integrable in the sense that $\{s = \text{const}\}$ are invariant tori.
3. V_m is $\mathbb{S}^1$-invariant in the sense that $[V_m, R] = 0$.

Step 2: The contact forms engendered by V_m are overtwisted.
Indeed, notice that

$$\alpha_m := V_m^b = \cos^2 s\,(F_m + G_m)\,d\phi_1 + \sin^2 s\,(F_m - G_m)\,d\phi_2$$

is a contact form. Moreover, it is $\mathbb{S}^1$-invariant: $L_R\alpha_m = 0$.
Giroux's first criterion Consider the characteristic surface

$$\Gamma_R := \left\{p \in \mathbb{S}^3 : R \text{ is tangent to } \xi_m \text{ at } p\right\}$$

Then α_m is tight if and only if $\Pi\,(\Gamma_R) = \emptyset, \Pi : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf fibration.

sphere case proof - v

For α_m, Γ_R consists of toroidal surfaces in $\mathbb{S}^3$:

$$\left\{s \in [0, \pi/2] : F_m\left(\cos^2 s\right) + \left(2\cos^2 s - 1\right) G_m\left(\cos^2 s\right) = 0\right\}$$

This set is nonempty and $\Pi\left(\Gamma_R\right) \neq \emptyset$ (a set of circles) $\Longrightarrow \alpha_m$ is overtwisted.

Step 3: Compute the Hopf invariant of the map $\varphi_m : \mathbb{S}^3 \to \mathbb{S}^2$:

$$\varphi_m(p) := \frac{1}{\left(f(p)^2 + f_1(p)^2 + f_2(p)^2\right)^{1/2}} \left(f(p), f_1(p), f_2(p)\right)$$

which is an integer $\in \pi_3\left(\mathbb{S}^2\right) = \mathbb{Z}$

Lemma

If $m \geqslant 1, V_m$ is homotopic through nonvanishing fields to:
(a) R if m is odd.
(b) R' if m is even.
Hopf invariant of R is 0, and of R' is -1 (Whitehead's formula).

sphere case - remarks

The construction for $m \leqslant -2$ is similar (in fact, V_m and $V_{-m}, m \geqslant 1$, are related by an orientation-reversing diffeo of $\mathbb{S}^3$).

Examples

$$V_2 = -\tfrac{1}{3}(3\cos 2s - 1)\partial_{\phi_1} - \tfrac{1}{3}(3\cos 2s + 1)\partial_{\phi_2},$$
$$V_3 = \left(\tfrac{3}{2} - 6\cos^2 s + 5\cos^4 s\right)\partial_{\phi_1} + \left(\tfrac{1}{2} - 4\cos^2 s + 5\cos^4 s\right)\partial_{\phi_2}.$$

The case of lowest eigenvalue $\lambda = 2$ (or $\lambda = -2$) is special: all corresponding Beltrami fields are isometric to R (resp. R'). Moreover, they exhibit a remarkable geometric rigidity:

Theorem (Gluck & Gu, 2001)

Let V be a Beltrami field on $\mathbb{S}^3$ with $|V| = 1$. Then $\lambda = \pm 2$ and V is isometric the Hopf (or anti-Hopf) field.

We provide a different (and simpler) proof using the Key Proposition above and standard classification results for harmonic morphisms.

Beltrami rigidity via harmonicity

- Prove: a curl eigenfield V with $\lambda > 2$ cannot have constant norm
  - Suppose that $V = fR + f_1X_1 + f_2X_2$ has const (unit) norm. Then the Gauss map is $\varphi_V = (f, f_1, f_2) : \mathbb{S}^3 \to \mathbb{S}^2$ and it is an eigenmap (harmonic map with constant energy density), has minimal fibres and V is tangent to them ($\nabla_V V = \frac{1}{2}\,\mathrm{grad}\,|V|^2$).
  - Weitzenbock formula for a harmonic map $\varphi : \mathbb{S}^3 \to \mathbb{S}^2$:

$$\tfrac{1}{2}\Delta|\mathrm{d}\varphi|^2 = |\nabla \mathrm{d}\varphi|^2 + 2|\mathrm{d}\varphi|^2 - 2|\Lambda^2 \mathrm{d}\varphi|^2 .$$

    in our case $\Rightarrow 2|\Lambda^2\mathrm{d}\varphi_V|^2 \geq 2\lambda(\lambda-2) > 0$, so $rank(d\varphi_V) = 2$
  - Apply the following to deduce that φ_V is a harmonic morphism

Paul Baird 1992

A harmonic map of rank 2 almost everywhere from a closed 3-manifold to a surface such that: (i)$Ric(E_1, E_1) = Ric(E_2, E_2) > 0$, (ii) the regular fibres are minimal, and (iii) $\mathrm{grad}\, e(\varphi)$ is horizontal, is horizontally conformal (a harmonic morphism).

