Variational problems for integral invariants of the second fundamental form of a map between Riemannian manifolds

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Contents

Sec. 0 - Previous research (1/2)

harmonic/biharmonic map

$(M^m, g_M), (N^n, g_N)$: Riemannian manifolds.

$\varphi : (M^m, g_M) \to (N^n, g_N)$; C^∞-map,

· φ is a <u>harmonic map</u>

$\overset{\text{def}}{\iff}$ φ is a critical point of $E(\varphi) = \dfrac{1}{2}\displaystyle\int_M |d\varphi|^2 d\mu_{g_M}$.

$\iff \tau(\varphi) = \mathrm{tr}_{g_M} \widetilde{\nabla} d\varphi = 0.$

· φ is a <u>biharmonic map</u>

$\overset{\text{def}}{\iff}$ φ is a critical point of $E_2(\varphi) = \dfrac{1}{2}\displaystyle\int_M |\tau(\varphi)|^2 d\mu_{g_M}$.

$\iff \tau_2(\varphi) = -\overline{\nabla}^* \overline{\nabla} \tau(\varphi) - \mathrm{tr}_{g_M}\left(R^N(d\varphi(\cdot), \tau(\varphi)) d\varphi(\cdot)\right) = 0.$

Sec. 0 - Previous research (2/2)

Definition (Howard 1993)

G/K: homogeneous space,

M: compact submanifold of G/K of type V_0.

Then

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M)\, d\mu_{g_M},$$

where h^M is the second fundamental form of M.

In this talk

We define integral invariants of the second fundamental form of a map φ and construct a family of energy functionals including $E_2(\varphi)$.

Here, we focus on some energy functionals among them and show their first variational formulae.

Setting

- $\mathrm{II}(\mathbb{E}^m, \mathbb{E}^n) := \{H : \mathbb{E}^m \times \mathbb{E}^m \to \mathbb{E}^n$; symmetric bilinear form$\}$.
- $G := O(m) \times O(n)$.
- G acts on $\mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$, that is

$$(gH)(u, v) := b\big(H(a^{-1}u, a^{-1}v)\big),$$

  where $u, v \in \mathbb{E}^m,\ g = (a, b) \in G,\ H \in \mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$.
- A function $\mathcal{P}$ on $\mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$ is G-invariant, that is for all $g \in G$

$$\mathcal{P}(gH) = \mathcal{P}(H).$$

- $\varphi : (M^m, g_M) \to (N^n, g_N)$; C^∞-map between Riemannian manifolds.
- The second fundamental form of the map φ is the symmetric bilinear form $\widetilde{\nabla} d\varphi : TM \times TM \to \varphi^{-1}TN$ defined by

$$\big(\widetilde{\nabla} d\varphi\big)(X, Y) := \overline{\nabla}_X\big(d\varphi(Y)\big) - d\varphi(\nabla_X Y) \qquad \big(X, Y \in \Gamma(TM)\big).$$

Note

For $x \in M$, we identify T_xM with $\mathbb{E}^m$ and $T_{\varphi(x)}N$ with $\mathbb{E}^n$.
Now there is a linear isomorphism
between $T_x^*M \odot T_x^*M \otimes T_{\varphi(x)}N$ and $\mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$.

That is, $(\widetilde{\nabla} d\varphi)_x \in T_x^*M \odot T_x^*M \otimes T_{\varphi(x)}N$ corresponds to $H_x := (h_{ij}^\alpha) \in \mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$, where

$$h_{ij}^\alpha = g_N\big((\widetilde{\nabla} d\varphi)_x(e_i, e_j), \xi_\alpha\big)$$

and, $\{e_i\}_{i=1}^m$ and $\{\xi_\alpha\}_{\alpha=1}^n$ are orthonormal basises of T_xM and $T_{\varphi(x)}N$.

Define an <u>invariant function</u> of the second fundamental form of φ as follows, for a G-invariant function $\mathcal{P}$ on $\mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$,

$$\mathcal{P}\big((\widetilde{\nabla} d\varphi)_x\big) := \mathcal{P}(H_x) \qquad (x \in M).$$

Definition

(M^m, g_M): m-dimensional Riemannian manifold,

(N^n, g_N): n-dimensional Riemannian manifold,

$\mathcal{P}$: G-invariant function on $\mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$.

