On the classification of biharmonic hypersurfaces in spheres

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- We are concerned with the hypersurfaces M^m in $(m+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{m+1}$; $\phi : M^m \to \mathbb{S}^{m+1}$.
- We denote by B the second fundamental form of M and by A its shape operator. The eigenvalue functions of A are called the principal curvatures

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m.$$

- M is called isoparametric, if all the λ_i's are constant functions.
- The scalar curvature s comes form the (intrinsic) curvature tensor of M, and from the Gauss equation we have

$$\begin{aligned} s &= \operatorname{trace}\mathrm{Ricci} = m(m-1) + \sum_{i \neq j} \lambda_i \lambda_j \\ &= m(m-1) + m^2 f^2 - |A|^2. \end{aligned}$$

- The mean curvature vector field H of M is given by

$$H = \frac{1}{m} \operatorname{trace} B = \left(\frac{1}{m} \sum_i \lambda_i \right) \eta.$$

- M is minimal if the mean curvature function $f = \frac{1}{m} \operatorname{trace} A = 0$.
- M is *CMC* if f is constant.

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Theorem 3.1 ([Simons – 1968])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ *be a compact minimal hypersurface. Then*

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- or $|A|^2 \equiv m$, i.e. $\phi(M)$ is one of the Clifford tori

$$\mathbb{S}^{m_1}\left(\sqrt{\frac{m_1}{m}}\right) \times \mathbb{S}^{m_2}\left(\sqrt{\frac{m_2}{m}}\right), \qquad m_1 + m_2 = m, \quad 1 \leq m_1 \leq m-1.$$

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Since M is minimal, then $|A|^2$ is constant if and only if the scalar curvature of M is constant. In this case, it follows that the range of $|A|^2$ has a gap, more precisely $|A|^2 \notin (0, m)$.

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(C1) Chern Conjecture. ([Chern – 1968])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a compact minimal hypersurface. If $|A|^2$ is constant, then the possible values for $|A|^2$ (or, equivalently, for the scalar curvature) form a discreet set. In particular, if $m \leq |A|^2 \leq 2m$, then $|A|^2 = m$ or $|A|^2 = 2m$.

(C2) Strong version of Chern Conjecture. ([Verstraelen – 1986])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a compact minimal hypersurface. If $|A|^2$ is constant, then M is isoparametric.

A natural generalization is that to non-compact hypersurfaces, i.e. a local version of the conjecture. In particular, this has been proposed by Bryant for the case $m = 3$:

Bryant Conjecture -1994

Let $\phi : M^3 \to \mathbb{S}^4$ be a minimal hypersurface. If $|A|^2$ is constant, then M is isoparametric.

(C3) Generalized Chern Conjecture.

Let $m \geq 3$ and $\phi : M^m \to \mathbb{S}^{m+1}$ be a CMC hypersurface. If $|A|^2$ is constant, then M is isoparametric.

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- For $m = 3$, the compact case was proved in [De Almeida, Brito – 1990, Chang – 1993].

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A very interesting result related to Generalized Chern Conjecture, was obtained:

Theorem 3.2 ([De Almeida, Brito, Scherfner, Weiss – 2020])

Let $m > 3$ and $\phi : M^m \to \mathbb{S}^{m+1}$ be a CMC hypersurface. If $|A|^2$ is constant and M has three distinct principal curvatures everywhere, then M is isoparametric.

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Theorem 4.1 ([Peng, Terng – 1983])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a compact minimal hypersurface. If $|A|^2$ is constant and if $m \leq |A|^2 \leq m + \frac{1}{12m}$, then $|A|^2 = m$, so $\phi(M)$ is a Clifford torus.

During the past three decades, there has been some important progress on the Chern conjecture.

- In 1991-1998, Yang and Cheng improved the pinching constant $\frac{1}{12m}$ to $\frac{m}{3}$.
- In 2007, Suh and Yang improved this pinching constant to $\frac{3m}{7}$.

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Theorem 4.2 ([Lei, Xu, Xu – 2017])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ *be a compact minimal hypersurface. If* $m \leq |A|^2 \leq m + \frac{m}{18}$*, then* $|A|^2 = m$ *and thus* $\phi(M)$ *is a Clifford torus.*

The pinching phenomenon for hypersurfaces of constant mean curvature in spheres is much more complicated than in the minimal hypersurface case.