  - Apply the classification of the harmonic morphisms to deduce that φ_V is essentially Hopf fibration so $\lambda = 2$, contradiction
- Prove: any eigenfield with eigenvalue 2 is isometric to the Hopf field

Part 3

1.
2.
3. Beltrami fields representatives on the 3-torus

the spectrum of the curl operator curl $= *\mathrm{d}$ on the flat 3-torus $\mathbb{T}^3$ is given by

$$\{\lambda = \pm|k| : k \in \mathbb{Z}^3\} \qquad \left(|k| := \sqrt{k_1^2 + k_2^2 + k_3^2}\right),$$

where an eigenvalue λ has the multiplicity $\sharp\{\mu \in \mathbb{Z}^3, |\mu| = |\lambda|\}$.

Theorem 2 (R.S. and D. Peralta-Salas)

For each eigenvalue λ of curl on $\mathbb{T}^3$, there exists a nonvanishing curl eigenfield V_λ which is homotopically trivial and whose associated contact structure is tight. Moreover, all tight contactomorphic classes are realized this way. Furthermore, there exist infinitely many eigenvalues $\{\lambda_\ell\}_{\ell \in \mathbb{N}^*}$ and corresponding eigenfields V_ℓ such that, for each ℓ, the contact structure engendered by V_ℓ is overtwisted.

torus case - proof i

- For any non-zero vector $b \in \mathbb{R}^3$, $b \perp k$, the vector field

$$V_k = \cos(k \cdot x)\, b + \frac{1}{|k|} \sin(k \cdot x)\, b \times k \qquad (2)$$

  is an eigenfield of the curl operator with eigenvalue $|k|$
- $|V_k| = |b|$ (constant norm), so V_k is nonvanishing and then it induces a contact structure on $\mathbb{T}^3$.
- All these contact structures are tight (our proof, simple).
- with $k = (0, 0, m)$, $m \in \mathbb{Z}$, and $b = (0, 1, 0)$, we find the standard family of contact structures on $\mathbb{T}^3$:

$$\eta_m = \sin(mx_3)\mathrm{d}x_1 + \cos(mx_3)\mathrm{d}x_2\,, \qquad m \in \mathbb{Z}\,, \qquad (3)$$

  corresponding to the integer part of the spectrum: $*\mathrm{d}\eta_m = m\eta_m$.

Tight classification on $\mathbb{T}^3$ [Y. Kanda, Comm. Anal. Geom. 1997]

- the contact forms η_m are tight and homotopically trivial, but they belong to distinct contactomorphic classes: there is no contactomorphism $(\mathbb{T}^3, \zeta_n) \to (\mathbb{T}^3, \zeta_m)$ if $n \neq m$.
- any tight contact structure on $\mathbb{T}^3$ is contactomorphic to one of these η_m

torus case - proof ii

For the second claim we consider the equivariant curl eigenfield

$$V = \frac{\partial f}{\partial x_2}\partial_{x_1} - \frac{\partial f}{\partial x_1}\partial_{x_2} + \lambda f \partial_{x_3},$$

where $f \equiv f(x_1, x_2)$ is a λ^2-eigenfunction of the Laplacian on $\mathbb{T}^2$.

Lemma (Peralta-Salas & R.S.)

There exists an infinite sequence of eigenvalues $\{\Lambda_\ell\}_{\ell \in \mathbb{N}^*}$ and corresponding eigenfunctions f_ℓ of the Laplacian on $\mathbb{T}^2$ such that, for each ℓ, the nodal set of f_ℓ is regular, disconnected, and contains a contractible connected component.