Then for a smooth map $\varphi : (M^m, g_M) \to (N^n, g_N)$, we define

$$I^{\mathcal{P}}(\varphi) := \int_M \mathcal{P}\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M}.$$

Remark

$I^{\mathcal{P}}(\varphi)$ is an invariant of a map φ.

That is, for all $f \in \mathrm{Isom}(M)$ and $g \in \mathrm{Isom}(N)$, the following formula holds

$$I^{\mathcal{P}}(g \circ \varphi \circ f^{-1}) = I^{\mathcal{P}}(\varphi).$$

Definition

For $H = (h_{ij}^{\alpha}) \in \mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$, define the polynomials $\underline{\mathcal{Q}_1}$ and $\underline{\mathcal{Q}_2}$ as

$$\mathcal{Q}_1(H) = \sum_{\alpha} \sum_{i,j} (h_{ij}^{\alpha})^2, \qquad \mathcal{Q}_2(H) = \sum_{\alpha} \Big(\sum_{i} h_{ii}^{\alpha} \Big)^2.$$

Proposition

The space of polynomials homogeneous of degree 2 invariant under G is spanned by $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Remark

If φ is a smooth map between Riemannian manifolds, then we have

$$\mathcal{Q}_1\big((\widetilde{\nabla} d\varphi)_x\big) = |\widetilde{\nabla} d\varphi|^2(x), \qquad \mathcal{Q}_2\big((\widetilde{\nabla} d\varphi)_x\big) = |\mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi)|^2(x).$$

Definition

For $\varphi \in C^\infty(M, N)$, the $\mathcal{Q}_1$-energy functional $I^{\mathcal{Q}_1}(\varphi)$ and the $\mathcal{Q}_2$-energy functional $I^{\mathcal{Q}_2}(\varphi)$ are defined by

$$I^{\mathcal{Q}_1}(\varphi) = \int_M \mathcal{Q}_1\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M} = \int_M |\widetilde{\nabla} d\varphi|^2 d\mu_{g_M},$$
$$I^{\mathcal{Q}_2}(\varphi) = \int_M \mathcal{Q}_2\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M} = \int_M |\mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi)|^2 d\mu_{g_M}.$$

Then φ is called a $\mathcal{Q}_1$-map if it is a critical point of $I^{\mathcal{Q}_1}(\varphi)$.
Also, then φ is called a $\mathcal{Q}_2$-map if it is a critical point of $I^{\mathcal{Q}_2}(\varphi)$.

The $\mathcal{Q}_2$-energy functional is equal to two times of the bienergy functional, actually

$$I^{\mathcal{Q}_2}(\varphi) = \int_M |\mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi)|^2 d\mu_{g_M} = \int_M |\tau(\varphi)|^2 d\mu_{g_M} = 2E_2(\varphi).$$

$\widetilde{\nabla}^2 d\varphi \in \Gamma\big(\bigotimes^3 T^*M \otimes \varphi^{-1}TN\big)$ and $\widetilde{\nabla}^3 d\varphi \in \Gamma\big(\bigotimes^4 T^*M \otimes \varphi^{-1}TN\big)$ are defined by

$$\begin{aligned}(\widetilde{\nabla}^2 d\varphi)(X,Y,Z) &:= \overline{\nabla}_X\big((\widetilde{\nabla} d\varphi)(Y,Z)\big) - (\widetilde{\nabla} d\varphi)(\nabla_X Y, Z)\\ &\quad - (\widetilde{\nabla} d\varphi)(Y, \nabla_X Z),\\ (\widetilde{\nabla}^3 d\varphi)(X,Y,Z,W) &:= \overline{\nabla}_X\big((\widetilde{\nabla}^2 d\varphi)(Y,Z,W)\big) - (\widetilde{\nabla}^2 d\varphi)(\nabla_X Y, Z, W)\\ &\quad - (\widetilde{\nabla}^2 d\varphi)(Y, \nabla_X Z, W) - (\widetilde{\nabla}^2 d\varphi)(Y, Z, \nabla_X W),\end{aligned}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$.

