Surfaces of prescribed linear Weingarten curvature in $\mathbb{R}^3$

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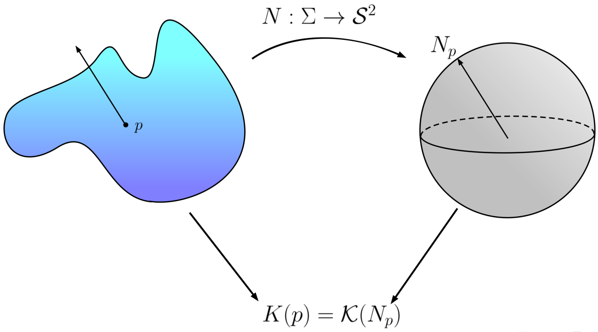
The Minkowski problem

Definition

Let be $\mathcal{K} \in C^1(\mathbb{S}^2)$. An immersed surface Σ in $\mathbb{R}^3$ has prescribed curvature in terms of its Gauss map if its Gauss curvature satisfies

$$K(p) = \mathcal{K}(N_p), \qquad \forall p \in \Sigma,$$

where $N : \Sigma \to \mathbb{S}^2$ is the Gauss map.



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- Originally posed and solved by Minkowski.
- An ovaloid of prescribed curvature $\mathcal{K}$ exists if and only if

$$\int_{\mathbb{S}^2} \frac{x}{\mathcal{K}(x)} = 0.$$

- Motivated further spectacular research: Alexandrov, Pogorelov, Nirenberg, Yau, Chen...

Prescribed mean curvature

Definition

Let be $\mathcal{H} \in C^1(\mathbb{S}^2)$. An immersed surface Σ in $\mathbb{R}^3$ has prescribed mean curvature in terms of its Gauss map if its mean curvature satisfies

$$H(p) = \mathcal{H}(N_p), \quad \forall p \in \Sigma,$$

where $N : \Sigma \to \mathbb{S}^2$ is the Gauss map.

- Existence of ovaloids: Alexandrov, Pogorelov, Hartman, Wintner, B. Guan, P. Guan, Gálvez, Mira.
- Complete surfaces: developed in the Ph.D. of the first author (supervised by Gálvez and Mira).
  - Structure of properly embedded surfaces.
  - Classification of rotational surfaces.
  - Stability.
- Half-space theorems.
- Existence and uniqueness of the Björling problem.

Surfaces treated in this talk

Definition

Let be $\Phi \in C^1(\mathbb{S}^2)$, $a, b \in \mathbb{R}$. *An immersed surface* Σ *in* $\mathbb{R}^3$ *has prescribed linear Weingarten curvature if*

$$2aH(p) + bK(p) = \Phi(N_p), \qquad \forall p \in \Sigma,$$

where $N : \Sigma \to \mathbb{S}^2$ *is the Gauss map. For short, we say that* Σ *is a* Φ*-surface.*

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Paramount particular case

- Linear Weingarten surfaces: $\Phi = c \in \mathbb{R}$.
- Generalize constant Gauss and mean curvature.
- Studied by Rosenberg, Sa Earp, Toubiana...

Main topic in this talk

Our objective is twofold

i) Classification of rotational Φ-surfaces. Some hypotheses for Φ needed

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- Case $\mathcal{H} = H_0 \in \mathbb{R}$.
  - For $H_0 = 0$ we have planes and catenoids.
  - For $H_0 \neq 0$ we have the Delaunay surfaces (sphere, cylinder, unduloids, nodoids).
- Case $\mathcal{K} = K_0 \in \mathbb{R}$. Distinguish between the sign of K_0.

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Hypotheses on Φ

Inspired by the study of prescribed mean and Gauss curvature surfaces, we assume:

- $\Phi \neq 0$.
- Φ is antipodally symmetric, i.e. $\Phi(X) = \Phi(-X), \ \forall X \in \mathbb{S}^2$.

