Generalised Selberg Integrals and Macdonald Polynomials

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Motivation

The main object of interest to us is the Selberg integral. It has played an important role in:

- Analytic number theory — the distribution of the Riemann zeros on the critical line
- Random matrices — the distribution of eigenvalues for certain classes of random matrices
- Conformal field theory — computation of conformal blocks and the AGT conjecture

The Hypergeometric Differential Equation

$$x(1-x)\frac{\mathrm{d}^2F}{\mathrm{d}x^2} + (c-(a+b+1)x)\frac{\mathrm{d}F}{\mathrm{d}x} - abF = 0.$$

Gauss solved this ODE as a series:

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

where

$$(a)_k = (a)(a+1)\cdots(a+k-1).$$

Euler solved it as an integral:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,\mathrm{d}t $$

where $\Gamma(x)$ is the gamma function

$$ \Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t, \qquad \mathrm{Re}(x) > 0. $$

Sending $x \to 1$ and using Gauss' summation formula

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{1}{k!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

one may deduce the integral formula

$$\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} = \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,\mathrm{d}t.$$

This is simply the beta integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$$

with $\alpha = b$ and $\beta = c-a-b$.

The Selberg integral

In 1944 Selberg proved the integral evaluation

$$\frac{1}{k!}\int\limits_{[0,1]^k}\prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\leqslant i<j\leqslant k}|t_i-t_j|^{2\gamma}\,\mathrm{d}t_1\cdots\mathrm{d}t_k$$
$$=\prod_{j=0}^{k-1}\frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(j\gamma)}{\Gamma(\alpha+\beta+(k+j-1)\gamma)\Gamma(\gamma)}.$$

Here k is a positive integer, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$ and $\mathrm{Re}(\gamma)>\dots$

Question:

- Can we generalise the previous approach to the Selberg integral?

Partitions

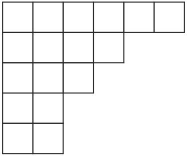
A partition of a positive integer n is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

Here k is the length of λ, and denoted $\ell(\lambda)$. For example there are five partitions of 4:

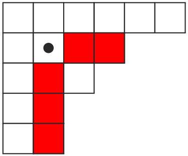
$$4 \qquad 3+1 \qquad 2+2 \qquad 2+1+1 \qquad 1+1+1+1$$

We represent these graphically as Young diagrams:



This is the partition $(6, 4, 3, 2, 2)$. It has length $\ell(\lambda) = 5$ and is a partition of 17.

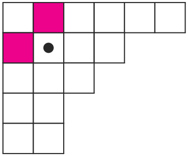
Given a partition λ and its Young diagram, we may define the arm and leg lengths of a square s.



For the bulleted square we have

$$a(s) = 2, \qquad l(s) = 3.$$

The arm colength and leg colength are then



so

$$a'(s) = l'(s) = 1.$$

Symmetric functions

We call a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n] =: \mathbb{Q}[X]$ symmetric if it is invariant permutation of the variables, i.e., for any $w \in \mathfrak{S}_n$,

$$f(x_1, \ldots, x_n) = f(x_{w(1)}, \ldots, x_{w(n)}).$$

The set of such polynomials forms a subring $\Lambda_{\mathbb{Q},n}$ of $\mathbb{Q}[X]$.

Examples are the power sums

$$p_r(X) = x_1^r + x_2^r + \cdots + x_n^r.$$

One extends this to partitions by

$$p_\lambda(X) = p_{\lambda_1}(X) \cdots p_{\lambda_{\ell(\lambda)}}(X).$$

We may define the Hall scalar product on $\Lambda_{\mathbb{Q},n}$ by imposing that the power sums are orthogonal:

$$\langle p_\lambda, p_\mu \rangle = 0, \qquad \lambda \neq \mu.$$

Macdonald polynomials

In 1988 Macdonald introduced a remarkable new basis for $\Lambda_{\mathbb{Q}(q,t)}$, denoted by $P_\lambda(X; q, t)$.

They generalise many well-known classes of symmetric functions such as the Schur functions and Hall–Littlewood polynomials.