Note By definition, they have the following symmetry

$$\begin{aligned}\big(\widetilde{\nabla}^2 d\varphi\big)(X,Y,Z) &= \big(\widetilde{\nabla}^2 d\varphi\big)(X,Z,Y),\\ \big(\widetilde{\nabla}^3 d\varphi\big)(X,Y,Z,W) &= \big(\widetilde{\nabla}^3 d\varphi\big)(X,Y,W,Z).\end{aligned}$$

Theorem

Let $\varphi : (M^m, g_M) \to (N^n, g_N)$ be a C^∞-map from compact Riemannian manifold to any Riemannian manifold. Consider a C^∞-variation $\{\varphi_t\}_{t\in I}$ of φ with variational vectorfield V. Then the following formula holds

$$\frac{d}{dt} I^{\mathcal{Q}_i}(\varphi_t)\bigg|_{t=0} = 2\int_M \langle W_i(\varphi), V\rangle \, d\mu_{g_M} \qquad (i = 1, 2)\ ,$$

where

$$W_1(\varphi) = \sum_{i,j} \Big\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_j, e_i, e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_j), d\varphi(e_i)\big) d\varphi(e_j) \Big\}$$

$$W_2(\varphi) = \sum_{i,j} \Big\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_i), d\varphi(e_j)\big) d\varphi(e_j) \Big\}$$

and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M^m, g_M).

Theorem

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where

$$W_1(\varphi) = \sum_{i,j} \left\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_j, e_i, e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_j), d\varphi(e_i)\big) d\varphi(e_j) \right\}$$

$$W_2(\varphi) = \sum_{i,j} \left\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_i), d\varphi(e_j)\big) d\varphi(e_j) \right\}$$

and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M^m, g_M).

Lemmas used to prove the theorem (for $\mathcal{Q}_1$-energy)

We consider a smooth variation $\{\varphi_t\}_{t\in I}$ of φ, that is we consider a smooth map Φ given by

$$\begin{aligned}&\Phi : M \times I \to N,\\ &\qquad (x,t) \mapsto \Phi(x,t) =: \varphi_t(x) \quad \text{s.t. } \varphi_0(x) = \varphi(x) \quad ({}^\forall x \in M),\end{aligned}$$

and denote by V its variational vector field, that is

$$V = \left(\frac{\partial}{\partial t}\bigg|_{t=0} \right) \in \Gamma(\varphi^{-1}TN).$$

Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field on a neighborhood U of $x \in M$, then $\left\{e_i, \frac{\partial}{\partial t}\right\}$ is a orthonormal frame field on the neighborhood $U \times I$ of $(x,t) \in M \times I$, and it holds that

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0, \quad \nabla_{\frac{\partial}{\partial t}} e_i = \nabla_{e_i} \frac{\partial}{\partial t} = 0 \quad (1 \leq i \leq m).$$

Under the setting above, for any variation $\{\varphi_t\}_{t\in I}$ of φ,
the following lemmas hold.

Lemma

$$\frac{d}{dt}I^{\mathcal{Q}_1}(\varphi_t) = 2\int_M \sum_{i,j}\left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_j, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_i, e_j)\right\rangle d\mu_{g_M}$$
$$- 2\int_M \sum_{i,j}\left\langle R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_j), (\widetilde{\nabla} d\Phi)(e_i, e_j)\right\rangle d\mu_{g_M}$$

Lemma

$$\int_M \sum_{i,j}\left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_j, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_i, e_j)\right\rangle d\mu_{g_M}$$
$$= \int_M \sum_{i,j}\left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\nabla}^3 d\Phi)(e_i, e_j, e_i, e_j)\right\rangle d\mu_{g_M}$$

Lemmas used to prove the theorem (for $\mathcal{Q}_2$-energy)

Lemma

$$\frac{d}{dt} I^{\mathcal{Q}_2}(\varphi_t) = 2\int_M \sum_{i,j} \left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M}$$
$$- 2\int_M \sum_{i,j} \left\langle R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i), (\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M}$$

Lemma

$$\int_M \sum_{i,j} \left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M}$$
$$= \int_M \sum_{i,j} \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\nabla}^3 d\Phi)(e_i, e_i, e_j, e_j) \right\rangle d\mu_{g_M}$$

Jiang[10] use the following lemmas.

