Harmonic maps and morphisms from g-natural metrics on tangent bundles

Giovanni Calvaruso

University of Salento

(joint works with M.T.K. Abbassi and D. Perrone)

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Table of contents

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The unit tangent sphere bundle over (M, g) is the hypersurface of TM defined by

$$T_1M = \{(x, u) \in TM : g_x(u, u) = 1\}.$$

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At any point $(x, u) \in TM$,

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

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The horizontal lift of a vector $X \in M_x$ is $X^h \in \mathcal{H}_{(x,u)}$, such that $d\pi(X^h) = X$.
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EXAMPLE

In the compact case, a vector field $V : (M, g) \to (TM, g^s)$ is a harmonic map if and only if it is parallel (Nouhaud 1977, Ishihara 1979).
The existence of a parallel vector field forces M to be locally reducible.

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As their name suggests, those metrics are constructed in a "natural" way from a Riemannian metric g over M ([Kowalski and Sekizawa, 1988], [Kolář, Michor and Slovák, 1993]).

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(a) the *Sasaki lift* g^s (positive definite) is given by

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The three lifts above permit to describe the whole class of g-natural metrics on TM. They are also the image of g under first order natural operators $D : S^2_+ T^* \rightsquigarrow (S^2 T^*)T$, which transform Riemannian metrics on M into (possibly degenerate) metrics on TM. [Kowalski and Sekizawa, 1988].

g-natural metrics on TM depend on six smooth functions $\alpha_i,\ \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$. Explicitly,

$$\begin{aligned}
G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\
&\quad + (\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\
G_{(x,u)}(X^h, Y^v) &= G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2) g_x(X, Y) \\
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G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u),
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for every $u,\ X,\ Y \in M_x$, where $r^2 = g_x(u, u)$.

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Setting

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(\mathrm{t})$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

G is Riemannian if and only if, for all $t \geq 0$,

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0.$$

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- the Cheeger–Gromoll metric g^{CG} [Cheeger and Gromoll, 1972] is obtained when
  $\alpha_2 = \beta_2 = 0$, $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$.

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- metrics of Cheeger-Gromoll type $g_{m,r}$ [Benyounes, Loubeau and Wood, 2007] are obtained for
  $\alpha_1(t) = \frac{1}{(1+t)^m}$, $\alpha_3 = 1 - \alpha_1$, $\alpha_2 = \beta_2 = 0$,
  $\beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m}$.

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- Oproiu metrics [Oproiu, 1999] are obtained when there exist two smooth functions $v, w : \mathbb{R}^+ \to \mathbb{R}$, such that

$$\begin{aligned} \alpha_1(t) &= \frac{1}{v(t/2)}, & \alpha_2 &= 0, & (\alpha_1+\alpha_3)(t) &= v(t/2), \\ \beta_1(t) &= -\frac{w(t/2)}{v(t/2)[v(t/2)+w(t/2)]}, & \beta_2 &= 0, & (\beta_1+\beta_3)(t) &= w(t/2). \end{aligned}$$

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- Kaluza–Klein metrics [Wood, 1990] are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.
- We defined the class of metrics of Kaluza–Klein type, which includes all previous examples, by the geometric condition of orthogonality between horizontal and vertical distributions: $\alpha_2 = \beta_2 = 0$.

TECHNICALITIES...

The Levi-Civita connection $\bar{\nabla}$ of (TM, G) is completely determined by $\bar{\nabla}_{X^h} Y^h$, $\bar{\nabla}_{X^h} Y^v$, $\bar{\nabla}_{X^v} Y^h$, $\bar{\nabla}_{X^v} Y^v$.

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$$\dots$$

for all vector fields X, Y on M and $(x, u) \in TM$, where

$$\begin{aligned} A(u; X, Y) &= A_1[R(X, u)Y + R(Y, u)X] + A_2[g_x(Y, u)X + g_x(X, u)Y] \\ &\quad + A_3 g_x(R(X, u)Y, u)u + A_4 g_x(X, Y)u + A_5 g_x(X, u) g_x(Y, u)u, \end{aligned}$$

with

$$\begin{aligned} A_1 &= -\frac{\alpha_1\alpha_2}{2\alpha}, \qquad A_2 = \frac{\alpha_2(\beta_1+\beta_3)}{2\alpha}, \qquad A_3 = \frac{\alpha_2\{\alpha_1[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]+\alpha_2(\beta_1\alpha_2-\beta_2\alpha_1)\}}{\alpha\phi}, \\ A_4 &= \frac{\phi_2(\alpha_1+\alpha_3)'}{\phi}, \\ A_5 &= \frac{\alpha\phi_2(\beta_1+\beta_3)'+(\beta_1+\beta_3)\{\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)\}}{\alpha\phi} \dots \end{aligned}$$