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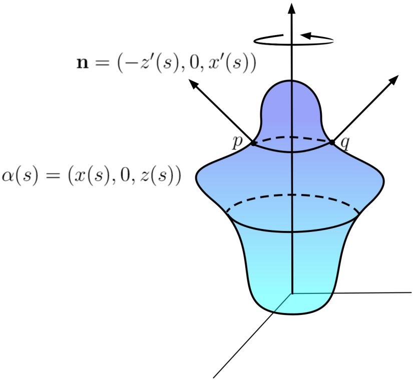
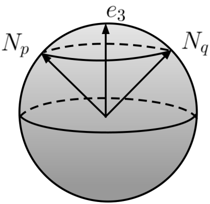
ii) Exhibit a complete classification for linear Weingarten surfaces. Depends on the sign of the discriminant $\Delta = a^2 + bc$.
   - Elliptic: $\Delta > 0$. Rosenberg, Sa Earp, Toubiana. Some examples are missed.
   - Hyperbolic: $\Delta < 0$. López. There is a mistake.

The differential equations





$$\Phi(N_p) = \Phi(N_q)$$

$$\Phi(X) = \phi(\langle X, e_3\rangle)$$

- $\phi \in C^1([-1,1])$ and even.
- $x' = \cos\theta, \;\; z' = \sin\theta.$
- $2H = \theta' + \dfrac{\sin\theta}{x}.$
- $K = \theta' \dfrac{\sin\theta}{x}.$

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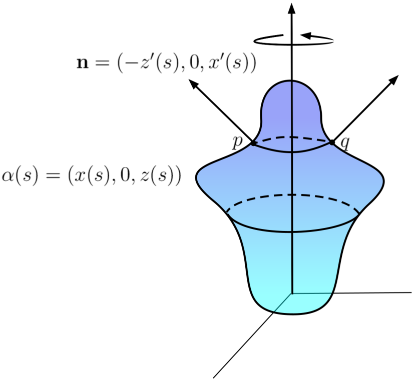

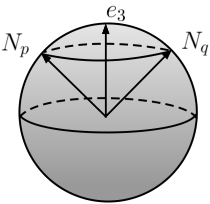

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The prescribed curvature equation

$$a\left(\theta' + \frac{\sin\theta}{x}\right) + b\frac{\theta'\sin\theta}{x} = \phi(\cos\theta).$$

The differential equations

- Solving θ' we arrive to

$$\theta' = \frac{x\phi(\cos\theta) - a\sin\theta}{ax + b\sin\theta},$$

wherever $ax + b\sin\theta \neq 0$.

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A phase plane analysis

The differential system

$$\begin{pmatrix} x \\ \theta \end{pmatrix}' = \begin{pmatrix} \cos\theta(s) \\ \dfrac{x\phi(\cos\theta) - a\sin\theta}{ax + b\sin\theta} \end{pmatrix}$$

- We define the phase plane as

$$\Theta := (0,\infty) \times (0, 2\pi) - \mathcal{S},$$
$$\Theta_1 = \Theta \cap \{\theta < \pi\}, \text{ symmetric w.r.t. } \theta = \pi/2$$
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where the compact arc

$$\mathcal{S} := \left\{(x,\theta) : \; x = \mathcal{S}(\theta) := \frac{-b\sin\theta}{a}\right\}$$

corresponds to singular points.

- The solutions $\gamma(s) = (x(s), \theta(s))$ are called orbits. Cauchy problem: for any $(x_0, \theta_0) \in \Theta$, there exists a unique orbit passing through it.

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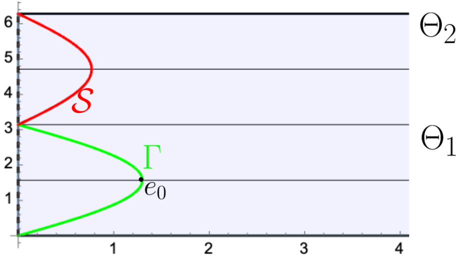
Properties of the phase plane

- $\theta' = 0$ iff $x\phi(\cos\theta) = a\sin\theta$.
- The compact arc $\Gamma := \Theta \cap \{x = \Gamma(\theta)\}$, where $\Gamma(\theta) := \frac{a\sin\theta}{\phi(\cos\theta)}$ corresponds to points of the profile curve α with vanishing curvature.
- The equilibrium $e_0 = (\frac{a}{\phi(0)}, \frac{\pi}{2}), a > 0$; $e_0 = (\frac{-a}{\phi(0)}, \frac{3\pi}{2}), a < 0$, corresponds to a CMC cylinder of vertical rulings.