Lemma

$$
\begin{aligned}
&\frac{d}{dt}E_2(\varphi_t)\\
&= 2\int_M \sum_{i,j} \left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) - (\widetilde{\nabla} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j)\right\rangle d\mu_{g_M}\\
&\quad - 2\int_M \sum_{i,j} \left\langle R^N \left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i), (\widetilde{\nabla} d\Phi)(e_j, e_j)\right\rangle d\mu_{g_M}
\end{aligned}
$$

Lemma

$$
\begin{aligned}
&\int_M \sum_{i,j} \left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) - (\widetilde{\nabla} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j)\right\rangle d\mu_{g_M}\\
&= \int_M \sum_{i,j} \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} - \overline{\nabla}_{\nabla_{e_i} e_i})(\widetilde{\nabla} d\Phi)(e_j, e_j)\right\rangle d\mu_{g_M}
\end{aligned}
$$

Sec. 2 - The first variational formulae of $\mathcal{Q}_1$ and $\mathcal{Q}_2$-energies (9/10)

Jiang[10] use the following lemmas.

Lemma

$$
\begin{aligned}
&\frac{d}{dt} E_2(\varphi_t) \\
= 2 &\int_M \sum_{i,j} \left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) - (\widetilde{\nabla} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M} \\
&- 2 \int_M \sum_{i,j} \left\langle R^N \left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i), (\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M}
\end{aligned}
$$

Lemma

$$
\begin{aligned}
&\int_M \sum_{i,j} \left\langle (\widetilde{\nabla}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) - (\widetilde{\nabla} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M} \\
= &\int_M \sum_{i,j} \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} - \overline{\nabla}_{\nabla_{e_i} e_i})(\widetilde{\nabla} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M}
\end{aligned}
$$

By comparing the first variational formulae of the bienergy and $\mathcal{Q}_2$-energy, we get the following proposition.

Proposition

Let $\varphi : (M^m, g_M) \to (N^n, g_N)$ be a C^∞-map between Riemannian manifolds. Then the following formula holds

$$-\overline{\nabla}^*\overline{\nabla}\tau(\varphi) = \sum_{i,j}(\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j),$$

where $-\overline{\nabla}^*\overline{\nabla} := \sum_k \left(\overline{\nabla}_{e_k}\overline{\nabla}_{e_k} - \overline{\nabla}_{\nabla_{e_k}e_k}\right)$ is the rough Laplacian and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M^m, g_M).

Sec. 3 - The Chern–Federer energy (1/6)

Definition

For $H = (h^{\alpha}_{ij}) \in \mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$,
the Chern–Federer polynomial $\mathrm{CF}(H)$ of degree two is defned by

$$\mathrm{CF}(H) := \mathcal{Q}_2(H) - \mathcal{Q}_1(H).$$

Definition

For $\varphi \in C^{\infty}(M, N)$,
the Chern–Federer energy functional $I^{\mathrm{CF}}(\varphi)$ is defined by

$$I^{\mathrm{CF}}(\varphi) = \int_M \mathrm{CF}\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M}.$$

Then φ is called a Chern–Federer map if it is a critical point of $I^{\mathrm{CF}}(\varphi)$.

Sec. 3 - The Chern–Federer energy (2/6)

Definition

For $H = (h^{\alpha}_{ij}) \in \mathrm{II}(\mathbb{E}^m, \mathbb{E}^n)$,
the Willmore–Chen polynomial $\mathrm{WC}(H)$ is defned by

$$\mathrm{WC}(H) := m\mathcal{Q}_1(H) - \mathcal{Q}_2(H).$$

Definition

For $\varphi \in C^{\infty}(M, N)$,
the Willmore–Chen energy functional $I^{\mathrm{WC}}(\varphi)$ is defined by

$$I^{\mathrm{WC}}(\varphi) = \int_M \mathrm{WC}\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M}.$$

Then φ is called a Willmore–Chen map if it is a critical point of $I^{\mathrm{WC}}(\varphi)$.

