Classification of surfaces with linear prescribed mean curvature (PMC)

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Joint work with Antonio Bueno

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Summary

Surfaces of prescribed mean curvature

Definition (Prescribed mean curvature)

Given $\mathcal{H} \in C^1(\mathbb{S}^2)$, an oriented surface Σ in $\mathbb{R}^3$ is a surface of *prescribed mean curvature* $\mathcal{H}$ if its mean curvature H_Σ satisfies

$$H_\Sigma(p) = \mathcal{H}(N_p) \quad \forall p \in \Sigma,$$

where $N : \Sigma \to \mathbb{S}^2 \subset \mathbb{R}^3$ stands for the Gauss map of Σ.

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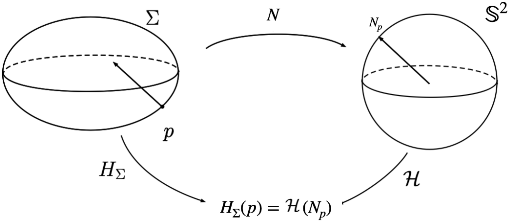

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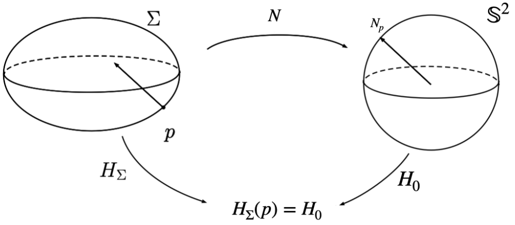

When $\mathcal{H} \equiv H_0$, Σ is a surface of constant mean curvature H_0.

Results of prescribed mean curvature surfaces/hypersurfaces

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- Bueno (2019)
  $\rightarrow$ Resolution of the Björling problem.
  $\rightarrow$ Obtention of half-space theorems for PMC surfaces.

Rotationally symmetric prescribed functions

Definition (Rotationally symmetric function)

Given a prescribed function $\mathcal{H} \in C^1(\mathbb{S}^2)$, it is said that $\mathcal{H}$ is *rotationally symmetric* if

$$\exists\, \mathfrak{h} \in C^1([-1,1]) \text{ s.t. } \mathcal{H}(x) = \mathfrak{h}(\langle x, e_3 \rangle),\ \forall x \in \mathbb{S}^2.$$

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- For such functions, we can consider PMC surfaces satisfying

$$H_\Sigma(p) = \mathcal{H}(N_p) = \mathfrak{h}(\langle N_p, e_3 \rangle) = \mathfrak{h}(\nu(p)), \quad \forall p \in \Sigma,$$

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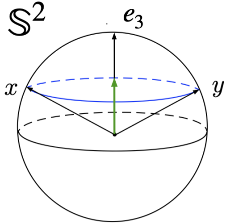

$$\langle x, e_3\rangle = \langle y, e_3\rangle \ \Rightarrow \mathcal{H}(x) = \mathcal{H}(y)$$

$\mathfrak{h}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Definition ($\mathfrak{h}$-surface in $\mathbb{H}^2 \times \mathbb{R}$)

An oriented surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ is an $\mathfrak{h}$-*surface* if

$$H_\Sigma(p) = \mathfrak{h}(\langle \eta_p, \partial_z \rangle) \quad \forall p \in \Sigma$$

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- If $\mathfrak{h} \equiv H_0 \in \mathbb{R}$, Σ is a CMC surface in $\mathbb{H}^2 \times \mathbb{R}$.
- If $\mathfrak{h}(y) = y$, Σ is a translating soliton of the mean curvature flow.

Main aim of this work

Purposes

The main purpose is to further investigate the theory of $\mathfrak{h}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ inspired by the well-known results for:

- CMC and minimal surfaces,
- Translating solitons, and
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Specifically:

1. To construct new examples of rotational $\mathfrak{h}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$.
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Main tool: study of the phase plane of the first order autonomous system satisfied by rotational $\mathfrak{h}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Summary

Profile curve of a rotational $\mathfrak{h}$-surface Σ

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- Let Σ be a *rotational $\mathfrak{h}$-surface* generated after rotating an a.l.p. curve

$$\alpha(s) = (\sinh(x(s)), 0, \cosh(x(s)), z(s)) \subset \mathbb{H}^2 \times \mathbb{R},$$

$x(s) > 0$, $s \in I \subset \mathbb{R}$, contained in a vertical plane passing through $(0, 0, 1, 0)$ around the vertical axis $\{(0, 0, 1)\} \times \mathbb{R}$.