Theorem 4.3 ([Xu – 1993])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a compact non-minimal CMC hypersurface. If $|A|^2 \leq \alpha$, then $|A|^2$ is constant and $\phi(M)$ is either a small hypersphere of radius $\frac{1}{\sqrt{1+f^2}}$, or a Clifford torus $\mathbb{S}^1\left(\frac{\beta}{\sqrt{1+\beta^2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{1+\beta^2}}\right)$.

Here,

$$\alpha = \alpha(m,f) = m + \frac{m^3}{2(m-1)}f^2 - \frac{m(m-2)}{2(m-1)}\sqrt{m^2f^4 + 4(m-1)f^2}, \tag{1}$$

$$\beta = \beta(m,f) = f + \sqrt{\frac{\alpha - mf^2}{m(m-1)}}. \tag{2}$$

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Theorem 4.4 ([Gu, Lei, Xu – 2018])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a complete non-minimal CMC hypersurface with $|A|^2$ constant. If $|A|^2 > \alpha$, where $m \geq 4$, then

$$|A|^2 > \alpha + B_m \frac{mf^2}{m-1},$$

where

$$B_m = \begin{cases} 0.2, & 4 \leq m \leq 20 \\ 0.196, & m > 20 \end{cases}. \tag{3}$$

We note that

- $m > \alpha \Leftrightarrow f < \frac{m-2}{m}$
- $m > \alpha + B_m \frac{mf^2}{m-1} \Leftrightarrow f < \gamma < \frac{m-2}{m}$.

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Biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds. Assume that M is compact and consider

- Bienergy functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \qquad E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau_2(\phi) &= -\Delta^\phi \tau(\phi) - \operatorname{trace}_g R^N(d\phi, \tau(\phi))d\phi \\ &= 0. \end{aligned}$$

Critical points of E_2 are called biharmonic maps.

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where

$$\Delta^{\phi} = -\operatorname{trace}_g \left(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\phi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
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Biharmonic submanifolds

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A submanifold $\phi : M^m \to N^n$ is biharmonic if and only if

$$\operatorname{trace} A_{\nabla^{\perp}_{\cdot} H}(\cdot) + \operatorname{trace} \nabla A_H + \operatorname{trace}\left(R^N(\cdot, H)\cdot\right)^{\top} = 0$$

and

$$\Delta^{\perp} H + \operatorname{trace} B\left(\cdot, A_H(\cdot)\right) + \operatorname{trace}\left(R^N(\cdot, H)\cdot\right)^{\perp} = 0.$$

- For hypersurfaces, we can see also [Ou – 2010].

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Examples of proper-biharmonic submanifolds

- In spaces of non-positive curvature, with just one exception (see [Ou, Tang – 2012]), we have only non-existence results, i.e., biharmonicity implies harmonicity (minimality); in particular in $\mathbb{R}^n$ we have only non-existence results.
- In spaces of positive curvature, especially in Euclidean spheres, we have many examples and classification results for proper-biharmonic submanifolds (see, for example [Balmuş, Montaldo, Oniciuc – 2012]).

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Biharmonic hypersurfaces in $\mathbb{S}^{m+1}$

Theorem 6.1

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a hypersurface. Then M is biharmonic if and only if

$$A(\operatorname{grad} f) = -\frac{m}{2} f \operatorname{grad} f$$

and

$$\Delta f = \left(m - |A|^2\right) f.$$

Biharmonic hypersurfaces in $\mathbb{S}^{m+1}$

- If $f \equiv const \neq 0$, then M^m is proper-biharmonic if and only if $|A|^2 \equiv m$.
- If $f \equiv 0$, hypersurfaces with $|A|^2 \equiv m$ were already classified in the famous paper [Chern, Do Carmo, Kobayashi – 1970].

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- The study of proper-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ with f constant, i.e., CMC, can be seen as a natural generalization of the above mentioned classical problem.