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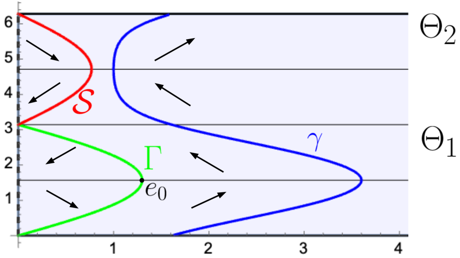
- The curves $\mathcal{S}$ and Γ and the lines $\theta = \pi/2, \theta = 3\pi/2$ divide Θ into monotonicity regions, where the coordinates of each orbit are monotonous.



The underlying PDE

- Let Σ be a Φ-surface, $\Sigma = \mathrm{graph}(u),\ u:\Omega\to\mathbb{R}$. Then, u is a solution of the PDE

$$a\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)+b\frac{det(D^2u)}{(1+|Du|^2)^2}=\Phi\left(\frac{(-Du,1)}{\sqrt{1+|Du|^2}}\right).$$

- This can be written as $\mathfrak{F}(u_x,u_y,u_{xx},u_{yy},u_{xy})=0$.
- The discriminant of this PDE is

$$\mathfrak{F}_{u_{xx}}\mathfrak{F}_{u_{yy}}-\frac{1}{4}\mathfrak{F}_{u_{xy}}^2=(1+u_x^2+u_y^2)^2(a^2+b\Phi).$$

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Definition

- *If $a^2 + b\Phi > 0$, Σ is of elliptic type.*
- *If $a^2 + b\Phi < 0$, Σ is of hyperbolic type.*
- *If $a^2 + b\Phi = 0$, Σ is of parabolic type. Leads to Φ being constant. A. Bueno, R. López, Radial solutions for equations of Weingarten type, J. Math. Anal. Appl.* ***517*** *(2023).*

Existence of radial solutions

Theorem (Bueno-López)

Let be $\Phi \in C^1(\mathbb{S}^2),\ a, b \in \mathbb{R}$.

- *If* $a^2 + b\Phi > 0$, *there exists a rotational* Φ*-surface intersecting orthogonally the axis of rotation.*
- *If* $a^2 + b\Phi < 0$, *there does not exist any* Φ*-surface intersecting orthogonally the axis of rotation.*

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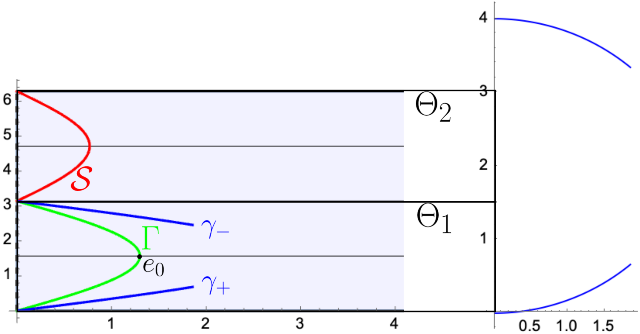

The elliptic case

$$2aH + bK = \Phi(N),\ a^2 + b\Phi > 0.$$

- We obtain a total of 10 rotational elliptic Φ-surfaces.
- Six common examples, and two different depending on $b > 0, b < 0$.

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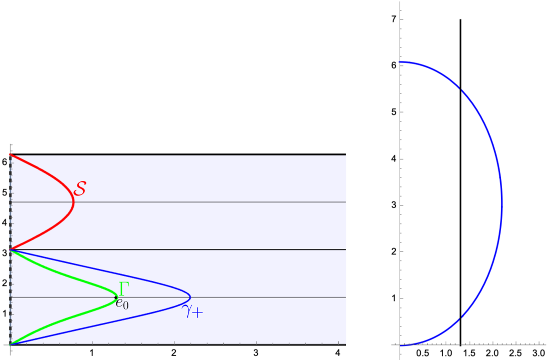
- Six common examples, and two different depending on $b > 0, b < 0$. We assume $a > 0$.
- In particular, we fully classify linear Weingarten surfaces ($\Phi = c$), settling this problem.

The elliptic case: $b > 0$

- The equilibrium generates a CMC cylinder.
- The coordinates of the orbit γ_+ cannot satisfy $x \to \infty, \theta \to \theta_0 \in (0, \pi/2]$.
- The orbit γ_+ cannot converge to e_0, since it has a center structure and the orbits close enough to it are ellipses.