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- The principal curvatures of Σ are

$$\kappa_1 = \kappa_\alpha = x'z'' - x''z', \qquad \kappa_2 = \frac{z'}{\tanh x}.$$

First order autonomous system

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- Now, assume that Σ is an $\mathfrak{h}$-surface for some $\mathfrak{h} \in C^1([-1,1])$, that is, $H_\Sigma(s) = \mathfrak{h}(x'(s))$.
- Then, after the change $x' = y$, the previous ODE transforms into the first order autonomous system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ \dfrac{1-y^2}{\tanh x} - 2\varepsilon\mathfrak{h}(y)\sqrt{1-y^2} \end{pmatrix} =: F_\varepsilon(x,y). \qquad (1)$$

Phase plane of rotational $\mathfrak{h}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Definition (Phase plane)

- The *phase plane* is the half-strip $\Theta_\varepsilon := (0, \infty) \times (-1, 1)$, $\varepsilon = \pm 1$, with coordinates (x, y) denoting:
  - x the distance to the axis of rotation,
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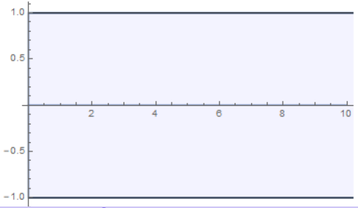
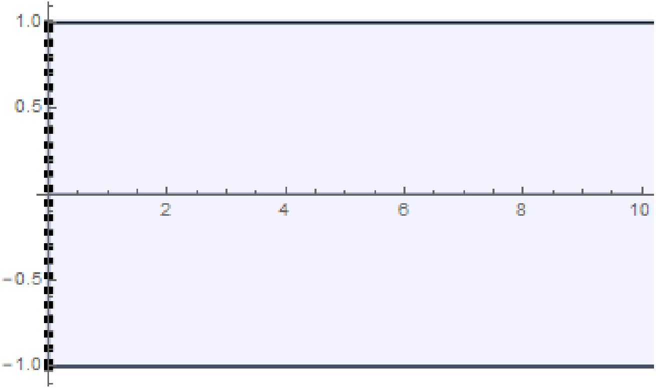


Θ_ε

Properties of the phase plane



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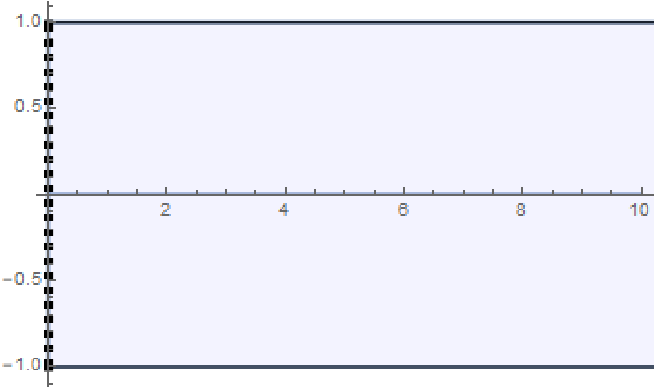

- An orbit cannot converge to a point $(0, y)$, $|y| < 1$.
- However, there exists an orbit with and endpoint at $(0, \pm 1)$. That is, a rotational $\mathfrak{h}$-surface only intersects the rotation axis orthogonally.