Biharmonic hypersurfaces in $\mathbb{S}^{m+1}$

The only known examples of proper-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ (see [Caddeo, Montaldo, Oniciuc – 2001, Jiang – 1986]) are:

- open parts of the small hypersphere of radius $\frac{1}{\sqrt{2}}$, i.e., $\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)$,
- open parts of the Clifford tori $S^{m_1}\left(\frac{1}{\sqrt{2}}\right)\times S^{m_2}\left(\frac{1}{\sqrt{2}}\right)$, with $m_1 \neq m_2$ and $m_1+m_2=m$

Moreover, it was proved that, under various additional geometric assumptions, the proper-biharmonic hypersurfaces have to be the above ones or (at least) they must be CMC. Consequently, the following two conjectures have been proposed in 2008 (see [Balmuş, Montaldo, Oniciuc – 2008]).

(B1) Conjecture 1.

Any proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is either an open part of $\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)$, or an open part of $S^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times S^{m_2}\left(\frac{1}{\sqrt{2}}\right)$, with $m_1 \neq m_2$ and $m_1 + m_2 = m$.

(B2) Conjecture 2.

Any proper-biharmonic submanifold in $\mathbb{S}^n$ is CMC.

Until now, *B1* was proved in several particular cases:

- $m = 2$ (see [Caddeo, Montaldo, Oniciuc – 2001]),
- $m = 3$ and the hypersurface M is complete (see [Balmuş, Montaldo, Oniciuc – 2010]),
- M^m has at most two distinct principal curvatures at any point (see [Balmuş, Montaldo, Oniciuc – 2008]),
- M^m is isoparametric (see [Ichiyama, Inoguchi, Urakawa – 2010]),
- M^m is CMC and has non-positive sectional curvature (see [Oniciuc – 2012]),
- M^m is compact and belongs to a hemisphere (see [Vieira – 2022]).

There is a deep link between the proof of *B1* knowing that *B2* is true and the *Generalized Chern Conjecture*, as we will explain below.

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- Further, as a non-minimal CMC hypersurfaces in $\mathbb{S}^{m+1}$ is proper-biharmonic if and only if $|A|^2 = m$, if the *Generalized Chern Conjecture* will be proved to be true, then our *B1* will follow immediately using *B2* and the results in [Ichiyama, Inoguchi, Urakawa – 2010]. Therefore, the proof of *B1* under the CMC hypothesis can be seen as a special case of *Generalized Chern Conjecture*.
- However, the *Generalized Chern Conjecture* seems very difficult to be proved in its full generality. We think that there are more chances to prove the Generalized Chern Conjecture in the special case when $|A|^2 = m$, equivalently, to prove *B1* using *B2*.

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We recall the famous conjecture

Chen Conjecture [B.-Y. Chen – 1991]

Any biharmonic submanifold in $\mathbb{R}^n$ is minimal.

Since any CMC biharmonic submanifold in $\mathbb{R}^n$ is minimal (see [Dimitrić – 1992]), the *Chen Conjecture* can be reformulated in a weaker form:

Chen Conjecture.

Any biharmonic submanifold in $\mathbb{R}^n$ is CMC.

(B2) Conjecture 2.

Any proper-biharmonic submanifold in $\mathbb{S}^n$ is CMC.

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As it is not clear how *B2* could imply *B1*, we propose an intermediary objective.

Open Problem.

Let M^m be a CMC proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$. Then, the set of all possible values of the mean curvature is discreet and, more precisely,

$$f \in \left\{ \frac{m-2r}{m} \quad \middle| \quad r \in \mathbb{N},\ 0 \leq r \leq s^* \right\},$$

where $s^* = s-1$, if $m = 2s$, and $s^* = s$, if $m = 2s+1$.

Remark.

Finally, one should further prove that, if $f = \frac{m-2r}{m}$, then M^m must be an open part of $S^r\left(\frac{1}{\sqrt{2}}\right) \times S^{m-r}\left(\frac{1}{\sqrt{2}}\right)$.

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First, we recall that a CMC proper-biharmonic hypersurface in the Euclidean unit sphere has $f \in (0,1]$. More precisely, we have

Theorem 8.1 ([Balmuş, Oniciuc – 2012])

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a CMC proper-biharmonic hypersurface with $m > 2$. Then $f \in \left(0, \frac{m-2}{m}\right] \cup \{1\}$. Moreover, $f = 1$ if and only if $\phi(M)$ is an open subset of the small hypersphere $\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)$, and $f = \frac{m-2}{m}$ if and only if $\phi(M)$ is an open subset of the standard product $\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{2}}\right)$.