1. The Beltrami field V_ℓ defined using f_ℓ has eigenvalue $\sqrt{\Lambda_\ell}$ and is nonvanishing.
2. The contact form $\eta_\ell := V_\ell^\flat$ is $\mathbb{S}^1$-invariant with respect to the action generated by $Z := \partial_{x_3}$. The projection onto $\mathbb{T}^2$ defined by Z is $\Pi(x_1, x_2, x_3) = (x_1, x_2)$.
3. For Giroux' characteristic surface $\Gamma_Z^\ell := \{p \in \mathbb{T}^3 : Z \text{ tangent to the contact distribution } \ker \eta_\ell \text{ at } p\}$ we have $\Pi\left(\Gamma_Z^\ell\right) =$ the nodal set of f_ℓ.

torus case - proof iii

- Giroux's second criterion: If $\mathbb{T}^2 \backslash \Pi\left(\Gamma_Z^\ell\right)$ has a component diffeomorphic to a disk, the contact structure defined by η_ℓ is tight only if $\Pi\left(\Gamma_Z^\ell\right)$ is connected.
  Since the nodal set of f_ℓ is disconnected, and its complement in $\mathbb{T}^2$ contains a disk, Giroux's criterion implies that η_ℓ is overtwisted.
- Remark 1: The proof of the existence of the eigenfunctions f_ℓ is not constructive (it is based on the inverse localization technique developed by Enciso, Peralta-Salas & Torres de Lizaur (2017), which allows us to transplant the nodal set of a monochromatic wave in $\mathbb{R}^2$ into the nodal set of an eigenfunction in $\mathbb{T}^2$ with high eigenvalue.).
- Remark 2: We cannot compute the Hopf invariant of the overtwisted contact structures obtained this way (they are not explicit).

Open problems

- Problem 1: Do there exist tight Beltrami fields on $\mathbb{S}^3$ with eigenvalue $\lambda \neq \pm 2$?
- Problem 2: Can any overtwisted contact structure be engendered by a Beltrami field on $\mathbb{S}^3$?
- Problem 3: Which overtwisted contact structures can be engendered by Beltrami field on $\mathbb{T}^3$?

Part 4

1.
2.
3.
4. ABC fields on the 3-torus & chaos.

Motivations:

- *all contact forms associated to "small" eigenvalue Beltrami's on $\mathbb{T}^3$ are tight?*
  (for the first eigenvalue $\lambda = 1$ this is true)
- *go beyond the well-understood example of ABC (Arnold-Beltrami-Childress) flow when finding chaos*

3-dim. steady flows with chaotic Lagrangian structure: infinitesimally close fluid particles following the streamlines may separate exponentially in time, while remaining in a bounded domain, and individual streamlines may appear to fill entire regions of space.

Thus the positions of fluid particles may become effectively unpredictable for long times.

Flows on the 3-torus

Standard ABC flow on $\mathbb{T}^3$ analysed (for the first time) in [3]

$$\begin{aligned} \dot{x} &= A\sin(z) + C\cos(y) \\ \dot{y} &= A\cos(z) + B\sin(x) \\ \dot{z} &= B\cos(x) + C\sin(y) \end{aligned} \tag{4}$$

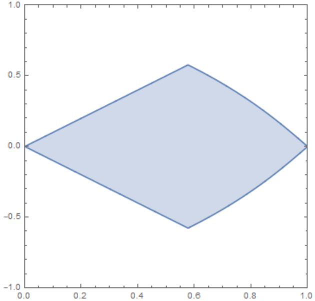
curl-eigenfield for $\lambda_1 = 1$
Our ABC-like flow on $\mathbb{T}^3$

$$\begin{aligned} \dot{x} &= -\frac{A}{\sqrt{2}}\sin(x+y) + \frac{B}{\sqrt{2}}\sin(x+z) + C\cos(y+z) \\ \dot{y} &= \frac{A}{\sqrt{2}}\sin(x+y) + B\cos(x+z) - \frac{C}{\sqrt{2}}\sin(y+z) \\ \dot{z} &= A\cos(x+y) - \frac{B}{\sqrt{2}}\sin(x+z) + \frac{C}{\sqrt{2}}\sin(y+z) \end{aligned} \tag{5}$$

curl-eigenfield for $\lambda_2 = \sqrt{2}$

Theorem (R.S. & M. Marciu)

On the flat 3-torus, every nonvanishing ABC-like Beltrami field (5) is transverse to a tight contact structure in the same contactomorphism class as the standard contact form $\eta_1 = \sin z\mathrm{d}x + \cos z\mathrm{d}y$.



Parameter space for ABC-like fields that do not vanish anywhere. This is path connected so all fields are deformable into the $(A, B, C) = (1, 0, 0)$-like field, which engenders a contact structure contactomorphic to η_1 (proof: explicit contact diffeo $\oplus$ 1-parameter homotopy through contact forms).