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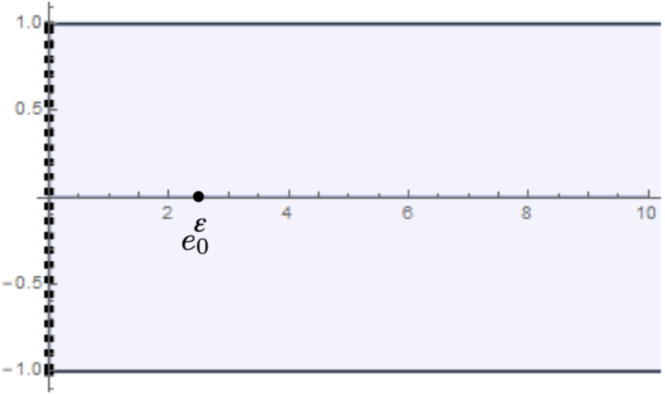

- If $\varepsilon\mathfrak{h}(0) > 1/2$, there is a unique equilibrium

$$e_0^\varepsilon = \left(\operatorname{arctanh}\left(\frac{1}{2\varepsilon\mathfrak{h}(0)}\right), 0\right) \quad \text{in } \Theta_\varepsilon$$

generating the right circular cylinder $\mathbb{S}^1(x_0^\varepsilon) \times \mathbb{R}$ of CMC $\mathfrak{h}(0)$.

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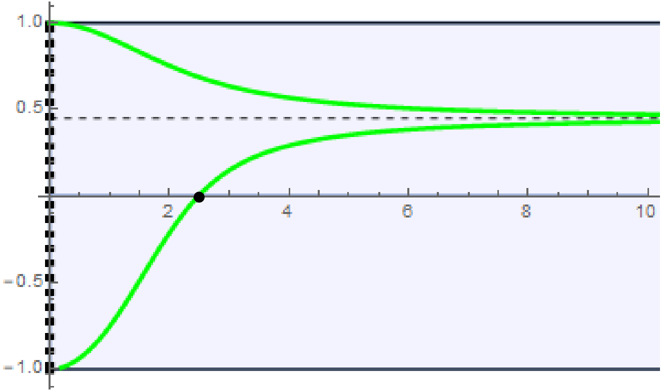

- The points of α with $\kappa_\alpha = 0$ are located in

$$\Gamma_\varepsilon(y) = \operatorname{arctanh}\left(\frac{\sqrt{1-y^2}}{2\varepsilon\mathfrak{h}(y)}\right).$$

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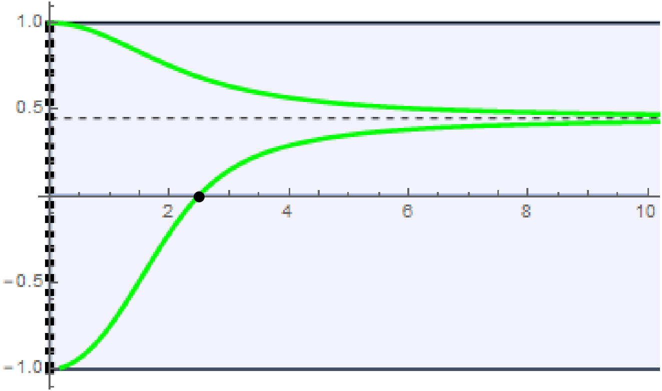

- The axis $y = 0$ and Γ_ε divide Θ_ε into connected components, where the coordinates $x(s)$ and $y(s)$ are monotonous.

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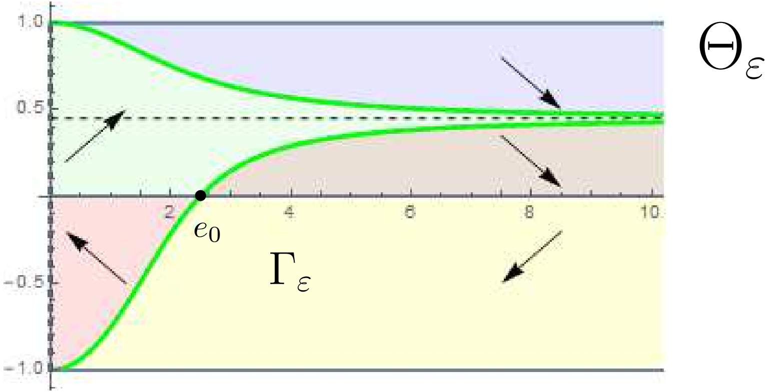

- At each monotonicity region, the motion of an orbit is uniquely determined.

