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Outflow Boundary Condition Leading to Minimal Energy Dissipation for an Incompressible Flow

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Abstract. A method for determining the boundary condition on artificial boundaries is presented. This method is formulated as an optimization problem for appropriate functional representing, for example, the dissipation of energy. We show that this functional attains its minimum on the set of solutions to the stationary Stokes (or Navier–Stokes) system with partially unspecified boundary condition. Thus, this method gives rise to a physically reasonable boundary condition which ensures the existence of a solution to the corresponding system. In particular, it is proved that the implicit boundary condition obtained for the Stokes system implies the modification of the "do-nothing" boundary condition for the symmetric velocity gradient. To the author's knowledge, methods and conclusions contained in this paper are new.

Introduction

We consider a flow of an incompressible fluid through a region with an artificial boundary (for example the outflow), such as the pipe flow. In order to complete the corresponding system of equations, one needs to introduce some boundary condition at the outlet, where the flow may be a priori unknown. The use of the popular "do-nothing" boundary condition has its downsides. Not only does it seem to have no physical justification, but also the well-posedness of the Navier–Stokes system with this boundary condition is unknown. See [*Heywood et al.*, 1996] for details.

In this paper we discuss the possibility of selecting the outflow boundary condition in such a way that the resulting flow minimizes a given functional representing the dissipation of energy, for example. This is physically reasonable from the point of view that for stable flows the dissipation of the energy should decrease over time to a minimum value which corresponds to a stationary flow. Also, the existence of a solution to the (Navier–)Stokes system with such a boundary condition is obtained automatically — we only need to show that the selected functional attains its minimum in the set of admissible solutions. The boundary conditions obtained in this way are implicit and in general it is not obvious whether these can be reduced to some convenient form. However, at least for the Stokes system, we show that such a reduction is possible and leads to some familiar boundary conditions.

Notation

We distinguish between scalar, vectorial and matrix quantities (and corresponding spaces) using different fonts as follows: a, $\boldsymbol{a}$ and $\mathbb{A}$, respectively. Throughout this paper, the symbol $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, $d \geq 2$, stands for an open, bounded, simply connected set with Lipschitz boundary $\partial\Omega$. As usual, the symbols $\boldsymbol{C}^\infty(\overline{\Omega})$, $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ represent the space of smooth functions, the Lebesgue space and the Sobolev space, respectively. The norm of $\boldsymbol{L}^2(\Omega)$ is denoted by $\|\cdot\|_2$ and the norm of $\boldsymbol{H}^1(\Omega)$ is defined by $\|\boldsymbol{\varphi}\|_{1,2} := (\|\boldsymbol{\varphi}\|_2^2 + \|\nabla\boldsymbol{\varphi}\|_2^2)^{\frac{1}{2}}$. Furthermore, let $\boldsymbol{C}_0^\infty(\Omega)$ be the space of smooth functions with compact support in Ω and let $\boldsymbol{C}_{0,\mathrm{div}}^\infty(\Omega)$ be its subspace consisting of functions with zero divergence. If $\emptyset \neq \Gamma \subset \partial\Omega$, then the space $\boldsymbol{H}_\Gamma^1(\Omega)$ is defined as the closure of the set $\{\boldsymbol{\varphi} \in \boldsymbol{C}^\infty(\overline{\Omega}) : \boldsymbol{\varphi} = 0 \text{ on } \Gamma\}$ in the norm $\|\cdot\|_{1,2}$. Moreover, we set $\boldsymbol{H}_0^1(\Omega) := \boldsymbol{H}_{\partial\Omega}^1(\Omega)$. Similarly, the space $\boldsymbol{H}_{\Gamma,\mathrm{div}}^1(\Omega)$ is the closure of the set $\{\boldsymbol{\varphi} \in \boldsymbol{C}^\infty(\overline{\Omega}) : \boldsymbol{\varphi} = 0 \text{ on } \Gamma,\ \mathrm{div}\,\boldsymbol{\varphi} = 0\}$ in the norm $\|\cdot\|_{1,2}$ and $\boldsymbol{H}_{0,\mathrm{div}}^1(\Omega) := \boldsymbol{H}_{\partial\Omega,\mathrm{div}}^1(\Omega)$.