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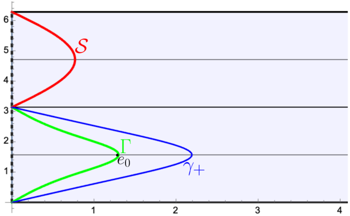
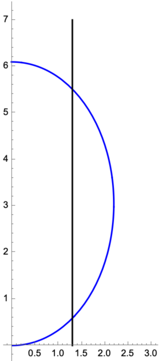
- The equilibrium generates a CMC cylinder.
- The orbit γ_+ of the radial solution closes by symmetry, generating a sphere.





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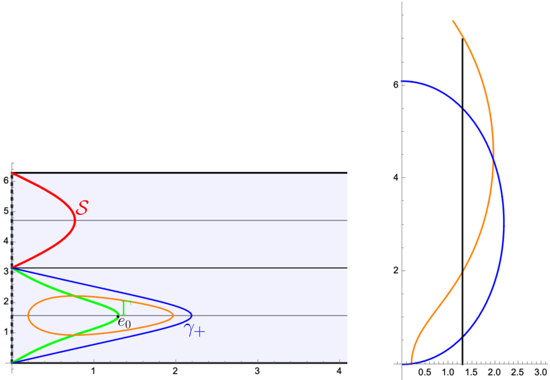
- The equilibrium generates a CMC cylinder.
- The orbit γ_+ of the radial solution closes by symmetry, generating a sphere.
- Any orbit passing through $(x_0, \pi/2)$ at the l.h.s. of Γ must be closed, generating an unduloid-type Φ-surface.



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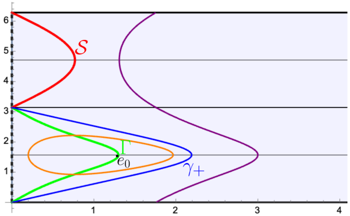
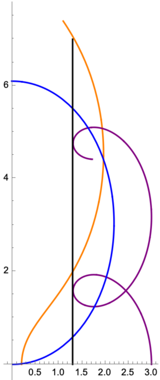
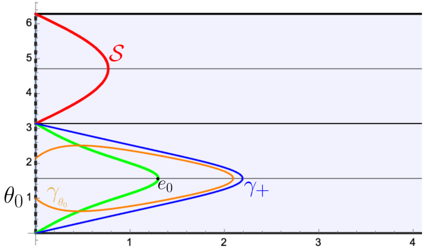
- Any orbit passing through $(x_0, 3\pi/2)$, $x_0 > b/a$, has no singular points and generates a nodoid-type Φ-surface.

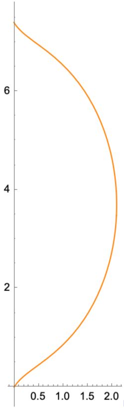




The elliptic case: $b > 0$

- There exists a unique orbit γ_{θ_0} passing through $(0, \theta_0)$.
- The orbit γ_{θ_0} intersects Γ, and cannot intersect γ_+ by uniqueness, hence is symmetric w.r.t. $\theta = \pi/2$.





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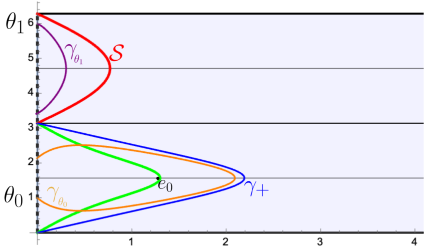
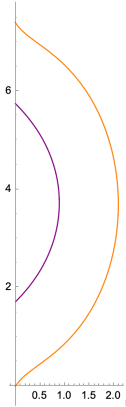
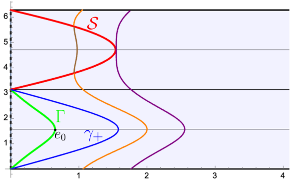
- There exists a unique orbit γ_{θ_1} passing through $(0, \theta_1)$.
- γ_{θ_1} cannot converge to $\mathcal{S}$, hence intersects $\theta = 3\pi/2$ and extends symmetrically.