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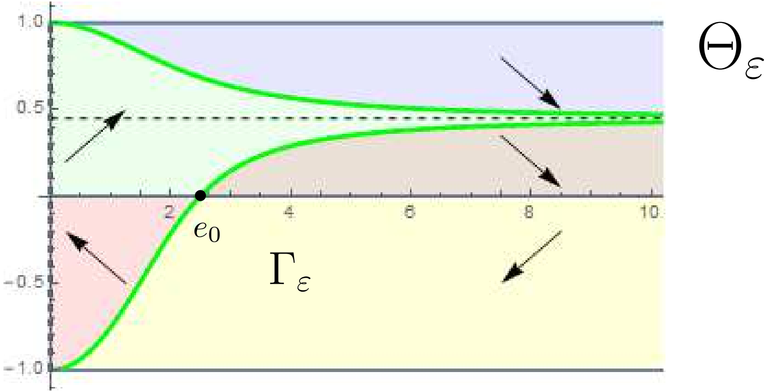

- If an orbit intersects Γ_ε, the function $y(s)$ has a local extremum.
- If an orbit intersects the axis $y = 0$, it does orthogonally.

Summary

$\mathfrak{h}$-bowls in $\mathbb{H}^2 \times \mathbb{R}$

Proposition 1 (Existence of $\mathfrak{h}$-bowls in $\mathbb{H}^2 \times \mathbb{R}$)

Let $\mathfrak{h}$ be a C^1 function on $[-1, 1]$, and suppose that $\exists y_* \in [0, 1]$ (resp. $y_* \in [-1, 0]$) s.t

$$2\varepsilon\mathfrak{h}(y_*) = \sqrt{1 - {y_*}^2}.$$

Then, there exists an upwards-oriented (resp. downwards-oriented) entire rotational $\mathfrak{h}$-graph Σ in $\mathbb{H}^2 \times \mathbb{R}$. Moreover:

1. either Σ is a *horizontal plane*,
2. or Σ is a *strictly convex graph*.

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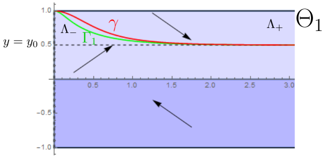
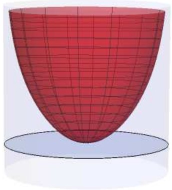
These $\mathfrak{h}$-surfaces will be called $\mathfrak{h}$-***bowls***.





The prescribed function of this figure is $\mathfrak{h}(y) = \sqrt{3}(y - 0.25)$.

$\mathfrak{h}$-catenoids in $\mathbb{H}^2 \times \mathbb{R}$

Proposition 2 (Existence of $\mathfrak{h}$-catenoids in $\mathbb{H}^2 \times \mathbb{R}$)

Let $\mathfrak{h}$ be a C^1 function on $[-1, 1]$, and suppose that

$$\mathfrak{h} \leq 0 \quad \text{and} \quad \mathfrak{h}(\pm 1) = 0.$$

Then, there exists a one-parameter family of properly embedded, rotational $\mathfrak{h}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ of strictly negative extrinsic curvature at every point, and diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Each example is a bi-graph over $\mathbb{H}^2 - \mathbb{D}_{\mathbb{H}^2}(x_0)$, where $\mathbb{D}_{\mathbb{H}^2}(x_0) = \{x \in \mathbb{H}^2 : |x|_{\mathbb{H}^2} < x_0\}$, for some $x_0 > 0$.

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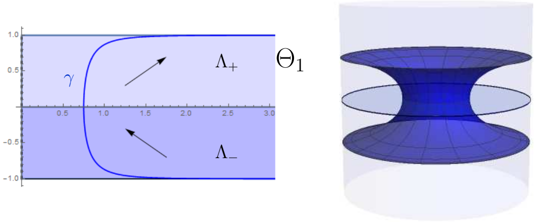

The prescribed function of this figure is $\mathfrak{h}(y) = y^2 - 1$.

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Characterizations of $\mathfrak{h}_\lambda$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$

The study of $\mathfrak{h}_\lambda$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is very natural since they are closely related to the theory of manifolds with density.