We shall everywhere assume that $\partial\Omega$ consists of two parts Γ, Γ_a of positive measure. On Γ, we prescribe the Dirichlet boundary conditions. For example, Γ can contain rigid walls or the inflow, if it is available. On the remaining part Γ_a, we do not prescribe anything explicitly. In typical situation, Γ_a is the artificial boundary such as the outflow, the inflow, or both. To avoid technical difficulties, we shall also assume that Γ_a is a smooth hypersurface in $\mathbb{R}^d$. This way, we can define the space $C_0^\infty(\Gamma_\mathrm{a})$ in a standard way.

Formulation of the optimization problem

The Stokes flow through Ω is described by the system

$$\begin{aligned} \operatorname{div}\mathbb{T}(\boldsymbol{v},p) &= 0 \quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{v} &= 0 \quad \text{in } \Omega \\ \boldsymbol{v} &= \tilde{\boldsymbol{v}}_0 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where

$$\mathbb{T}(\boldsymbol{v},p) := -p\mathbb{I} + 2\nu\mathbb{D}\boldsymbol{v}, \quad \nu > 0, \quad \text{and} \quad \mathbb{D}\boldsymbol{v} := \tfrac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$$

and $\tilde{\boldsymbol{v}}_0$ is such that there exists its divergence-free extension $\boldsymbol{v}_0 \in \boldsymbol{H}^1(\Omega)$ (the construction of $\boldsymbol{v}_0$ can be found in [*Ladyzhenskaya*, 1969, Chapter I, Problem 2.1]). Our goal is to find the optimal boundary condition on Γ_a in the sense that the resulting flow will dissipate the least amount of energy.

We shall say that a function $\boldsymbol{v}$ is a weak solution to system (1) if $\boldsymbol{v} \in \boldsymbol{v}_0 + \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ and

$$\int_\Omega \mathbb{D}\boldsymbol{v} \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for all} \quad \boldsymbol{\psi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega). \tag{2}$$

Then the the set of all weak solutions of (1) will be denoted by $\mathcal{S}$. This will be the domain of our optimization problem. Note that (2) is equivalent to

$$\int_\Omega \nabla\boldsymbol{v} \cdot \nabla\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for all} \quad \boldsymbol{\psi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega). \tag{3}$$

Note also that the function p does not appear explicitly in the definition of the weak solution. It is determined (up to an additive constant) by the following simple version of De Rahm's lemma, for which we refer to [*Temam*, 1979, Chapter I, Remark 1.9] and references there.

Lemma 1. *If a continuous linear functional $\boldsymbol{f}$ on $\boldsymbol{H}^1_0(\Omega)$ satisfies*

$$\boldsymbol{f}(\boldsymbol{\varphi}) = 0 \quad \textit{for every} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega),$$

then there exists $p \in L^2(\Omega)$ with $\|p\|_2 \le c \sup_{\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1_0(\Omega)}=1} |\boldsymbol{f}(\boldsymbol{\psi})|$ satisfying

$$\boldsymbol{f}(\boldsymbol{\varphi}) = \int_\Omega p \operatorname{div}\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \quad \textit{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1_0(\Omega).$$

In the view of Lemma 1, we shall also call the couple $(\boldsymbol{v},p)$ a weak solution of (1) if $\boldsymbol{v} \in \boldsymbol{v}_0 + \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$, $p \in L^2(\Omega)$ and

$$2\nu\int_\Omega \mathbb{D}\boldsymbol{v} \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = \int_\Omega p \operatorname{div}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \quad \text{for all} \quad \boldsymbol{\psi} \in \boldsymbol{H}^1_0(\Omega).$$

The set $\mathcal{S}$ is non-empty, for it contains, for example, the solutions of the classical Stokes problem corresponding to some compatible Dirichlet conditions on Γ_a (see the proof of Theorem 4 for the reference).

Finally, let

$$F(\boldsymbol{\varphi}) := \int_\Omega |\mathbb{D}\boldsymbol{\varphi}|^2 \,\mathrm{d}\boldsymbol{x} \quad \text{and} \quad G(\boldsymbol{\varphi}) := \int_\Omega |\nabla\boldsymbol{\varphi}|^2 \,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{\varphi} \in \mathcal{S}, \tag{4}$$

where F represents the dissipation of energy (or the entropy production — recall the balance of entropy for the Newtonian fluids), whereas G will be considered only for the comparison.

Existence of a solution

For reader's convenience, we prove here the following modification of Korn's inequality, which will be used to estimate F from below.