- We mention that Theorem 8.1 can be reobtained, under the additional assumption M^m compact, from Theorem 4.3.
- All the pinching results related to Chern Conjecture, with only one exception, lead us, for our biharmonic problem, either to the same conclusion as in Theorem 8.1, or they give us no information.
- The only pinching result which can be useful for us and improve Theorem 8.1 is Theorem 4.4.

Our result is an improvement of Theorem 8.1. We show that there is a larger gap for f than $\left(\frac{m-2}{m},1\right)$. More precisely, considering $m \geq 4$ and denoting

$$\gamma = (m-2)\sqrt{\frac{m-1}{m^2(m-1)+B_m\left(B_m+m^2\right)}} \in \left(\frac{m-3}{m},\frac{m-2}{m}\right), \tag{4}$$

where

$$B_m = \begin{cases} 0.2, & 4 \leq m \leq 42 \\ 0.199, & 43 \leq m \leq 65 \\ 0.198, & 66 \leq m \leq 149 \\ 0.197, & m \geq 150 \end{cases}, \tag{5}$$

we have the following result.

Theorem 8.2 ([S.N. – 2022])

Let M^m be a complete CMC proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$. If $m \geq 4$ and the mean curvature $f \in \left[\gamma, \frac{m-2}{m}\right]$, then $f = \frac{m-2}{m}$ and $M = \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{2}}\right)$.

A direct consequence of Theorem 8.2 is the next result, which gives the new gap for f.

Corollary 8.3 ([S.N. – 2022])

Let M^m be a complete CMC proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$, with $m \geq 4$. Then

$$f \in (0, \gamma) \cup \left\{\frac{m-2}{m}\right\} \cup \{1\}.$$

Moreover, $f = \frac{m-2}{m}$ if and only if $M = \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{2}}\right)$, and $f = 1$ if and only if $M = \mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)$.

Notations:

- For our objective, it will be more convenient to replace the principal curvatures λ_i of M by
$$\lambda_i = \sqrt{m(1-f^2)}\mu_i + f.$$
Now, the advantage of using μ_i's is that
$$\sum_{i=1}^{m} \mu_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \mu_i^2 = 1. \tag{6}$$
- We also need to consider on the hypersurface M^m three functions ϕ, η and σ given by
$$\phi = \sum_{i=1}^{m} \mu_i^3 + \frac{m-2}{\sqrt{m(m-1)}}, \quad \eta = \sqrt{\frac{m}{m-1}}\mu_1 + 1, \quad \sigma = \sqrt{\sum_{i=2}^{m} \left(\mu_i + \frac{\mu_1}{m-1}\right)^2}. \tag{7}$$

Intermediate results for Theorem 8.2

Lemma 8.4 ([Gu, Lei, Xu – 2018])

If $m \geq 3$, the real numbers ϕ, η and σ satisfy

$$\frac{\sqrt{m(m-1)}}{m-2}\phi \geq \eta \geq \frac{\sigma^2}{2} \tag{8}$$

and

$$\phi\sqrt{m(m-1)} \geq \eta\left[3m-3(m+1)\eta-2\sigma\sqrt{m(m-1)}\right]. \tag{9}$$

Intermediate results for Theorem 8.2

In order to state the next lemma, we consider a positive number defined by

$$\begin{aligned} \alpha_0 &= \alpha - mf^2 = \\ &= m + \frac{m^3}{2(m-1)}f^2 - \frac{m(m-2)}{2(m-1)}\sqrt{m^2f^4 + 4(m-1)f^2} - mf^2. \end{aligned} \tag{10}$$

Lemma 8.5 ([Gu, Lei, Xu – 2018])

If $m \geq 3$, then the following equality holds

$$(m-2)f\sqrt{\frac{m}{m-1}\alpha_0} = m\left(f^2+1\right) - \alpha_0. \tag{11}$$

Intermediate results for Theorem 8.2

From the definitions of σ and η we get a link between the difference $\mu_2 - \mu_1$ and a quantity which contains σ, η and m. This we will be useful to prove the third lemma.