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Summary

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  (c) If $\lambda < \sqrt{2}/2$, then every orbit close enough to e_0 converges asymptotically to it *directly*, i.e. **without spiraling around**.

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For $\mathfrak{h}_\lambda$-surfaces, $\Gamma_\varepsilon(y) = \operatorname{arctanh}\left(\frac{\sqrt{1-y^2}}{2\varepsilon(y+\lambda)}\right)$.

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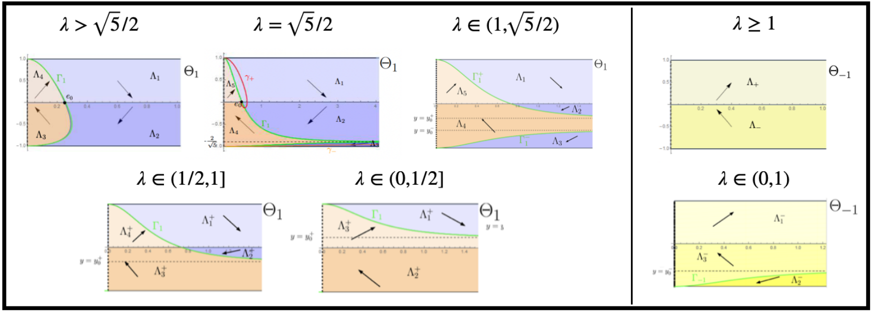

Rotational $\mathfrak{h}_\lambda$-surfaces intersecting the rotation axis

Theorem 1

Let be Σ_+ the complete, rotational $\mathfrak{h}_\lambda$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ intersecting the rotation axis with upwards orientation. Then:

1. For $\lambda > 1/2$, Σ_+ is properly embedded, simply connected and converges to the flat CMC cylinder C_λ of radius $\operatorname{arg\,tanh}\left(\frac{1}{2\lambda}\right)$. Moreover:
   - (a) If $\lambda > \sqrt{2}/2$, Σ_+ intersects C_λ infinitely many times.
   - (b) If $\lambda = \sqrt{2}/2$, Σ_+ intersects C_λ a finite number of times and is a graph outside a compact set.
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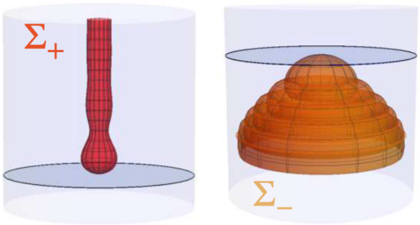
Analogously, let be Σ_- with downwards orientation. Then:

3. For $\lambda > \sqrt{5}/2$, Σ_- is properly immersed, simply connected and has unbounded distance to the rotation axis.
4. For $\lambda \leq \sqrt{5}/2$, Σ_- is an entire graph. Moreover, if $\lambda = 1$, Σ_- is a horizontal plane. Otherwise, Σ_- has positive Gauss-Kronecker curvature.

Rotational $\mathfrak{h}_\lambda$-surfaces intersecting the rotation axis

Case $\lambda > \sqrt{5}/2$



- For $\lambda > 1/2$, Σ_+ is properly embedded, simply connected and converges to C_λ intersecting it infinitely many times (as $\lambda > \sqrt{2}/2$).
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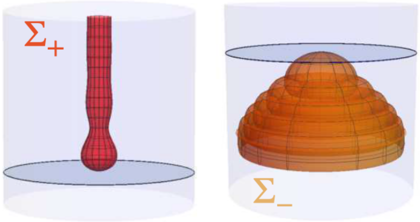

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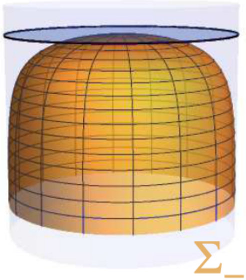

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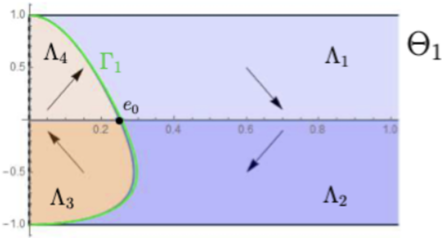