The P_λ are orthogonal under a q,t-deformation of the scalar product:

$$\langle P_\lambda(X;q,t), P_\mu(X;q,t)\rangle_{q,t} = 0, \qquad \lambda \neq \mu.$$

Assume $q \in \mathbb{C}$ such that $|q| < 1$. Define the infinite q-shifted factorial

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$

Equivalent to the orthogonality of the Macdonald polynomials is the Cauchy identity

$$\sum_\lambda P_\lambda(X;q,t)P_\lambda(Y;q,t)\prod_{s\in\lambda}\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} = \prod_{i,j\geqslant 1}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

The last ingredient in our approch to generalised Selberg integrals is the q-integral. For $0 < q < 1$ define this by

$$\int_0^1 f(t)\,\mathrm{d}_q t = (1-q)\sum_{i=0}^{\infty} q^i f(q^i).$$

A simple q-beta integral is then

$$\int_0^1 t^{\alpha-1}\frac{(tq;q)_\infty}{(tq^\beta;q)_\infty}\,\mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

where the q-gamma function is given by

$$\Gamma_q(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty}.$$

This q-integral is actually a heavily specialised version of the Cauchy identity for Macdonald polynomials, as it is equivalent to the q-binomial theorem

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

The approach

Cauchy-type identity for Macdonald polynomials

$$\Downarrow$$

Multidimensional q-Selberg integral

$$\Downarrow$$

Generalised Selberg integral

The relationship between symmetric functions and Selberg integrals has long been known. Indeed Macdonald conjectured the following evaluation, which was proved by Kadell:

$$\frac{1}{k!}\int\limits_{[0,1]^k} \tilde{P}_\lambda^{(1/\gamma)}(t)\prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\leqslant i<j\leqslant k}|t_i-t_j|^{2\gamma}\,\mathrm{d}t_1,\dots,\mathrm{d}t_k$$

$$=\prod_{i=1}^{k}\frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)\Gamma(\gamma i)}{\Gamma(\alpha+\beta+(k+i-2)\gamma+\lambda_i)\Gamma(\gamma)}.$$

Here $P_\lambda^{(1/\gamma)}$ is a Jack polynomial, obtained from the Macdonald polynomials by setting $(q,t)=(q,q^\gamma)$ and sending $q\to 1$.

In their studies of conformal field theory, Alba, Fateev, Litvinov and Tarnopolsky (2011) discovered the even more general integral formula

$$\int\limits_{[0,1]^k} \tilde{P}_\lambda^{(1/\gamma)}(\boldsymbol{t})\tilde{P}_\mu^{(1/\gamma)}[\boldsymbol{t}+\beta/\gamma-1]\prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\leqslant i<j\leqslant k}|t_i-t_j|^{2\gamma}\,\mathrm{d}\boldsymbol{t}$$
$$=\prod_{i=1}^{n}\frac{\Gamma(\alpha+\lambda_i+\gamma(n-i))\Gamma(\beta+\gamma(n-i))\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+\lambda_i+\gamma(n-i-1))\Gamma(1+\gamma)}$$
$$\times\prod_{i,j=1}^{n}\frac{\Gamma(\alpha+\beta+\lambda_i+\mu_j+\gamma(2n-i-j-1))}{\Gamma(\alpha+\beta+\lambda_i+\mu_j+\gamma(2n-i-j))}.$$

Using the full Cauchy identity for Macdonald polynomials a proof of this integral and a q-analogue are possible.

Higher rank cases

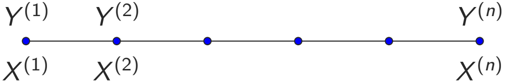
The left-hand side of the Cauchy identity contains two Macdonald polynomials on different alphabets, yet indexed by the same partition:

$$\sum_{\lambda} P_\lambda(X) P_\lambda(Y) \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}$$

We think of this as associating two alphabets X, Y to the Dynkin diagram of $\mathfrak{sl}_2$:

$$\begin{array}{c} Y \\ \bullet \\ X \end{array}$$

We therefore think of a higher rank Cauchy identity as having two alphabets associated to each vertex:



Indeed we are interested in evaluating the sums of the form

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(n)}} \prod_{s=1}^{n} P_{\lambda^{(s)}}(X^{(s)};q,t)P_{\lambda^{(s)}}(Y^{(s)};q,t)\prod_{i=1}^{n-1} f^{(s)}_{\lambda^{(s)},\lambda^{(s+1)}}$$

where f is some function representing the edges of the Dynkin diagram.