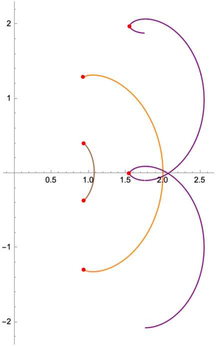




The elliptic case: $b > 0$

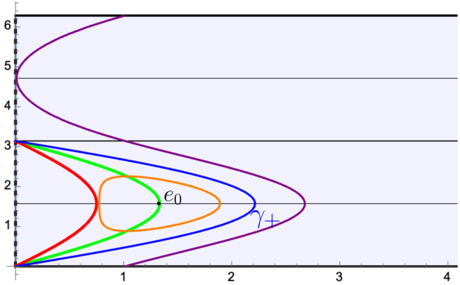
- The orange orbits vary between γ_+ and the orbits of the nodoids, having the purple as limit.
- The brown orbits foliate the remaining of the region bounded by $\mathcal{S}$.
- The corresponding profile curves all have singular points as limit points.
- This concludes the case $b > 0$.





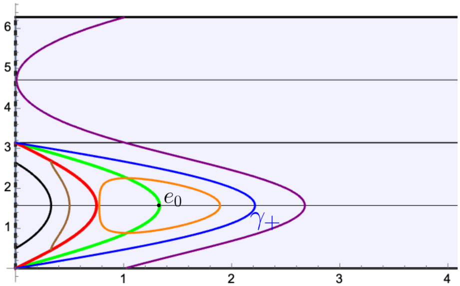
The elliptic case: $b < 0$

- Since $b < 0$, $\mathcal{S}$ lies in Θ_1.
- The fact that $a^2 + b\Phi > 0$ implies that $\mathcal{S}$ is at the l.h.s. of Γ.
- The existence of the complete orbits carries over verbatim.
- The orbits of the unduloids and nodoids converge to limit orbits having singular points.



The elliptic case: $b < 0$

- We focus on the inner region bounded by $\mathcal{S}$.
- The orbits passing through any $(x_0, \pi/2)$, $x_0 < -b/a$ have been already described.



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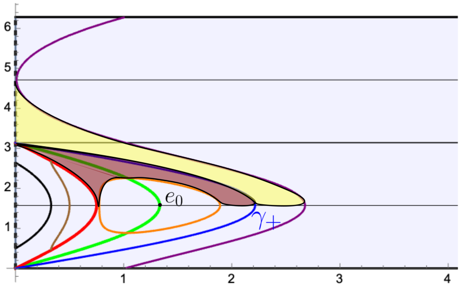
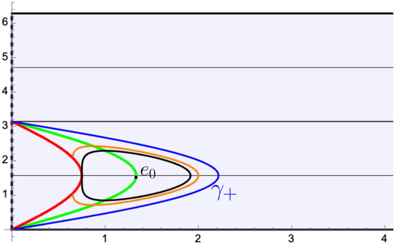
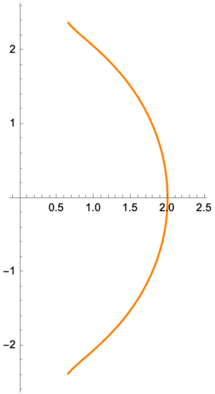
- The orbits passing through any $(x_0, \pi/2)$, $x_0 < -b/a$ have been already described.
- We miss to cover these regions, having γ_+ as common boundary.



The elliptic case: $b < 0$

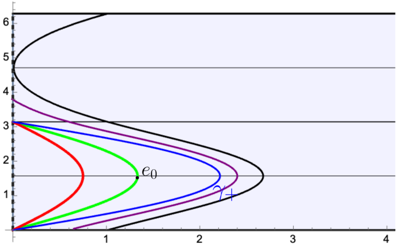
- Fix some $(x_0, \pi/2)$ between γ_+ and the limit unduloid orbit.
- The orbit passing through $(x_0, \pi/2)$ tends to close and converges to $\mathcal{S}$.
- The corresponding profile curve has two singular points as limit points.

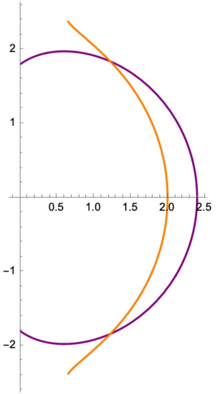




The elliptic case: $b < 0$

- Fix some $(x_0, \pi/2)$ between γ_+ and the limit nodoid orbit.
- The orbit passing through $(x_0, \pi/2)$ must intersect $\theta = \pi$ and reach some $(0, \theta_1)$, $\theta_1 \in (\pi, 3\pi/2)$.
- The corresponding profile curve has two cusp points.
- This concludes the case $b < 0$.





