Magnetic Jacobi fields in almost contact metric manifolds

Differential Geometry workshop, Vienna 2022

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... are given by a second order nonlinear differential equation

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Geodesics and magnetic curves

... are given by a second order nonlinear differential equation

Geodesics: γ in a Riemannian manifold (M, g): **kinetic energy**

$$E(\gamma) = \int_a^b \frac{1}{2} |\gamma'(s)|^2 \, ds$$

Let ω be the **potential 1-form**. Consider the **Landau Hall functional**

$$LH(\gamma) = \int_a^b \left(\frac{1}{2} \langle \gamma'(t), \gamma'(t) \rangle + \omega(\gamma'(t)) \right) dt.$$

Magnetic curves

The critical points of the LH functional are solutions of the equation $\frac{d}{d\epsilon}LH(\gamma_\epsilon)\Big|_{\epsilon=0} = 0$, that is

$$\frac{d}{d\epsilon}LH(\gamma_\epsilon)\Big|_{\epsilon=0} = -\int_a^b g\big(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V\big)dt = 0,$$

Magnetic curves

The critical points of the LH functional are solutions of the equation $\frac{d}{d\epsilon} LH(\gamma_\epsilon)\Big|_{\epsilon=0} = 0$, that is

$$\frac{d}{d\epsilon} LH(\gamma_\epsilon)\Big|_{\epsilon=0} = -\int_a^b g\big(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V\big)dt = 0,$$

which is equivalent to

$$\nabla_{\gamma'}\gamma' - \phi(\gamma') = 0$$

known as the **Lorentz equation**.

Background

(M, g) Riemannian manifold; ($\dim M = n \geq 2$)

Lorentz force ϕ: $g(\phi(X), Y) = 2d\omega(X, Y)$, X, Y tangent to M

Background

(M, g) Riemannian manifold; $(\dim M = n \geq 2)$

magnetic field: F - closed 2-form on M $F = 2d\omega$

Lorentz force ϕ: $g(\phi(X), Y) = F(X, Y)$, X, Y tangent to M

Background

(M, g) Riemannian manifold; $(\dim M = n \geq 2)$
magnetic field: F - closed 2-form on M $\qquad F = 2d\omega$
Lorentz force ϕ: $g(\phi(X), Y) = F(X, Y)$, X, Y tangent to M

A smooth curve γ in (M, g, F) is called
magnetic curve/trajectory/geodesic of (M, g, F)
if its velocity vector field γ' satisfies the **Lorentz equation**:

$$\nabla_{\gamma'}\gamma' = \phi(\gamma')$$

Magnetic Jacobi fields

Magnetic Jacobi fields

A second variational formula for the integral LH:

$$\frac{D^2}{ds^2}W - R(\dot{\gamma}, W)\dot{\gamma} - \phi\left(\frac{D}{ds}W\right) - (\nabla_W \phi)\,\dot{\gamma} = 0.$$

R : the Riemannian curvature tensor of M.

Almost contact metric manifolds

A (φ, ξ, η) structure:

- a field φ of endomorphisms of tangent spaces,
- a vector field ξ and
- a 1-form η

satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

When (M, φ, ξ, η) is endowed with a **compatible** Riemannian metric g

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \text{ for all } X, Y \in \mathfrak{X}(M),$$

then M is said to have an *almost contact metric structure*, and $(M, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*.

Sasakian manifolds

the fundamental 2*-form*:

$$\Omega(X,Y) = g(X,\varphi Y), \text{ for all } X,Y \in \mathfrak{X}(M), \tag{1}$$

If $\Omega = d\eta$, then (M,φ,ξ,η,g) is called a *contact metric manifold*.

A *Sasakian manifold* is defined as a **normal** contact metric manifold.

Characterization:

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \quad \text{for any } \; X,Y \in \mathfrak{X}(M).$$

Cosymplectic manifolds

$(M, \varphi, \xi, \eta, g)$ with $d\eta = 0$ and $d\Omega = 0$ is said to be an *almost cosymplectic* manifold.

If an almost cosymplectic structure is normal, we get a *cosymplectic* manifold.