Flow visualisation by Poincaré sections

The classical ABC-flow (4), with the choice $A = 1, B = \sqrt{\frac{2}{3}}, C = \frac{1}{\sqrt{3}}$, can be visualised via Poincaré sections displaying 'chaotic' and 'ordered' regions (cf.[3]):

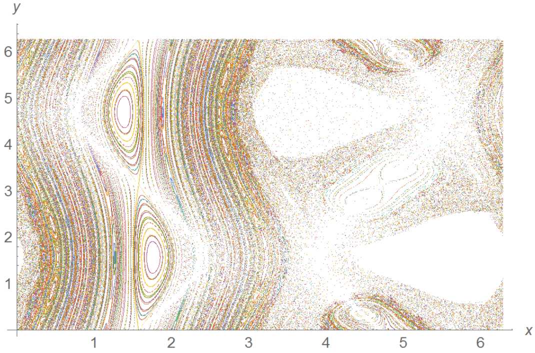


Figure: Poincaré section of the standard ABC flow through the plane $z = 0$

Flow visualisation by Poincaré sections

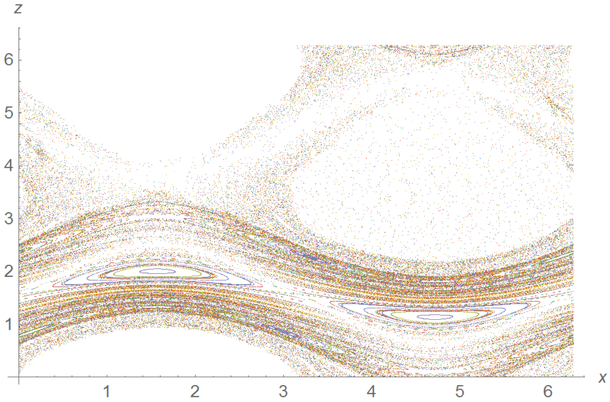


Figure: Poincaré section of the standard ABC flow through the plane $y = 0$

Flow visualisation by Poincaré sections

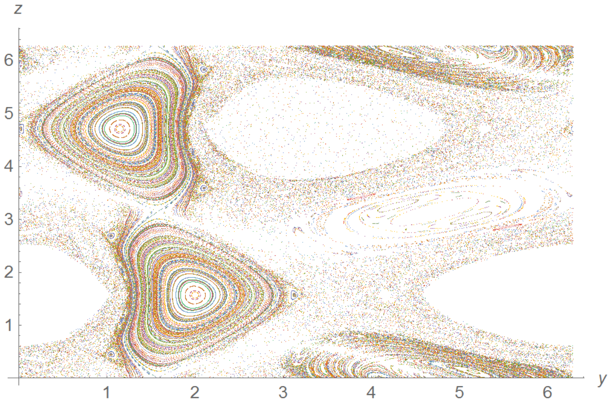
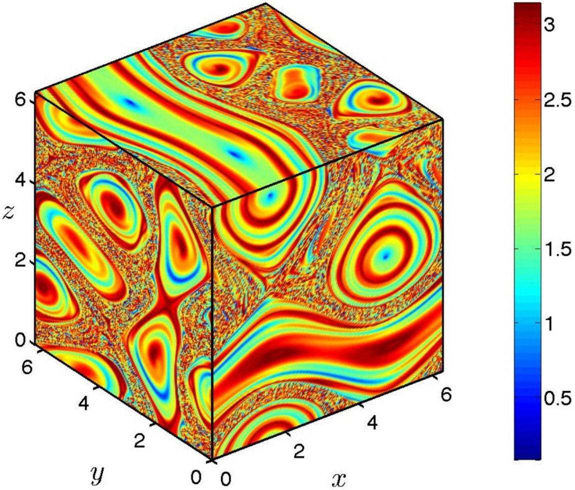


Figure: Poincaré sections of the standard ABC flow through the plane $x = 0$

see also `http://ameli.github.io/lcs/` for LCS visualisation



6
4
2
0
z
6
4
2
0
y
0
2
4
6
x
3
2.5
2
1.5
1
0.5

Flow visualisation by Poincaré sections

The Poincaré sections for the ABC-like flow (5) for the same values of the parameters A, B and C (for which there exists stagnation points!).