A smooth map $\varphi : (M', g') \to (M, g)$ between two Riemannian manifolds induces the decomposition of the tangent space at a point $x \in M'$ as

$$M'_x = H^\varphi_x \oplus V^\varphi_x,$$

where V^φ_x :=ker$(d\varphi_x)$ and $H^\varphi_x = (V_x)^\perp$. V^φ_x and H^φ_x are respectively called the *vertical* and *horizontal* spaces at the point x w.r.to φ.

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$\varphi : (M', g') \to (M, g)$ is said to be *horizontally (weakly) conformal* if, for every point $x \in M'$, either $d\varphi_x = 0$ or $d\varphi_x$ is surjective and $g(d\varphi_x(X), d\varphi_x(Y)) = \lambda^2(x) g'(X, Y)$ for any $X, Y \in H^{\varphi}_x$, for some $\lambda(x) > 0$.

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- *horizontally homothetic* when φ is horizontally conformal with dilation function λ such that grad(λ) is vertical, that is, its projection on H^φ vanishes;
- a *Riemannian submersion up to scale* when φ is horizontally conformal with constant dilation function $\lambda = k > 0$. In this case, φ is a Riemannian submersion after a suitable homothetic change of the metric on either the domain or the codomain.

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In order to decide whether π is horizontally conformal, we first clarify the relationship between the decomposition $(TM)_{(x,u)} = H^{\pi}_{(x,u)} \oplus V^{\pi}_{(x,u)}$ and the canonical decomposition $(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

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$$H^{\pi}_{(x,u)} = \left\{ X^h - \frac{\alpha_2}{\alpha_1} X^v + \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\alpha_1\phi_1} g(X,u)u^v \; : \; X \in M_x \right\}.$$

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In particular, $H^{\pi}_{(x,u)} \equiv \mathcal{H}_{(x,u)}$ at each point $(x,u) \in TM$ if and only if $\alpha_2 = \beta_2 = 0$, that is, when G is of Kaluza-Klein type.

Theorem

$\pi : (TM, G) \to (M, g)$ is horizontally conformal if and only if

$$(*) \quad \alpha_1(\beta_1 + \beta_3)\phi_1 + \alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) - \alpha_1\beta_2\phi_2 = 0.$$

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Consequences:

$\pi : (TM, G) \to (M, g)$

- is a Riemannian submersion up to scale if and only if (*) holds and $\alpha = k^2\alpha_1$. Hence, Riemannian g-natural metrics G on TM for which $\pi : (TM, G) \to (M, g)$ is a Riemannian submersion up to scale depend on four arbitrary functions and a positive constant $k = \lambda$.

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$$(*) \quad \alpha_1(\beta_1 + \beta_3)\phi_1 + \alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) - \alpha_1\beta_2\phi_2 = 0.$$

In this case, the dilation function is given by $\lambda = \sqrt{\frac{\alpha_1(\alpha_1+\alpha_3)-\alpha_2^2}{\alpha_1}}$.

Consequences:

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- is a Riemannian submersion up to scale if and only if (*) holds and $\alpha = k^2\alpha_1$. Hence, Riemannian g-natural metrics G on TM for which $\pi : (TM, G) \to (M, g)$ is a Riemannian submersion up to scale depend on four arbitrary functions and a positive constant $k = \lambda$.
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(ii) is a Riemannian submersion up to scale if and only if $\alpha_1 + \alpha_3 = k^2$ and $\beta_1 + \beta_3 = 0$ (in particular, for all metrics of Cheeger-Gromoll type $h_{m,r}$).

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The fibres of $\pi : (TM, G) \to (M, g)$ are totally geodesic if and only if there exists some real constant κ, such that, for all $t \geq 0$,

$$\alpha_2(t) = \frac{\kappa}{\sqrt{\phi_1}}(t \cdot \alpha_1'(t) + \alpha_1(t)), \qquad \beta_2(t) = \frac{\kappa}{\sqrt{\phi_1}}(\beta_1(t) - \alpha_1'(t)).$$

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All metrics of Kaluza-Klein type satisfy the above conditions (for $\kappa = 0$).