Characterization:

$$\nabla \varphi = 0.$$

Magnetic Jacobi fields in cosymplectic manifolds

Contact magnetic fields in cosymplectic manifolds

Magnetic field: $F = -q\Omega$, $q \in \mathbb{R}$
Lorentz force: $\phi = q\varphi$ is **uniform**

Magnetic Jacobi field equation:

$$\frac{D^2}{ds^2}W - R(\dot\gamma, W)\dot\gamma - \phi\left(\frac{D}{ds}W\right) - \cancel{(\nabla_W \phi)\,\dot\gamma} = 0.$$

paralellism of ϕ

Contact magnetic fields in cosymplectic manifolds

Magnetic field: $F = -q\Omega$, $q \in \mathbb{R}$
Lorentz force: $\phi = q\varphi$

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$$\frac{D^2}{ds^2}W - R(\dot\gamma, W)\dot\gamma - \phi\left(\frac{D}{ds}W\right) = 0.$$

Magnetic Jacobi Fields in $\mathbb{E}^3$

Magnetic curves in $\mathbb{E}^3$:

$$\ddot{\gamma}(s) = V(s) \times \dot{\gamma}(s), \quad \text{where} \quad V(s) = V(\gamma(s)) \text{ is divergence free.}$$

The *Lorentz force* : $\phi : \mathfrak{X}(\mathbb{E}^3) \to \mathfrak{X}(\mathbb{E}^3), \quad \phi X = V \times X, \quad \forall X \in \mathfrak{X}(\mathbb{E}^3).$

Take the Killing vector field $V_0 = \dfrac{\partial}{\partial z}$. $\qquad V = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}.$

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Take the Killing vector field $V_0 = \dfrac{\partial}{\partial z}$.

Magnetic trajectories are helices with axis V_0:

$$\gamma(t) = (x_0 + a\cos t,\ y_0 + a\sin t, z_0 + bt), \quad (x_0, y_0, z_0) \in \mathbb{E}^3, \quad a, b \in \mathbb{R}.$$

Magnetic Jacobi Fields in $\mathbb{E}^3$

Magnetic Jacobi equation:

$$W''(s) - V(s) \times W'(s) - (\nabla_W V) \times \dot{\gamma}(s) = 0,$$

where we take $V = q\dfrac{\partial}{\partial z}, \qquad q \in \mathbb{R} \setminus \{0\}.$

The magnetic Jacobi equation :

$$W''(s) - q\frac{\partial}{\partial z} \times W'(s) = 0,$$

Magnetic Jacobi Fields in $\mathbb{E}^3$

Theorem (–, Nistor : 2021)

Let $\gamma(s)$ be a normal magnetic curve corresponding to the Killing vector field $q\dfrac{\partial}{\partial z}$ in $\mathbb{E}^3$. Then, the magnetic Jacobi fields along γ are given by:

(i) $W(s) = W_0 + as\dfrac{\partial}{\partial z}$,

(ii) $W(s) = W_0 + \dfrac{\sin qs}{q}v_0 - \cos qs\ \phi v_0 + as\dfrac{\partial}{\partial z}$,

where W_0 is a constant vector in $\mathbb{R}^3$, v_0 is a constant vector orthogonal to $\dfrac{\partial}{\partial z}$ and $a \in \mathbb{R}$.

Examples of magnetic Jacobi fields in $\mathbb{E}^3$

Initial conditions	**magnetic Jacobi field** $W(s)$
$W(0) = (0,0,1)$, $W'(0) = (0,0,0)$	$(0,0,1)$
$W(0) = (0,0,\lambda)$, $W'(0) = (0,0,1)$	$(0,0,s+\lambda)$, $\lambda \in \mathbb{R}$
$W(0) = (0,0,0)$, $W'(0) = (\cos\psi, \sin\psi, 0)$, $\psi \in \mathbb{R}$	$\dfrac{\sin qs}{q}(\cos\psi, \sin\psi, 0) + \dfrac{1-\cos qs}{q}(-\sin\psi, \cos\psi, 0)$

Examples of magnetic Jacobi fields in $\mathbb{E}^3$

Remark. The expression of the magnetic trajectory γ is not, seemingly, explicitly involved. However, the function $\langle W'(s), \dot{\gamma} \rangle$ is constant.

Examples of magnetic Jacobi fields in $\mathbb{E}^3$

Change the Killing vector field: $V = q\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

Then the magnetic Jacobi equation becomes

$$W''(s) - V(s) \times W'(s) - q\left(\frac{\partial}{\partial z} \times W(s)\right) \times \dot{\gamma}(s) = 0$$

(the expression of γ appears explicitly)

Solve this ODE!!: γ is complicated!!)

Examples of magnetic Jacobi fields in $\mathbb{E}^3$

Change the Killing vector field: $V = q\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

not cosymplectic

Then the magnetic Jacobi equation becomes

$$W''(s) - V(s) \times W'(s) - q\left(\frac{\partial}{\partial z} \times W(s)\right) \times \dot{\gamma}(s) = 0$$

(the expression of γ appears explicitly)

Remark. The function $\langle W'(s), \dot{\gamma}\rangle$ is still constant.

Arbitrary cosymplectic manifolds

Proposition (General result)

On an arbitrary Riemannian manifold (M, g) endowed with a magnetic field, the (unit) speed vector $\dot{\gamma}$ is **always** a magnetic Jacobi field along the magnetic curve γ.