Let $\varphi : (M^m, g_M) \to (N^n, g_N)$ be a C^∞-map,
α, β constant numbers such that $\alpha^2 + \beta^2 \neq 0$.
When the map φ satisfies that the following equation

$$\alpha W_1(\varphi) + \beta W_2(\varphi) = 0,$$

we call it an $(\alpha\mathcal{Q}_1 + \beta\mathcal{Q}_2)$-map. For an $(\alpha\mathcal{Q}_1 + \beta\mathcal{Q}_2)$-map φ,

1. φ is a $\mathcal{Q}_1$-map $\Leftrightarrow (\alpha, \beta) = (1, 0)$;
2. φ is a $\mathcal{Q}_2$-map $\Leftrightarrow (\alpha, \beta) = (0, 1)$;
3. φ is a <u>Chern–Federer map</u> $\Leftrightarrow (\alpha, \beta) = (-1, 1)$;
4. φ is a <u>Willmore–Chen map</u> $\Leftrightarrow (\alpha, \beta) = (m, -1)$.

Sec. 3 - The Chern–Federer energy (4/6)

Proposition

A smooth map $\varphi : (M^m, g_M) \to (N^n, g_N)$ is a Chern–Federer map if and only if

$$0 = \sum_{i,j} \Big\{ \big(\nabla R^N\big)\big(d\varphi(e_i), d\varphi(e_i), d\varphi(e_j)\big) d\varphi(e_j) - \big(\widetilde{\nabla} d\varphi\big)\big(e_i, R^M(e_i, e_j)e_j\big) \\ - d\varphi\big((\nabla R^M)(e_i, e_i, e_j)e_j\big) + 2R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_i), d\varphi(e_j)\big) d\varphi(e_j) \\ + 2R^N\big(d\varphi(e_i), (\widetilde{\nabla} d\varphi)(e_i, e_j)\big) d\varphi(e_j) \Big\},$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M^m, g_M).

$\rightsquigarrow$ The Euler–Lagrange equation of the $I^{\mathrm{CF}}(\varphi)$ is a second-order partial differential equation for φ.

We use the following two lemmas to prove the previous proposition.

For a smooth map $\varphi : M \to N$ and $X, Y, Z, W \in \Gamma(TM)$,
the following equation holds:

Lemma

$$
\begin{aligned}
&\big(\widetilde{\nabla}^2 d\varphi\big)(X, Y, Z) - \big(\widetilde{\nabla}^2 d\varphi\big)(Y, X, Z) \\
&= R^N\big(d\varphi(X), d\varphi(Y)\big)d\varphi(Z) - d\varphi(R^M(X, Y)Z)
\end{aligned}
$$

Lemma

$$
\begin{aligned}
&\big(\widetilde{\nabla}^3 d\varphi\big)(X, Y, Z, W) - \big(\widetilde{\nabla}^3 d\varphi\big)(X, Z, Y, W) \\
&= \big(\nabla R^N\big)\big(d\varphi(X), d\varphi(Y), d\varphi(Z)\big)d\varphi(W) + R^N\big((\widetilde{\nabla} d\varphi)(X, Y), d\varphi(Z)\big)d\varphi(W) \\
&\quad + R^N\big(d\varphi(Y), (\widetilde{\nabla} d\varphi)(X, Z)\big)d\varphi(W) + R^N\big(d\varphi(Y), d\varphi(Z)\big)(\widetilde{\nabla} d\varphi)(X, W) \\
&\quad - (\widetilde{\nabla} d\varphi)\big(X, R^M(Y, Z)W\big) - d\varphi\big((\nabla R^M)(X, Y, Z)W\big)
\end{aligned}
$$

Note

We express the Chern–Federer energy functional as follows:

$$\begin{aligned} I^{\mathrm{CF}}(\varphi) &= I^{\mathcal{Q}_2-\mathcal{Q}_1}(\varphi) \\ &= \int_M \mathcal{Q}_2\big((\widetilde{\nabla} d\varphi)_x\big) - \mathcal{Q}_1\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M} \\ &= \int_M \sum_\alpha \big(\sum_i h_{ii}^\alpha\big)^2 - \sum_\alpha \sum_{i,j} (h_{ij}^\alpha)^2 d\mu_{g_M} \\ &= \int_M \sum_\alpha \sum_{i,j} \det \begin{pmatrix} h_{ii}^\alpha & h_{ij}^\alpha \\ h_{ij}^\alpha & h_{jj}^\alpha \end{pmatrix} d\mu_{g_M}. \end{aligned}$$

Sec. 3 - The Chern–Federer energy (6/6)