Lemma 2. *There exists $c > 0$ such that*

$$\|\boldsymbol{\varphi}\|_{1,2} \le c\|\mathbb{D}\boldsymbol{\varphi}\|_2 \quad \textit{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1_\Gamma(\Omega). \tag{5}$$

Proof. The usual form of Korn's inequality reads as

$$\|\boldsymbol{\varphi}\|_{1,2} \le c(\|\mathbb{D}\boldsymbol{\varphi}\|_2 + \|\boldsymbol{\varphi}\|_2) \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega) \tag{6}$$

and for some $c > 0$ (see, for example, [*Ciarlet*, 2010, Theorem 2.1]). Now suppose that (5) does not hold. Then there exists a sequence $\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}} \subset \boldsymbol{H}^1_\Gamma(\Omega)$, such that $\|\boldsymbol{\varphi}_n\|_{1,2} = 1$ and

$$\|\boldsymbol{\varphi}_n\|_{1,2} > n\|\mathbb{D}\boldsymbol{\varphi}_n\|_2 \tag{7}$$

for all $n \in \mathbb{N}$. Since $\boldsymbol{H}^1_\Gamma(\Omega)$ is reflexive, there is a subsequence (not relabeled) and $\boldsymbol{\varphi}$ satisfying

$$\boldsymbol{\varphi}_n \rightharpoonup \boldsymbol{\varphi} \quad \text{in} \quad \boldsymbol{H}^1_\Gamma(\Omega), \quad n \to \infty. \tag{8}$$

This, together with (7) implies

$$\|\mathbb{D}\boldsymbol{\varphi}_n\|_2 \to \|\mathbb{D}\boldsymbol{\varphi}\|_2 = 0, \quad n \to \infty, \tag{9}$$

hence $\mathbb{D}\boldsymbol{\varphi} = 0$ a.e. in Ω. Then the identity

$$\partial_i\partial_j\psi_k = \partial_i(\mathbb{D}\boldsymbol{\psi})_{jk} + \partial_j(\mathbb{D}\boldsymbol{\psi})_{ki} - \partial_k(\mathbb{D}\boldsymbol{\psi})_{ij}$$

valid for all $\boldsymbol{\psi} \in \boldsymbol{C}^\infty(\overline{\Omega})$ and $1 \le i, j, k \le d$ shows (by approximating $\boldsymbol{\varphi}$ with smooth functions) that every second and, consequently, also every higher distributional derivative of $\boldsymbol{\varphi}$ is zero. Therefore, $\boldsymbol{\varphi}$ is of the form $\boldsymbol{\varphi}(\boldsymbol{x}) = \mathbb{W}\boldsymbol{x} + \boldsymbol{b}$ for almost every $\boldsymbol{x} \in \Omega$ and for some skew-symmetric matrix $\mathbb{W}$ and vector $\boldsymbol{b}$. Then, using the fact that $\boldsymbol{\varphi} \in \boldsymbol{H}^1_\Gamma(\Omega)$ with $|\Gamma| > 0$, we get $\boldsymbol{\varphi} = 0$ a.e. in Ω. Thus, from (9), (8) and compact embedding $\boldsymbol{H}^1_\Gamma(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^2(\Omega)$, we obtain

$$\|\boldsymbol{\varphi}_n\|_{1,2} \le c(\|\mathbb{D}\boldsymbol{\varphi}_n\|_2 + \|\boldsymbol{\varphi}_n\|_2) \to 0 \quad \text{as} \quad n \to \infty,$$

a contradiction. □

Now we can proceed with the main results of this paper concerning the optimization problem stated above.

Theorem 3. *The functionals F and G attain its minima on $\mathcal{S}$.*

Proof. The proof will be done for F. For G the proof is easier — we just replace $\mathbb{D}$ by ∇ and we do not use Lemma 2.

Take some minimizing sequence of F, that is, some $\{\boldsymbol{v}_k\}_{k\in\mathbb{N}} \subset \mathcal{S}$, such that

$$\lim_{k\to\infty} F(\boldsymbol{v}_k) = \inf_{\mathcal{S}} F < \infty \tag{10}$$