Arbitrary cosymplectic manifolds

Proposition (conservation law)

Let W be a magnetic Jacobi field along the contact magnetic curve γ in a cosymplectic manifold of arbitrary dimension. Then, the function $g(\frac{D}{ds}W(s), \dot{\gamma}(s))$ is constant.

Arbitrary cosymplectic manifolds

Proposition (conservation law)

Let W be a magnetic Jacobi field along the contact magnetic curve γ in a cosymplectic manifold of arbitrary dimension. Then, the function $g(\frac{D}{ds}W(s), \dot{\gamma}(s))$ is constant.

Proposition (–, Nistor : 2021)

Let W_1 and W_2 be two magnetic Jacobi fields along the contact magnetic curve γ in a cosymplectic manifold of arbitrary dimension. Then, the function $g(\frac{D}{ds}W_1(s), W_2(s)) - g(\frac{D}{ds}W_2(s), W_1(s)) + qg(W_1(s), \varphi W_2(s))$ is constant.

Cosymplectic manifolds of dimension 3

Proposition (–, Nistor : 2021)

The characteristic vector field ξ of a cosymplectic manifold M^3 is a magnetic Jacobi field along any normal contact magnetic curve.

Proposition (–, Nistor : 2021)

Let γ be a contact magnetic curve on the cosymplectic three dimensional manifold M^3, such that γ is not an integral curve of ξ. Then, $\varphi\dot{\gamma}$ is a magnetic Jacobi field along γ if and only if M^3 is a cosymplectic space form $M^3(c)$ with $c = 0$.

Proof. $\mathcal{J}_F(\varphi\dot{\gamma}) = \frac{r}{2}\sin^2\theta\varphi\dot{\gamma}.$

Magnetic Jacobi fields in 3-dimensional cosymplectic space forms

1. The characteristic vector field ξ as magnetic field: $\dot{\gamma}(s) = \xi(s)$.

Theorem (–, Nistor : 2021)

Let γ be an integral curve of ξ in a cosymplectic manifold $(M^3, \varphi, \xi, \eta, g)$ and let $F = -q\phi$ be the magnetic field of strength q.
The magnetic Jacobi field along γ is given by:

either $W(s) = W_0(s) + (f_0 + f_1 s)\xi(\gamma(s)) + \sin qs v_0(s) - \cos qs \varphi v_0(s)$,

or $W(s) = W_0(s) + (f_0 + f_1 s)\xi(\gamma(s))$, where

$v_0(s)$ is a vector field parallel along $\gamma(s)$ lying in the contact distribution $\ker \eta$
W_0 is a linear combination, with constant coefficients, of $v_0(s)$ and $\varphi v_0(s)$.

Magnetic Jacobi fields in 3-dimensional cosymplectic space forms

② The case when $\dot{\gamma}(s) \nparallel \xi$; in particular, γ can be a Legendre curve, $\dot{\gamma} \perp \xi$.
Use the basis $\{\dot{\gamma}, \varphi\dot{\gamma}, \xi\}$ to decompose

$$W(s) = A(s)\dot{\gamma}(s) + B(s)\varphi\dot{\gamma}(s) + C(s)\xi(s), \quad A, B, C \in C^{\infty}(I)$$

Magnetic Jacobi fields in 3-dimensional cosymplectic space forms

② The case when $\dot{\gamma}(s) \nparallel \xi$; in particular, γ can be a Legendre curve, $\dot{\gamma} \perp \xi$.
Use the basis $\{\dot{\gamma}, \varphi\dot{\gamma}, \xi\}$ to decompose

$$W(s) = A(s)\dot{\gamma}(s) + B(s)\varphi\dot{\gamma}(s) + C(s)\xi(s), \quad A, B, C \in C^{\infty}(I)$$

the magnetic Jacobi equation is equivalent to

$$\begin{cases} A''(s) - qB'(s) = 0, \\ B''(s) + qA'(s) + cB(s)\sin^2\theta = 0, \\ C''(s) + q\cos\theta B'(s) = 0. \end{cases}$$

Solve the system

EQ1 yields $A'(s) = qB(s) + c_0$, $c_0 \in \mathbb{R}$.

Use EQ2 we get: $B''(s) + \mu B(s) + qc_0 = 0$, where $\mu := q^2 + c\sin^2\theta$

Solve the system

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Use EQ2 we get: $B''(s) + \mu B(s) + qc_0 = 0$, where $\mu := q^2 + c\sin^2\theta$

The sign of μ decides!

Solve the system

EQ1 yields $A'(s) = qB(s) + c_0$, $c_0 \in \mathbb{R}$.