The hyperbolic case

$$2aH + bK = \Phi, \qquad a^2 + b\Phi < 0$$

The hyperbolic case

$$2aH + K = \Phi, \qquad \Phi < -a^2$$

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Theorem

Let be $a \in \mathbb{R}$ and $\Phi \in C^1(\mathbb{S}^2)$ rotationally symmetric and even. Then, rotational Φ-surfaces are described in terms of a parameter $x_0 > 0$, $x_0 \neq 1/a$. Moreover,

i) *If $\Phi \leq -2a^2$, for every $x_0 > 1/a$ the rotational Φ-surface is complete.*

ii) *If $\Phi > -2a^2$, for $x_0 > 1/a$ there are both complete and non-complete rotational Φ-surfaces.*

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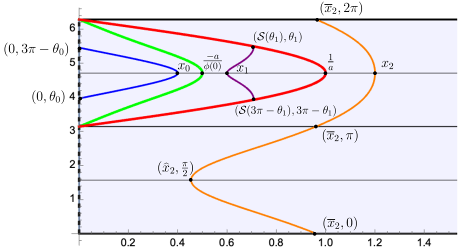
Case $\Phi = c$

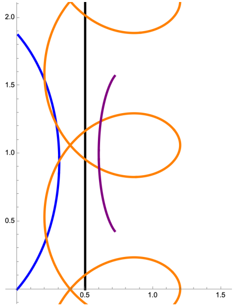
- Studied by López.
- Claimed that the surfaces were complete for every $x_0 > 1/a$, regardless of a and c.

The hyperbolic case

Let be $x_0 > 0$ and consider the orbit passing through $(x_0, 3\pi/2)$

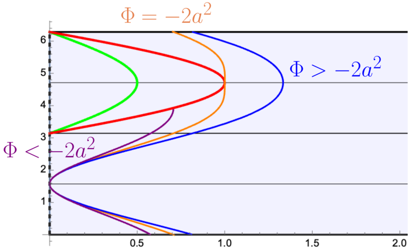
- The curve Γ is at the l.h.s. of $\mathcal{S}$ since $\Phi < -a^2$.
- If $x_0 < -a/\phi(0) \to$ peaked sphere of positive Gauss curvature.
- If $x_0 = -a/\phi(0) \to$ CMC cylinder.
- If $x_0 \in (-a/\phi(0), 1/a) \to$ annulus of negative Gauss curvature.





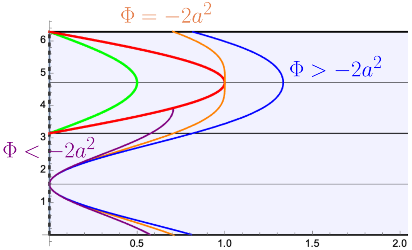
The hyperbolic case

- We study the behavior of the orbit passing through $(0, \pi/2)$, depending on Φ and $-2a^2$.
- If $\Phi \leq -2a^2$, for every $x_0 > 1/a$ the rotational Φ-surface is complete.
- If $\Phi > -2a^2$, for $x_0 > 1/a$ there are complete and non-complete rotational Φ-surfaces.



The hyperbolic case

- We study the behavior of the orbit passing through $(0, \pi/2)$, depending on Φ and $-2a^2$.
- If $\Phi \leq -2a^2$, for every $x_0 > 1/a$ the rotational Φ-surface is complete.
- If $\Phi > -2a^2$, for $x_0 > 1/a$ there are complete and non-complete rotational Φ-surfaces.
- This concludes the classification of rotational Φ-surfaces of hyperbolic type.
- In particular, we correct the mistake for linear Weingarten surfaces.



Further research

- The elliptic case. Motivated by the work of Rosenberg-Sa Earp on special Weingarten surfaces.
  - Structure of properly embedded Φ-surfaces.
  - Height and curvature estimates.
  - Non-existence of properly embedded Φ-surfaces of finite topology and one end.
- More general Weingarten relation, non-necessarily linear

$$W(H, K, N) = 0.$$

  More complex phase plane analysis. Recent research: Gálvez-Mira.
- Generalize to further ambient manifolds.

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- H. Rosenberg, R. Sa Earp, The geometry of properly embedded special surfaces in $\mathbb{R}^3$; e.g., surfaces satisfying $aH + bK = 1$, where a and b are positive, *Duke Math.J.* **73** (1994), 291–306.
- R. Sa Earp, E. Toubiana, Classification des surfaces de type Delaunay, *Amer. J. Math.* **121** (1999), 671–700.

Thank you for your attention!