(recall that $\mathcal{S} \ne \emptyset$). Let us denote $\boldsymbol{u}_k := \boldsymbol{v}_k - \boldsymbol{v}_0 \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$, $k \in \mathbb{N}$. Using (4), Hölder inequality, Lemma 2 and the fact that $\|\mathbb{D}\boldsymbol{\varphi}\|_2 \le \|\nabla\boldsymbol{\varphi}\|_2 \le \|\boldsymbol{\varphi}\|_{1,2}$ for all $\boldsymbol{\varphi} \in \boldsymbol{H}^1$, we get that F is coercive, i.e.,

$$F(\boldsymbol{v}_k) = \int_\Omega |\mathbb{D}\boldsymbol{v}_k|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega |\mathbb{D}\boldsymbol{u}_k|^2 \,\mathrm{d}\boldsymbol{x} + 2\int_\Omega \mathbb{D}\boldsymbol{u}_k \cdot \mathbb{D}\boldsymbol{v}_0 \,\mathrm{d}\boldsymbol{x} + \int_\Omega |\mathbb{D}\boldsymbol{v}_0|^2 \,\mathrm{d}\boldsymbol{x}$$
$$\geq c_1 \|\nabla \boldsymbol{u}_k\|_2^2 - c_2 \|\nabla \boldsymbol{u}_k\|_2 \|\boldsymbol{v}_0\|_{1,2}$$

for some positive constants c_1, c_2. Thus, using (10), we deduce that the sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$, is bounded in $\boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ and, consequently, there exist a subsequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ (not relabeled) and $\boldsymbol{u}$, such that

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega). \tag{11}$$

By Lemma 2, the mapping $(\boldsymbol{f}, \boldsymbol{g}) \mapsto \int_\Omega \mathbb{D}\boldsymbol{f} \cdot \mathbb{D}\boldsymbol{g} \,\mathrm{d}\boldsymbol{x}$ defines an inner product in $\boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$. Therefore, the property (11) implies, for every $\boldsymbol{\psi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega)$, that

$$0 = \int_\Omega \mathbb{D}\boldsymbol{v}_k \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \mathbb{D}\boldsymbol{v} \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \quad \text{as} \quad k \to \infty,$$

where $\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{v}_0$, and so it follows that $\boldsymbol{v} \in \mathcal{S}$. It is also obvious that the functional F is lower semi-continuous with respect to the weak topology of $\boldsymbol{H}^1_\Gamma(\Omega)$. From that, we get

$$F(\boldsymbol{v}) \leq \liminf_{k\to\infty} F(\boldsymbol{v}_k) = \inf_{\mathcal{S}} F \leq F(\boldsymbol{v})$$

and the proof is finished. □

The previous lemma proves the existence of a weak solution to the problem

$$\begin{aligned} -\operatorname{div}(\mathbb{D}\boldsymbol{v}) &= -\nabla p \quad && \text{in } \Omega \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega \\ \boldsymbol{v} &= \tilde{\boldsymbol{v}}_0 && \text{on } \Gamma \\ \boldsymbol{v} & \text{ minimizes } F. \end{aligned} \tag{12}$$

The additional constraint $F(\boldsymbol{v}) = \min_{\mathcal{S}} F$ can be thus seen as an implicitly given outflow boundary condition. We remark that everything that is stated above for the Stokes system holds analogically also for the Navier–Stokes system, but we omit the corresponding discussion to keep this paper as short as possible.

The optimal outflow boundary condition

Now we may ask what the implicit boundary condition $F(\boldsymbol{v}) = \min_{\mathcal{S}} F$ actually means for the flow or, even better, whether this condition can be made explicit. In our setting, this is indeed possible as is shown by the next theorem.

Theorem 4. *If* $(\boldsymbol{v}, p)$ *is a weak solution to* (12)*, then there is a constant* $c_0 \in \mathbb{R}$ *such that*

$$-p\boldsymbol{n} + 2\nu(\mathbb{D}\boldsymbol{v})\boldsymbol{n} = c_0\boldsymbol{n} \quad \textit{a.e. on } \Gamma_{\mathrm{a}},$$

or, equivalently,

$$\mathbb{T}(\boldsymbol{v}, p + c_0)\boldsymbol{n} = 0 \quad \textit{a.e. on } \Gamma_{\mathrm{a}}.$$

Proof. Suppose that $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ satisfies

$$\int_\Omega \mathbb{D}\boldsymbol{\varphi} \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for all} \quad \boldsymbol{\psi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega) \tag{13}$$