Use EQ2 we get: $B''(s) + \mu B(s) + qc_0 = 0$, where $\mu := q^2 + c\sin^2\theta$

$\mu = 0$:

$$B(s) = -qc_0\frac{s^2}{2} + c_1 s + c_2, \quad c_0, c_1, c_2 \in \mathbb{R}$$

$$A(s) = -q^2 c_0\frac{s^3}{6} + c_1 q\frac{s^2}{2} + (c_0 + c_2 q)s + c_3,$$

$$C(s) = q^2\cos\theta c_0\frac{s^3}{3} - q\cos\theta c_1\frac{s^2}{2} + (c_4 - c_2 q\cos\theta)s + c_5, \quad c_3, c_4, c_5 \in \mathbb{R}.$$

Solve the system

EQ1 yields $A'(s) = qB(s) + c_0$, $c_0 \in \mathbb{R}$.

Use EQ2 we get: $B''(s) + \mu B(s) + qc_0 = 0$, where $\mu := q^2 + c\sin^2\theta$

$\mu > 0$: $\mu = k^2$ $(k > 0)$.

$$
\begin{aligned}
A(s) &= \frac{qc_1}{k}\sin ks - \frac{qc_2}{k}\cos ks + \frac{c_0 c\sin^2\theta}{k^2}s + c_3, \\
B(s) &= c_1\cos ks + c_2\sin ks - \frac{qc_0}{k^2}, \quad c_0, c_1, c_2, c_3 \in \mathbb{R}, \\
C(s) &= -\frac{q\cos\theta}{k}(c_1\sin ks - c_2\cos ks) + c_4 s + c_5, \quad c_4, c_5 \in \mathbb{R}.
\end{aligned}
$$

Solve the system

EQ1 yields $A'(s) = qB(s) + c_0$, $c_0 \in \mathbb{R}$.

Use EQ2 we get: $B''(s) + \mu B(s) + qc_0 = 0$, where $\mu := q^2 + c\sin^2\theta$

$\mu < 0 \quad \mu = -k^2$ ($k > 0$).

$$A(s) = \frac{qc_1}{k}\sinh ks + \frac{qc_2}{k}\cosh ks - \frac{c_0 c\sin^2\theta}{k^2}s + c_3,$$

$$B(s) = c_1\cosh ks + c_2\sinh ks + \frac{qc_0}{k^2}, \quad c_0, c_1, c_2, c_3 \in \mathbb{R},$$

$$C(s) = -\frac{q\cos\theta}{k}(c_1\sinh ks + c_2\cosh ks) + c_4 s + c_5, \quad c_4, c_5 \in \mathbb{R}.$$

Jacobi magnetic fields on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

Theorem (–, Nistor : 2021)

Let $W(s) = (\bar{W}(s), a(s))$ be a magnetic Jacobi field along the normal contact magnetic curve $\gamma(s) = (\bar{\gamma}(s), t(s))$ in the product manifold $M^3 = N \times \mathbb{R}$, where N denotes the 2-sphere $\mathbb{S}^2$ or the hyperbolic plane $\mathbb{H}^2$.

Then $\bar{W}$ is a magnetic Jacobi field along $\bar{\gamma}$ on N and a is an affine function. The converse also holds.

Magnetic Jacobi fields in Sasakian manifolds

Sasakian space forms of dimension 3

Proposition (Inoguchi, – : 2022)

The Reeb vector field ξ is a magnetic Jacobi field along any normal contact magnetic curve in a Sasakian 3-manifold M.

Proposition (Inoguchi, – : 2022)

Let γ be a normal contact magnetic curve on a 3-dimensional Sasakian space form. Then $\varphi\gamma'$ is a magnetic Jacobi field along γ if and only if either it is an integral curve of the Reeb vector field ξ, or the holomorphic sectional curvature of M is 1.

Sasakian space forms of dimension 3

γ is an integral curve of ξ
$\{\xi(\gamma(s)), E(s), \varphi\, E(s)\}$: o.n. and parallel basis along γ

magnetic Jacobi fields:

$$W(s) = f(s)\xi(\gamma(s)) + a(s)E(s) + b(s)\varphi\, E(s)$$

$$f'' = 0,$$
$$a'' + (q+1)a + qb' = 0,$$
$$b'' + (q+1)b - qa' = 0.$$

Sasakian space forms of dimension 3

γ' is not colinear to ξ

$W(s) = f(s)\xi(\gamma(s)) + a(s)\gamma'(s) + b(s)\varphi\gamma'(s)$

$$f'' + (2 + q\cos\theta)b' = 0,$$
$$a'' - (q + 2\cos\theta)b' = 0,$$
$$b'' + qa' + b(c-1)\sin^2\theta - 2f' = 0.$$

Problem solved!

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