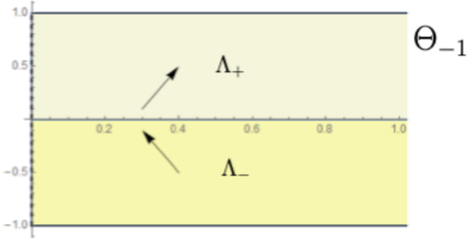

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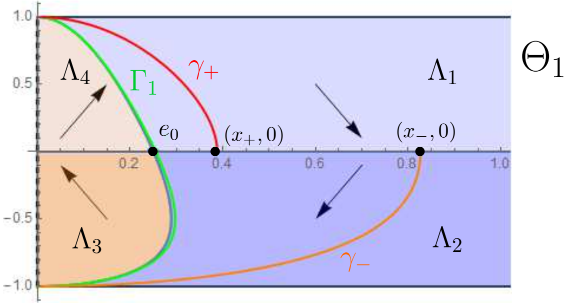

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Lemma (Behavior of γ_+ and γ_-)

1. γ_+ and γ_- intersect the axis $y = 0$ orthogonally at $(x_+, 0)$ and $(x_-, 0)$, resp., with x_+ and x_- greater than $\mathrm{arctanh}(1/(2\lambda))$.
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   Moreover, γ_+ can not go directly to e_0. Analogous for γ_-.

2. $x_+ \neq x_-$ since *there do not exist closed $\mathfrak{h}_\lambda$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$*. Moreover, if $x_- < x_+$, γ_- must converge to e_0 as $s \to -\infty$ and it contradicts the inward spiral structure of e_0.

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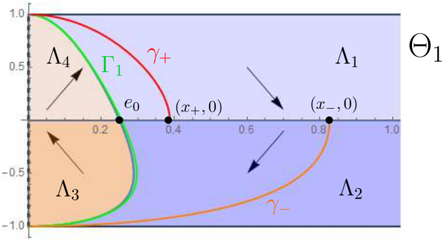
Once we have checked that the initial behavior of γ_+ and γ_- is the represented in the next Figure,



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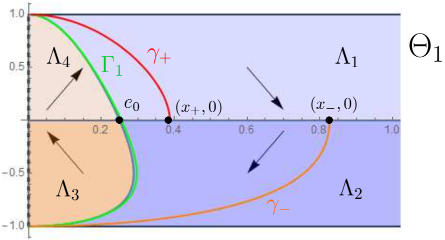
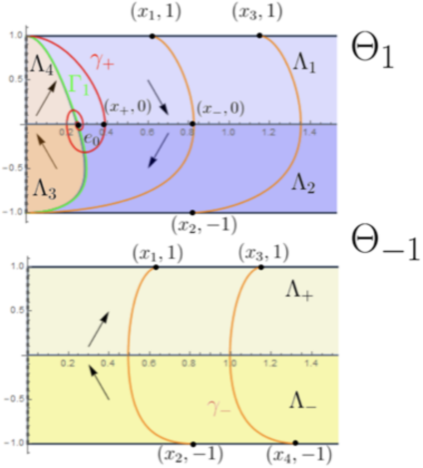


we continue analyzing γ_+ and γ_- graphically as follows.



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Third step. Let us draw the profile curves associated to γ_+ and γ_- and the corresponding $\mathfrak{h}_\lambda$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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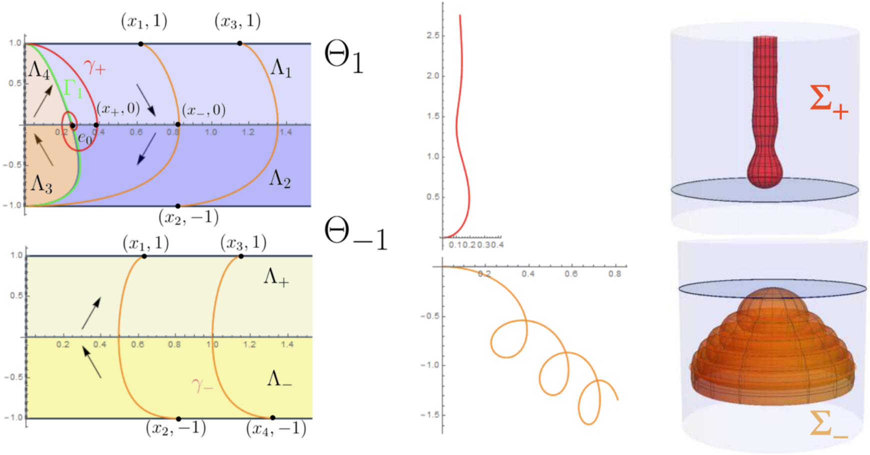