(such functions $\boldsymbol{\varphi}$ indeed exist as is discussed below when deriving (17)). Then, using $\boldsymbol{v} \in \mathcal{S}$, we also know that

$$\int_\Omega \mathbb{D}(\boldsymbol{v} + \varepsilon\boldsymbol{\varphi}) \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for all} \quad \boldsymbol{\psi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega) \quad \text{and for any} \quad \varepsilon \in \mathbb{R},$$

hence $\boldsymbol{v}+\varepsilon\boldsymbol{\varphi} \in \mathcal{S}$ is an appropriate candidate for testing the optimality condition in (12). Thus, we get

$$F(\boldsymbol{v}) \leq F(\boldsymbol{v} + \varepsilon\boldsymbol{\varphi}), \tag{14}$$

or, equivalently, (using (4))

$$0 \leq 2\varepsilon \int_\Omega \mathbb{D}\boldsymbol{v} \cdot \mathbb{D}\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} + \varepsilon^2 \int_\Omega |\mathbb{D}\boldsymbol{\varphi}|^2 \,\mathrm{d}\boldsymbol{x} \tag{15}$$

for every $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ satisfying (13) and for all $\varepsilon \in \mathbb{R}$. In inequality (15), we consider the cases $\varepsilon > 0$ and $\varepsilon < 0$ separately, divide by ε and then we let $\varepsilon \to 0\pm$. This way we obtain

$$\int_\Omega \mathbb{D}\boldsymbol{v} \cdot \mathbb{D}\boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega) \quad \text{satisfying} \quad (13). \tag{16}$$

Now let $\boldsymbol{w} \in \boldsymbol{H}^1_\Gamma(\Omega)$ be such that $\int_{\partial\Omega} \boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}S = 0$ (here in the surface integral, the symbol $\boldsymbol{w}$ stands for the trace of $\boldsymbol{w}$). Then, by [*Temam*, 1979, Chapter I, Theorem 2.4] and by the equivalence of (2) and (3), there exists an unique solution $\boldsymbol{v_w} \in \boldsymbol{H}^1_{\Gamma,\mathrm{div}}(\Omega)$ to the Stokes system

$$\begin{aligned} \int_\Omega \mathbb{D}\boldsymbol{v_w} \cdot \mathbb{D}\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for every} \quad \boldsymbol{\psi} \in \boldsymbol{H}^1_{0,\mathrm{div}}(\Omega),\\ \boldsymbol{v_w} - \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega). \end{aligned} \tag{17}$$

Thus, we may apply (16) with $\boldsymbol{\varphi} = \boldsymbol{v_w}$, which together with (17) and the weak formulation of (12) gives

$$2\nu \int_\Omega \mathbb{D}\boldsymbol{v}\cdot\mathbb{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = 2\nu\int_\Omega \mathbb{D}\boldsymbol{v}\cdot\mathbb{D}\boldsymbol{v_w}\,\mathrm{d}\boldsymbol{x} + 2\nu\int_\Omega \mathbb{D}\boldsymbol{v}\cdot\mathbb{D}(\boldsymbol{w}-\boldsymbol{v_w})\,\mathrm{d}\boldsymbol{x} = \int_\Omega p \,\mathrm{div}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x},$$

hence

$$\int_\Omega \mathbb{T}(\boldsymbol{v},p)\cdot\mathbb{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = 0. \tag{18}$$

Now let us assume that there exists the trace of $\mathbb{T}(\boldsymbol{v},p)$ (in fact there exists normal component of the trace of $\mathbb{T}(\boldsymbol{v},p)$ in the sense of distributions, since $\mathbb{T}(\boldsymbol{v},p)$ is an integrable solenoidal function). Then, it follows from (18), (12) and integration by parts that

$$0 = \int_\Omega \mathbb{T}(\boldsymbol{v},p)\cdot\mathbb{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_\mathrm{a}} \mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\cdot\boldsymbol{w}\,\mathrm{d}S \;\text{ for all }\; \boldsymbol{w}\in\boldsymbol{H}^1_\Gamma(\Omega) \;\text{ s.t. }\; \int_{\Gamma_\mathrm{a}} \boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}S = 0. \tag{19}$$

Since, at Γ_a, there is no restriction on the tangential part of the trace of $\boldsymbol{w} \in \boldsymbol{H}^1_\Gamma(\Omega)$, we deduce from (19) that

$$(\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n})_\tau = 0 \quad \text{a.e. on } \Gamma_\mathrm{a}, \tag{20}$$

or, equivalently

$$((\mathbb{D}\boldsymbol{v})\boldsymbol{n})_\tau = 0 \quad \text{a.e. on } \Gamma_\mathrm{a},$$

where $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$. Furthermore, using (20) in (19), we also get

$$\int_{\Gamma_a} (\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\cdot\boldsymbol{n})(\boldsymbol{w}\cdot\boldsymbol{n})\,\mathrm{d}S = 0 \quad \text{for all} \quad \boldsymbol{w}\in \boldsymbol{H}^1_\Gamma(\Omega) \quad \text{s.t.} \quad \int_{\Gamma_a} \boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}S = 0. \tag{21}$$