Theorem

A smooth map $\varphi : (M^m, g_M) \to (N^n, g_N)$ is a Chern–Federer map if and only if

$$C(\mu + \nu) = 0,$$

where C is defined by

$$C := \det \begin{pmatrix} C_{12} & C_{13} \\ C_{24} & C_{34} \end{pmatrix},$$

where C_{ij} is a contraction, and μ, ν are $(0,4)$-type tensor fields valued on $\varphi^{-1}TN$ which are defined by

$$\mu(X_1, X_2, X_3, X_4) := (\widetilde{\nabla}^3 d\varphi)(X_1, X_2, X_3, X_4)$$

and

$$\nu(X_1, X_2, X_3, X_4) := R^N\big((\widetilde{\nabla} d\varphi)(X_3, X_4), d\varphi(X_1)\big)d\varphi(X_2),$$

where $X_1, X_2, X_3, X_4 \in \Gamma(TM)$.

Setting

- (M^m, g_M): Riemannian manifold.
- $N^n(c)$: Riemannian space form of constant curvature $c \in \mathbb{R}$.
- $\varphi : (M^m, g_M) \to N^n(c)$; isometric immersion.
- $\mathcal{H}$: the mean curvature vector field, h: the second fundamental form.
- Q: the Ricci operator of (M^m, g_M).

Theorem

Then φ is a Chern–Federer map if and only if

$$-d\varphi(\mathrm{tr}_{g_M}(\nabla Q)) + 2cm(m-1)\mathcal{H} - \mathrm{tr}_{g_M} h(Q(\text{-}), \text{-}) = 0,$$

equivalently,

$$(\top) : \mathrm{tr}_{g_M}(\nabla Q) = 0, \quad (\perp) : 2cm(m-1)\mathcal{H} - \mathrm{tr}_{g_M} h(Q(\text{-}), \text{-}) = 0,$$

where $(\top)$ and $(\perp)$ denote the tangent component and the normal component, respectively.

Example

We consider a Euclidean n-space $\mathbb{E}^n$ as a target space (N^n, g_N).
If (M^m, g_M) is a Ricci-flat Riemannian manifold, then an arbitary isometric immersion $\varphi : (M^m, g_M) \to \mathbb{E}^n$ is a Chern–Federer map.

Example (for curves)

$I \subset \mathbb{R}$: open interval.
Then an arbitrary curve $\gamma : I \to (N^n, g_N)$ is a Chern–Federer map.

Proposition (for surfaces)

Let $\varphi : (M^2, g_M) \to N^n(c)$ be an isometric immersion and K the sectional curvature of (M^2, g_M). Then φ is a Chern-Federer map if and only if

(i) K is constant and φ is minimal, or (ii) $K = 2c$.

For hypersurfaces (especially isoparametric hypersurfaces)

Definition

When an isometric immersion $\varphi : (M^m, g_M) \to (N^n, g_N)$ is a Chern–Federer map, we call the image a Chern–Federer submanifold in (N^n, g_N), and the map φ to be Chern–Federer.

Theorem

Let $M^m \subset N^{m+1}(c)$ be an isoparametric hypersurface in a Riemannian space form. Then M^m is Chern–Federer if and only if it satisfies that

$$c(m-1)(\mathrm{tr}A) - (\mathrm{tr}A)(\mathrm{tr}A^2) + (\mathrm{tr}A^3) = 0$$

Sec. 4 - Some examples (4/6)

In the case of a unit sphere $\mathbb{S}^{m+1}(1)$,
we show some examples of Chern–Federer homogeneous hypersurfaces, which are also isoparametric.

g : the number of distinct principal curvatures of isoparametric hypersurfaces

- $[g = 1]$. The classification is the following totally umbilical hypersurfaces

$$\mathbb{S}^m(r) = \left\{ (x, \sqrt{1 - r^2}) \in \mathbb{E}^{m+2} \mid \|x\|^2 = r^2 \right\} \subset \mathbb{S}^{m+1}(1) \quad (0 < r \leq 1).$$

From this, we obtain:

Proposition

The isoparametric hypersurface of the above equation is Chern–Federer if and only if
$r = 1$ (totally geodesic one), or $r = 1/\sqrt{2}$ (proper biharmonic one).