Rotational $\mathfrak{h}_\lambda$-surfaces non-intersecting the rotation axis

Theorem 2

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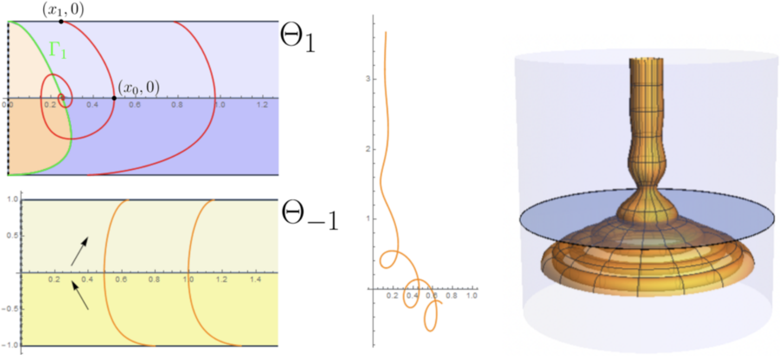
2. If $\lambda \leq 1/2$, then both ends are graphs outside compact sets.

Rotational $\mathfrak{h}_\lambda$-surfaces non-intersecting the rotation axis

Case $\lambda > \sqrt{5}/2$



- If $\lambda > 1/2$, then one end converges to C_λ with the same asymptotic behavior as in item *1* in Theorem 1, and:
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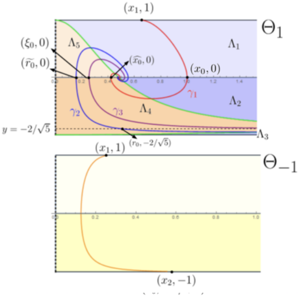

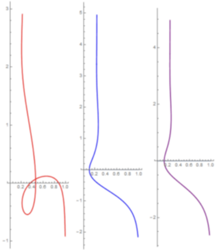

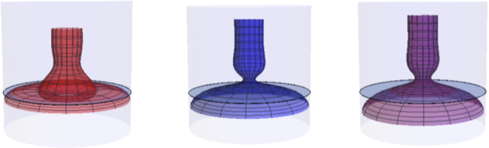

- If $\lambda > 1/2$, then one end converges to C_λ with the same asymptotic behavior as in item *1* in Theorem 1, and:
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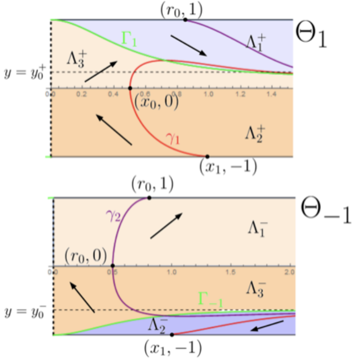
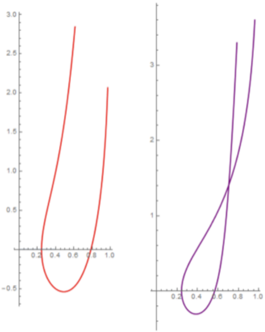
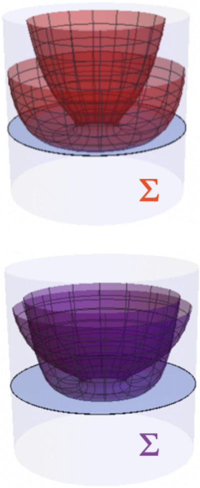
Case $\lambda \leq 1/2$







- If $\lambda \leq 1/2$, then both ends are graphs outside compact sets.

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Thank you for your attention!