Using the trace theorem, for every $\varphi \in C_0^\infty(\Gamma_a)$ there is $\boldsymbol{w}\in \boldsymbol{H}^1_\Gamma(\Omega)$ such that the trace of $\boldsymbol{w}$ times $\boldsymbol{n}$ is φ (extended by zero to $\partial\Omega$). This, together with (21) yields

$$\int_{\Gamma_a} T\varphi\,\mathrm{d}S = 0 \quad \text{for all} \quad \varphi\in C_0^\infty(\Gamma_a) \quad \text{s.t} \quad \int_{\Gamma_a} \varphi\,\mathrm{d}S = 0, \tag{22}$$

where we abbreviated $T := \mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\cdot\boldsymbol{n}$. Let $\psi,\eta \in C_0^\infty(\Gamma_a)$ be such that $\int_{\Gamma_a}\eta\,\mathrm{d}S > 0$. Then the function

$$\varphi := \psi\eta - \frac{\int_{\Gamma_a}\psi\eta\,\mathrm{d}S}{\int_{\Gamma_a}\eta\,\mathrm{d}S}\eta$$

again belongs to $C_0^\infty(\Gamma_a)$ and $\int_{\Gamma_a}\varphi\,\mathrm{d}S = 0$. Consequently, using the properties of φ and (22), we obtain

$$\int_{\Gamma_a}\left(T - \fint_{\Gamma_a} T\,\mathrm{d}S\right)\psi\eta\,\mathrm{d}S = \frac{\int_{\Gamma_a}\psi\eta\,\mathrm{d}S}{\int_{\Gamma_a}\eta\,\mathrm{d}S}\int_{\Gamma_a}\left(T - \fint_{\Gamma_a} T\,\mathrm{d}S\right)\eta\,\mathrm{d}S$$

where $\fint$ denotes the mean value of an integral. Now we are going to use this identity for a sequence of functions $0 \le \eta_k \in C_0^\infty(\Gamma_a)$, $k\in\mathbb{N}$, satisfying $\eta_k \uparrow 1$ as $k\to\infty$ pointwise in Γ_a. This way, if we apply the dominated convergence theorem, we get

$$\int_{\Gamma_a}\left(T - \fint_{\Gamma_a} T\,\mathrm{d}S\right)\psi\,\mathrm{d}S = \left(\fint_{\Gamma_a}\psi\,\mathrm{d}S\right)\int_{\Gamma_a}\left(T - \fint_{\Gamma_a} T\,\mathrm{d}S\right)\mathrm{d}S = 0.$$

Since $\psi\in C_0^\infty(\Gamma_a)$ was arbitrary, we may infer that $T = \fint_{\Gamma_a} T\,\mathrm{d}S$ a.e. on Γ_a, which means that there exists a constant $c_0\in\mathbb{R}$ such that

$$\mathbb{T}(\boldsymbol{v},p)\boldsymbol{n}\cdot\boldsymbol{n} = c_0 \quad \text{a.e. on } \Gamma_a. \tag{23}$$

Obviously, conditions (20) and (23) are together equivalent to

$$\mathbb{T}(\boldsymbol{v},p+c_0)\boldsymbol{n} = 0 \quad \text{a.e. on } \Gamma_a$$

and the proof is finished. □

Our final theorem shows that if we use G instead of F, we obtain essentially the "do-nothing" boundary condition.

Theorem 5. *If $(\boldsymbol{v},p)$ is a weak solution to (12) with F replaced by G, then there is a constant $c_1\in\mathbb{R}$ such that*

$$-p\boldsymbol{n} + \nu(\nabla\boldsymbol{v})\boldsymbol{n} = c_1\boldsymbol{n} \quad \textit{a.e. on } \Gamma_a.$$

Proof. We use the equivalence of (2) and (3) in the weak formulation of (12). Then the proof follows the same line as the proof of Theorem 4: We can just replace all occurrences of $\mathbb{D}$ and $\mathbb{T}(\boldsymbol{v},p)$ by ∇ and $-p\mathbb{I}+\nu\nabla\boldsymbol{v}$, respectively. □

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