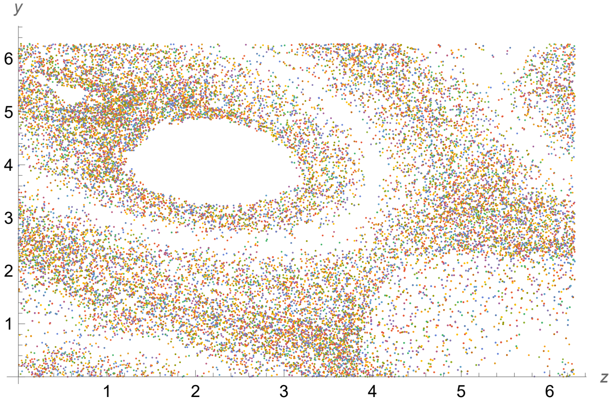
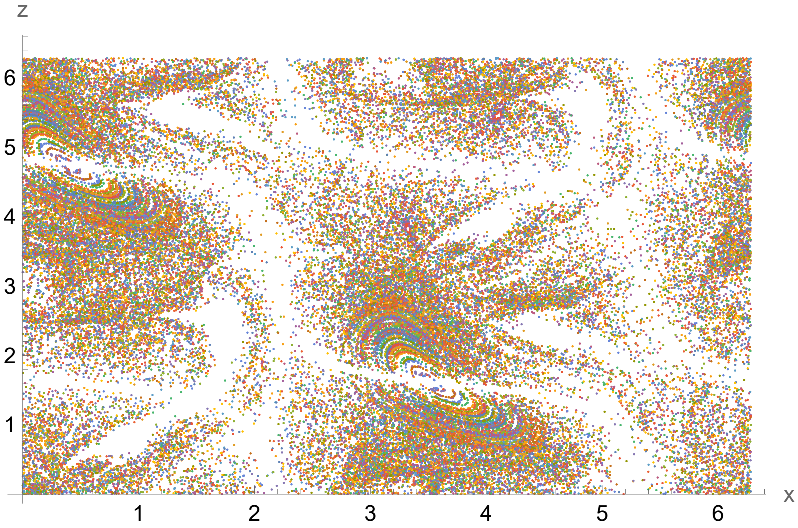


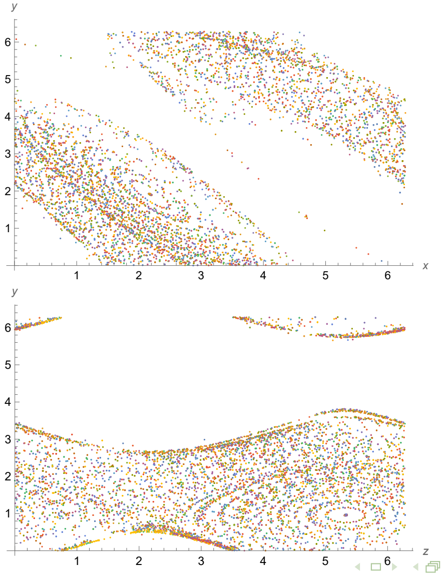
Figure: Poincaré section of ABC-like flow through the plane $x = 0$

Flow visualisation by Poincaré sections

Poincaré section of ABC-like flow through the plane $x + y = 0$:



For the parameters $(A, B, C) = (1, 0.4, 0.01)$ our ABC-like flow have no stagnation points and looks like:



introducing new coordinates $\tilde{x} = x, \tilde{y} = x + y, \tilde{z} = x + z$ (which amounts to a push-forward of v through a volume preserving diffeomorphism of $\mathbb{T}^3$) and rescaling the coefficients A, B, C with $\sqrt{3/2}$, the dynamical system (5) takes the form

$$\begin{aligned} \dot{\tilde{x}} &= -\tfrac{1}{\sqrt{3}}(A\sin(\tilde{y}) - B\sin(\tilde{z})) + C\sqrt{\tfrac{2}{3}}\cos(2\tilde{x} - \tilde{y} - \tilde{z}) \\ \dot{\tilde{y}} &= B\cos(\tilde{z} - \varphi_0) + C\cos(2\tilde{x} - \tilde{y} - \tilde{z} - \varphi_0) \\ \dot{\tilde{z}} &= A\cos(\tilde{y} + \varphi_0) + C\cos(2\tilde{x} - \tilde{y} - \tilde{z} + \varphi_0), \end{aligned} \tag{6}$$

so that the first integral in the case $C = 0$ is independent of $\tilde{x}$:

$$\widetilde{H}(\tilde{y}, \tilde{z}) = A\sin(\tilde{y} + \varphi_0) - B\sin(\tilde{z} - \varphi_0). \tag{7}$$

The function $\widetilde{H}$ has 4 critical points $(\tilde{y}_0, \tilde{z}_0) \in [0, 2\pi]^2$, two of which being of saddle type: $\left(\frac{\pi}{2} - \varphi_0, \frac{\pi}{2} + \varphi_0\right)$ and $\left(\frac{3\pi}{2} - \varphi_0, \frac{3\pi}{2} + \varphi_0\right)$. The values taken by $\widetilde{H}$ at these two points are $\pm(A - B)$, respectively. The level curves of $\widetilde{H}$ are pictured below for a specific choice of parameters.