A *harmonic map* $f : (M', g') \to (M, g)$ between Riemannian manifolds is a critical point of the energy functional

$$\mathcal{E}(f, \Omega) := \frac{1}{2} \int_{\Omega} ||df||^2 dv_{g'},$$

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Thus, a map $\varphi : (M', g') \to (M, g)$ is a harmonic morphism if, for any open set U of M with $\varphi^{-1}(U) \neq \emptyset$ and any harmonic function f on $(U, g|_U)$, the map $f \circ \varphi$ is a harmonic function on $(\varphi^{-1}(U), g'|_{\varphi^{-1}(U)})$.

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A fundamental characterization states that *a smooth map is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal* [Fuglede 1978, Ishihara 1979].

The following fundamental characterization holds:

Theorem [Baird and Eells, 1981]

Let $\varphi : (M, g) \to (N, h)$ be a smooth nonconstant horizontally weakly conformal map between Riemannian manifolds of dimensions $m, n \geq 1$ respectively.
Then φ is harmonic (and so, a harmonic morphism) if and only if, at every regular point, the mean curvature vector field μ^{V^φ} of the fibres and the gradient of the dilation λ of φ are related by

$$(n-2)\left(\mathrm{grad}\,\ln\lambda\right)^{H^\varphi} + (m-n)\mu_{V^\varphi} = 0.$$

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(a) is a harmonic morphism if and only if the functions α_i, β_i defining the metric G satisfy

$$\begin{cases} \alpha_1(\beta_1+\beta_3)\phi_1 + \alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) - \alpha_1\beta_2\phi_2 = 0, \\ \frac{(n-2)\phi_2(\alpha_1'\alpha - \alpha'\alpha_1)}{\alpha_1\alpha\phi} + \frac{(n-1)(\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1'))}{\alpha_1\phi} + \frac{2\phi_1\phi_2' - \phi_2\phi_1'}{\phi_1\phi} = 0. \end{cases}$$

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As a consequence, Riemannian g-natural metrics G on the tangent bundle TM, for which $\pi : (TM, G) \to (M, g)$ is a harmonic morphism, form a large class, which depends on four arbitrary smooth functions.

In the special case when $\alpha_2 = \beta_2 = 0$, we have the following results.

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(a) The property that the canonical projection $\pi : (TM, g^{CG}) \to (M, g)$ is a harmonic morphism was previously proved in [Gudmundsson and Kappos, 2002].

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Remarks

(a) The property that the canonical projection $\pi : (TM, g^{CG}) \to (M, g)$ is a harmonic morphism was previously proved in [Gudmundsson and Kappos, 2002].

(b) Kaluza-Klein metrics do not exhaust the class of Riemannian g-natural metrics for which $\pi : (TM, G) \to (M, g)$ is a harmonic morphism.

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At any point $(x, u) \in T_1M$ the tangent space splits as

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where $\mathcal{T}_{(x,u)}$ is spanned by tangential lifts.
Given $X \in M_x$, its *tangential lift*, w.r.to with respect to a g-natural metric G inducing $\tilde{G}$ on T_1M, is the tangential projection of X^v to (x, u) with respect to the unit normal vector

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a+c+d)\phi}}[-bu^h + (a+c+d)u^v],$$

where a, b, c, d are real constants and $\phi := a(a+c+d) - b^2$. Explicitly,

$$X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\, N^G_{(x,u)} = X^v - \sqrt{\tfrac{\phi}{a+c+d}} g_x(X, u)\, N^G_{(x,u)}.$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

g-natural metrics on T_1M are then completely determined by

$$\begin{aligned}
\tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X,Y) + d\, g_x(X,u)g_x(Y,u),\\
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$\tilde{G}$ is Riemannian if and only if

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$\tilde{G}$ is of Kaluza-Klein type if $b = 0$, a Kaluza-Klein metric if $b = d = 0$.

With regard to lifts of geodesics, we have the following.

Proposition

The horizontal lift of any geodesic of (M, g) is a geodesic of $(T_1M, \tilde{G})$ if and only if either $\tilde{G}$ is a Kaluza-Klein metric or (M, g) is flat and $d = 0$.

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The canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is given by $\pi_1 = \pi \circ i$, where $i : (T_1M, \tilde{G}) \hookrightarrow (TM, G)$ is the inclusion map.

Fix any point $(x, u) \in TM$. Then, it is easy to check that $\mathcal{T}_{(x,u)} \equiv V^{\pi_1}_{(x,u)}$. However, $\mathcal{H}_{(x,u)}$ needs not be orthogonal to $\mathcal{T}_{(x,u)}$.