$$\mu_2 - \mu_1 \geq \left(1 - \eta - \sigma\sqrt{\frac{m-1}{m}}\right)\sqrt{\frac{m}{m-1}}. \tag{12}$$

Lemma 8.6 ([S.N. – 2022])

Let $m \geq 4$. *If*

$$\phi \leq \frac{B_m}{2}\sqrt{\frac{m}{m-1}}, \tag{13}$$

then $2\sigma + 3\eta < \frac{3}{4}$ *and*

$$\mu_2 - \mu_1 > \frac{2}{3 - 3^{-10}}\sqrt{\frac{m}{m-1}}. \tag{14}$$

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Sketch of the proof of Theorem 8.2

Steps of the proof:

1. we prove that the smallest distinct principal curvature has constant multiplicity 1, so λ_1 is smooth on M.
2. we compute and estimate $\Delta\lambda_1$
3. using Omori-Yau maximum principle, we conclude that A is parallel, and then we get $f = \frac{m-2}{m}$.
4. using Okumura Lemma, we obtain that $M = \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{2}}\right)$.

Sketch of the proof of Theorem 8.2

Step 1: Proof of smoothness of λ_1 on M

- for any CMC hypersurface in $\mathbb{S}^{m+1}$ (see [Nomizu, Smyth – 1969]), we have on M:

$$\frac{1}{2}\Delta|A|^2 = -|\nabla A|^2 + m^2 f^2 + |A|^2\left(|A|^2 - m\right) - mf\,\mathrm{trace}\,A^3. \tag{15}$$

- using the biharmonicity hypothesis, i.e., $|A|^2 = m$, and the definition of ϕ, from (15), we obtain, on M,

$$|\nabla A|^2 + mf\phi\sqrt{m_0^3} = m_0\left[m_0 - m\left(f^2+1\right) + (m-2)f\sqrt{\frac{m}{m-1}m_0}\,\right], \tag{16}$$

  where $m_0 = m - mf^2$.
- finding convenient upper bound for the term in the right hand side of (16), we prove

$$\phi < \frac{B_m}{2}\sqrt{\frac{m}{m-1}}.$$

Sketch of the proof of Theorem 8.2

Step 1 (continue):

Now, we can apply Lemma 8.6 and achieve $\mu_2 > \mu_1$, which is equivalent to $\lambda_2 > \lambda_1$ on M^m.

Therefore, since the smallest principal curvature λ_1 of M has (constant) **multiplicity** 1 on M, it follows that it is **smooth** on M and there exists a local smooth unit vector field E_1 such that $A(E_1) = \lambda_1 E_1$ (see [Nomizu – 1973]).

We note that, as λ_1 has multiplicity 1, we expect to obtain $M = \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{2}}\right)$.

Then, we can find a local expression of $\Delta\lambda_1$ but, in order to work with, it is more convenient to fix arbitrarily a point p and consider $\{e_1, \ldots, e_m\}$ an orthonormal basis which diagonalize the shape operator A such that $e_1 = E_1(p)$.

Let

$$b_{ijk} = \langle (\nabla A)(e_i, e_j), e_k \rangle$$

be the components of the totally symmetric tensor $\langle (\nabla A)(\cdot, \cdot), \cdot \rangle$.

Sketch of the proof of Theorem 8.2

Step 2: Compute and estimate $\Delta\lambda_1$

- Since M^m is a *CMC* hypersurface in $\mathbb{S}^{m+1}$, it was shown in [Gu, Lei, Xu – 2018] that, at p,

$$\Delta\lambda_1 = mf + \left(|A|^2 - m\right)\lambda_1 - mf\lambda_1^2 - 2\sum_{\substack{i=1\\k\geq 2}}^{m} \frac{b_{i1k}^2}{\lambda_1 - \lambda_k}.$$

- Since M^m is proper-biharmonic and therefore $|A|^2 = m$, we have, at p,

$$\Delta\lambda_1 = mf - mf\lambda_1^2 - 2\sum_{\substack{i=1\\k\geq 2}}^{m} \frac{b_{i1k}^2}{\lambda_1 - \lambda_k}. \tag{17}$$