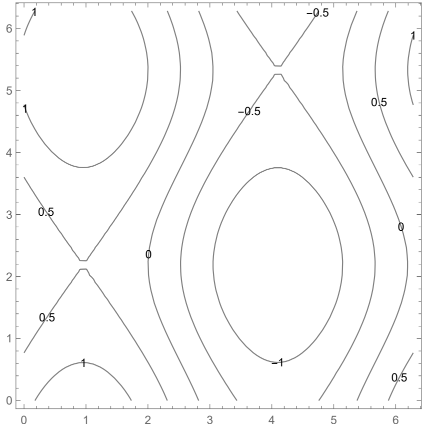
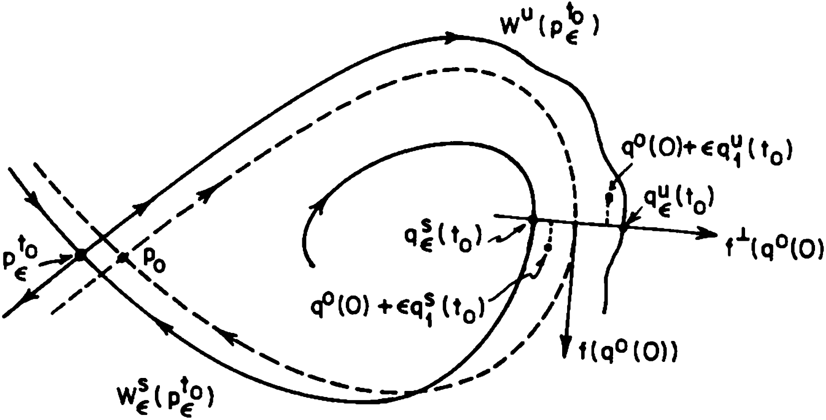


Figure: Level curves of $\widetilde{H}$ for $A = 1, B = 1/2$ in the plane $(\tilde{y}, \tilde{z})$

for the unperturbed system (i.e. for $C = 0$), we can explicitly compute the homoclinic orbit $(\tilde{y}^h(t), \tilde{z}^h(t))$ connecting to itself the hyperbolic point $\left(\frac{3\pi}{2} - \varphi_0, \frac{3\pi}{2} + \varphi_0\right)$, and the corresponding evolution in the $\tilde{x}$-direction.
According to [Mezic & Wiggins, *On the integrability and perturbation of 3-dim fluid flows with symmetry*, J. Nonlinear Sci. (1994)], the Melnikov distance function is given by

$$\mathcal{M}(t_0) = \int_{-\infty}^{\infty} \Big\{ - B\cos(\tilde{z}^h(\tau) - \varphi_0)\cos\left(2\tilde{x}^h(\tau + t_0) - \tilde{y}^h(\tau) - \tilde{z}^h(\tau) + \varphi_0\right) + A\cos(\tilde{y}^h(\tau) + \varphi_0)\cos\left(2\tilde{x}^h(\tau + t_0) - \tilde{y}^h(\tau) - \tilde{z}^h(\tau) - \varphi_0\right)\Big\}\mathrm{d}\tau$$

that we can evaluate (only) numerically and see that has simple zeros.
According to the discussion in Mezic & Wiggins, the presence of simple zeros implies that our ABC-like flow exhibits "Smale horseshoe" chaos when C is close to zero.


$W^u(p_\epsilon^{t_0})$
$q^0(0)+\epsilon q_1^u(t_0)$
$q_\epsilon^u(t_0)$
$q_\epsilon^s(t_0)$
$f^\perp(q^0(0)$
$p_\epsilon^{t_0}$
p_0
$q^0(0)+\epsilon q_1^s(t_0)$
$f(q^0(0))$
$W_\epsilon^s(p_\epsilon^{t_0})$