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$$\mu_{V^{\pi_1}} = -\frac{b}{a(a+c+d)}\, u^h$$

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and so, it is collinear to the geodesic flow vector field $\tilde{\xi} = u^h$.
In particular, the fibres of $(T_1M, \tilde{G})$ are minimal if and only if $\tilde{G}$ is of Kaluza-Klein type (that is, $b = 0$).

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In particular, *metrics of Cheeger-Gromoll type on T_1M are Riemannian submersions up to scale.*

The following characterization holds for the harmonicity of $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$.

Theorem

Let $\tilde{G}$ be an arbitrary Riemannian g-natural metric on the unit tangent sphere bundle T_1M of a Riemannian manifold (M, g).

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- **(iii)** $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic map.
- **(iv)** $\tilde{G}$ is a metric of Kaluza-Klein type.

With regard to harmonic morphisms, we have the following characterization.

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Let (M, g) be a Riemannian manifold of dimension $n > 1$ and $(T_1M, \tilde{G})$ its unit tangent bundle, equipped with an arbitrary Riemannian g-natural metric $\tilde{G}$.

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In particular, $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic morphism when $\tilde{G}$ is of Cheeger-Gromoll type.

ADDENDUM: canonical projection from $TM - \{0\}$ to T_1M

The radial projection

$$\varphi : \mathbb{R}^n - \{0\} \to \mathbb{S}^{n-1}, \ \ x \mapsto \frac{x}{||x||}$$

leads in a natural way to investigate the properties of the canonical projection

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Moreover, $\Phi : (TM - \{0\}, G) \to (T_1M, \tilde{G})$ is horizontally conformal (and so, a harmonic morphism) if and only if

$$\alpha_3(t) = \left(\frac{a+c}{a} t - 1\right) \alpha_1(t), \qquad \beta_3(t) = \frac{d}{a}\alpha_1(t) - \beta_1(t).$$

In this case, the dilation function is given by $\lambda(x, u) = \sqrt{\frac{a}{t\alpha_1(t)}}$, where $t = ||u||^2$, and Φ is horizontally homothetic.

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(c) For $G, \tilde{G}$ of Kaluza-Klein type, $\Phi : (TM - \{0\}, G) \to (T_1M, \tilde{G})$ is a Riemannian submersion up to scale, with dilation $\lambda = k > 0$, if and only if $\alpha_1(t) = \frac{a}{k^2 t}$.

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$$(\alpha_3 + t\beta_3)' = (n-1)[\beta_1 - \alpha_1' - (\alpha_1 + \alpha_3)']$$

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- either G is of Kaluza-Klein type,
- or (M, g) is an Einstein manifold, with Ricci operator $Qu = \kappa u$ for all u, and
$$2(\alpha_2 + t\beta_2)' = \kappa\alpha_2 - (n-1)\beta_2.$$

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$$\frac{1}{\phi}\left[\phi_2\phi_1(\phi_1+\phi_3)' + \phi_2(\phi+\phi_3)\phi_1' - 2\phi_1(\phi_1+\phi_3)\phi_2'\right]$$
$$+\frac{n-1}{\alpha}\Big[\ \phi_2\alpha' - \phi_1[\alpha_2(\beta_1+\beta_3) - (\alpha_1{+}_3)\beta_2] - \phi_2[(\alpha_1+\alpha_3)\beta_1 - \alpha_2\beta_2]\Big] = 0,$$
$$\frac{1}{\phi}\left[(\phi_2^2-\phi)(\phi_1+\phi_3)' + (\phi+\phi_3)^2\phi_1' - 2\phi_2(\phi_1+\phi_3)\phi_2'\right]$$
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Corollary

g^S is harmonic w.r.to Riemannian g-natural metrics G of Kaluza-Klein type satisfying

$$\beta_1 = \frac{(\alpha(\alpha_1+\alpha_3))'}{\alpha_1+\alpha_3}, \qquad \beta_3 = \frac{1-\kappa}{\kappa t}\phi_1 - \frac{1}{t}\alpha_3, \qquad \kappa > 0.$$

The tension fields of $\mathrm{id}_{\tilde{g^s}\tilde{G}} : (T_1M, \tilde{g^s}) \to (T_1M, \tilde{G})$ and $\mathrm{id}_{\tilde{G}\tilde{g_s}} : (T_1M, \tilde{G}) \to (T_1M, \tilde{g^s})$ are respectively given by

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Vielen Dank für Ihre freundliche Aufmerksamkeit!