- By some computations, using also equation (16), one gets, at p

$$\begin{aligned}\Delta\lambda_1 \quad = \quad & \eta\sqrt{m_0}\left\{-(\eta-2)(m-1)f\sqrt{m_0} + \left[m_0 - m\left(f^2+1\right)\right]\sqrt{\frac{m-1}{m}}\right\} - \\ & - \quad \sqrt{\frac{m-1}{mm_0}}\,|\nabla A|^2 - m_0\phi f\sqrt{m(m-1)} - \frac{2}{\sqrt{m_0}}\sum_{\substack{i=1\\k\geq 2}}^{m} \frac{b_{i1k}^2}{\mu_1 - \mu_k}.\end{aligned} \tag{18}$$

Sketch of the proof of Theorem 8.2

Step 2 (continue):

- Using Lemma 8.6, $1+B_m \leq 6/5$ and some convenient estimations, one gets at p,

$$\Delta\lambda_1 \leq -3^{-11}\sqrt{\frac{m-1}{mm_0}}\,|\nabla A|^2 - \frac{2}{5}m_0\phi f\sqrt{m(m-1)} \leq 0. \tag{19}$$

- As the point p was arbitrarily fixed, we conclude that $\Delta\lambda_1 \leq 0$ on M.

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Sketch of the proof of Theorem 8.2

Step 3: Omori-Yau maximum principle

Definition 8.7

Let M^m be a Riemannian manifold. We say that M admits Omori-Yau maximum principle for the Laplacian if for any $u \in C^2(M)$ with $u^* = \sup_M u < \infty$, there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M$ that satisfies

$$i)\ u(p_k) > u^* - \frac{1}{k}; \quad ii)\ |\nabla u(p_k)| < \frac{1}{k}; \quad iii)\ \Delta u(p_k) > -\frac{1}{k}, \quad k \in \mathbb{N}.$$

Theorem 8.8 ([Omori – 1967])

Any complete Riemannian manifold whose Ricci curvature has a lower bound admits Omori-Yau maximum principle for the Laplacian.

Sketch of the proof of Theorem 8.2

Step 3 (continue):
Since $|A|^2 = m$, we obtain $\mathrm{Ricci}(X,X) \geq -2m(m-1)$, for any $X \in C(TM)$, $|X| = 1$. Knowing also that M is complete and λ_1 is smooth and bounded by $\sqrt{m}$, it follows that there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M$ such that

$$(\Delta\lambda_1)(p_k) > -\frac{1}{k}.$$

But we have seen that $\Delta\lambda_1 \leq 0$ at any point of M, so,

$$\lim_{k\to\infty} (\Delta\lambda_1)(p_k) = 0,$$

and, moreover, from (19), we deduce

$$\lim_{k\to\infty} |\nabla A|^2 (p_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} \phi(p_k) = 0.$$

Now, using (16), it follows that

$$|\nabla A| \equiv 0 \quad \text{and} \quad \phi \equiv 0.$$

Then, it easy to see that

$$f = \frac{m-2}{m}.$$

Sketch of the proof of Theorem 8.2

Step 4: Okumura Lemma

Lemma 8.9 ([Okumura – 1974])

Let $\mu_1, \mu_2, \cdots, \mu_m$ be real numbers such that $\sum_{i=1}^m \mu_i = 0$. Then

$$-\frac{m-2}{\sqrt{m(m-1)}}\left(\sum_{i=1}^m \mu_i^2\right)^{3/2} \leq \sum_{i=1}^m \mu_i^3 \leq \frac{m-2}{\sqrt{m(m-1)}}\left(\sum_{i=1}^m \mu_i^2\right)^{3/2}.$$

Moreover, equality holds on the right-hand (respectively, left-hand) side if and only if $(m-1)$ of the μ_i's are non-positive (repectively, non-negative) and equal.

Finally, from $\phi \equiv 0$, we see that we have equality in the Okumura Lemma (in the left-hand side), and, it is not difficult to prove that

$$\lambda_1 = -1 \qquad \text{and} \qquad \lambda_2 = \lambda_3 = \cdots = \lambda_m = 1.$$

Thus, M is the Clifford torus $\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m-1}\left(\frac{1}{\sqrt{2}}\right)$. $\square$

Further work

We recall that

$$\gamma \in \left(\frac{m-3}{m}, \frac{m-2}{m} \right).$$

Our objective is to find a better value of γ, closer to $\frac{m-4}{m}$!

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Thank you for your attention!