Sec. 4 - Some examples (5/6)

- $[g = 2]$. The classification is the following Clifford hypersurfaces

$$\mathbb{S}^p(r_1) \times \mathbb{S}^{m-p}(r_2) \subset \mathbb{S}^{m+1}(1) \quad (r_1^2 + r_2^2 = 1).$$

We denote the distinct principal curvatures of the above by λ_1, λ_2.
Then by setting

$$\lambda := \lambda_1 = \cot t \quad \left(0 < t < \frac{\pi}{2}\right),$$

we have

$$\lambda_2 = \cot \left(t + \frac{\pi}{2}\right) = -\frac{1}{\cot t} = -\frac{1}{\lambda}.$$

From this, we obtain:

Proposition

The isoparametric hypersurface of the above equation is Chern–Federer if and only if λ satisfies that

$$p(p-1)\lambda^6 - p(2m-p-1)\lambda^4 + (m-p)(m+p-1)\lambda^2 - (m-p)(m-p-1) = 0$$

Sec. 4 - Some examples (6/6)

$$p(p-1)\lambda^6 - p(2m-p-1)\lambda^4 + (m-p)(m+p-1)\lambda^2 - (m-p)(m-p-1) = 0$$

The solutions for this equation are:

(i) when $m = 2$ and $p = 1$, then $\lambda = 1$ (minimal one)

(ii) when $m \geq 3$ and $p = 1$,
then $\lambda = 1$ (biharmonic one) or $\lambda = \sqrt{\frac{m-2}{2}}$ (non biharmonic)

(iii) when $m \geq 3$ and $p \geq 2$,
then $\lambda = 1$ (biharmonic one) or $\sqrt{\frac{p(m-p) \pm \sqrt{p(m-p)(m-1)}}{p(p-1)}}$ (non biharmonic)

Future outlook on research

We are currently working on the following issues:

- Advance research from the perspective of variational problems.
- Clarify geometric properties of the integral invariants of a map.
- Interpret the Euler-Lagrange equation of Chern-Federer maps from the viewpoint of integrable systems.

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Thank you very much for your attention !

Reference I

[1] C. B. Allendoerfer and A. Weil, *The Gauss–Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc. **53** (1943), 101–129.

[2] R. L. Bryant, *Minimal surfaces of constant curvatures in S^n*, Trans. Amer. M. S. **290** (1985), 259–271.

[3] T. E. Cecil and P. J. Ryan, *Geometry of hypersurfaces*, Springer Monographs in Mathematics. Springer, New York, 2015.

[4] B.-Y. Chen, *An invariant of conformal mappings*, Proc. Amer. Math. Soc. **40** (1973), 563–564.

[5] B.-Y. Chen, *Some conformal invariants of submanifolds and their applications*, Boll. Un. Mat. Ital. (4) **10** (1974), 380–385.

[6] B.-Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific, (2011).

Reference II

[7] B.-Y. Chen, *Recent developments in δ-Casorati curvature invariants*, Turkish J. Math. **45** (2021), no. 1, 1–46.

[8] R. Howard, *The kinematic formula in Riemannian homogeneous spaces*, Mem. Amer. Math. Soc., No.509, 1993.

[9] T. Ichiyama, J. Inoguchi and H. Urakawa, *Bi-harmonic maps and bi-Yang-Mills fields*, Note Mat. **28** (2009), [2008 on verso], suppl. 1, 233–275.

[10] G. Jiang, *2-harmonic maps and their first and second variational formulas*. Translated from the Chinese by Hajime Urakawa. Note Mat. **28** (2009), [2008 on verso], suppl. 1, 209–232.

Reference III

[11] H. J. Kang, T. Sakai and Y. J. Suh, *Kinematic formulas for integral invariants of degree two in real space forms*, Indiana Univ. Math. J. **54** (2005), no. 5, 1499–1519.

[12] Y. Kitagawa, *Periodicity of the asymptotic curves on flat tori in* S^3, J. Math. Soc. Japan **40** (1988), no. 3, 457–476.

[13] Y. Kitagawa, *Isometric deformations of flat tori in the 3-sphere with nonconstant mean curvature*, Tohoku Math. J. (2) **52** (2000), no. 2, 283–298.

[14] K. Kenmotsu, *On minimal immersion of* R^2 *into* S^n, J. Math. Soc. Japan **28** (1976), 182–191.

[15] H. Weyl, *On the volume of tubes*, Amer. J. Math., **61** (1939), 461–472.