Knizhnik–Zamolodchikov equations

Question: Why should such higher rank integrals exist?

Answer: The Knizhnik–Zamolodchikov (KZ) equations:

$$\kappa \frac{\partial u}{\partial z} = \frac{\Omega}{z-w} u,$$
$$\kappa \frac{\partial u}{\partial w} = \frac{\Omega}{w-z} u.$$

These are a system of partial differential equations based on Lie algebras. Indeed $u(z,w)$ takes values in $V_\lambda \otimes V_\mu$ where V_λ, V_μ are highest weight modules for a simple Lie algebra $\mathfrak{g}$.

For $\mathfrak{g} = \mathfrak{sl}_2$ the Selberg integral arrises as a solution analagous to the case of the beta integral and the hypergeometric differential equation.

The Mukhin–Varchenko Conjecture

The case of the Selberg integral lead Mukhin and Varchenko to conjecture the existence of a generalised Selberg integral for each simple Lie algebra $\mathfrak{g}$.

Conjecture (Mukhin–Varchenko (2000))

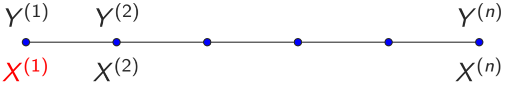
If the space of singular vectors of weight $\lambda + \mu - \sum k_i \bar{\alpha}_i$ is one-dimensional, then there exists some domain of integration D such that the integral

$$\int_D \left|\Phi(t)\right|^{1/\kappa} \mathrm{d}t$$

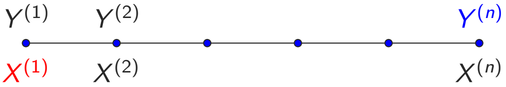
evaluates as a product of gamma functions. The function $\Phi(t)$ is the specialised master function.

Neither the domain D or the form the product of gamma functions takes are specified by the conjecture.

The Mukhin–Varchenko conjecture has a satisfactory only in the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$, due to Warnaar (2009). The proof relies on a rank n Cauchy-type identity where each alphabet is specialised, except for $X^{(1)}$ which is finite:



By adapting Warnaar's technique, it is possible to prove a rank n Cauchy identity for which the alphabet $Y^{(n)}$ is arbitrary:



This extra freedom allows the extension of the integral of Alba *et. al* to rank n.

Theorem ($\mathfrak{sl}_{n+1}$ Alba–Fateev–Litvinov–Tarnopolsky Integral)

Let n be a positive integer and $0 \leqslant k_1 \leqslant \cdots \leqslant k_n$ nonnegative integers. Suppose $\alpha_1, \ldots, \alpha_n, \beta, \gamma \in \mathbb{C}$ are such that

$$Re(\alpha_1), \ldots, Re(\alpha_n), \mathrm{Re}(\beta) > 0$$

(plus some more complicated conditions involving γ). Then

$$\begin{aligned}
\int\limits_{C_\gamma^{k_1,\ldots,k_n}} & \tilde{P}_\lambda^{(1/\gamma)}(t^{(1)}) \tilde{P}_\mu^{(1/\gamma)}[t^{(n)} + \beta/\gamma - 1] \\
& \times \prod_{s=1}^{n} \Bigg[\prod_{1 \leqslant i < j \leqslant k_s} |t_i^{(s)} - t_j^{(s)}|^{2\gamma} \prod_{i=1}^{k_s} (t_i^{(s)})^{\alpha_s - 1} (1 - t_i^{(s)})^{\beta_s - 1} \Bigg] \\
& \times \prod_{s=1}^{n-1} \Bigg(\prod_{i=1}^{k_s} \prod_{j=1}^{k_{s+1}} |t_i^{(s)} - t_j^{(s+1)}|^{-\gamma} \Bigg) \mathrm{d}t
\end{aligned}$$

$=$ *product of gamma functions.*

In the above $\beta_1, \ldots, \beta_{n-1} = 1$ and $\beta_n = \beta$.