

SKEW SYMPLECTIC AND ORTHOGONAL CHARACTERS THROUGH LATTICE PATHS

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ABSTRACT. The skew Schur functions admit many determinantal expressions. Chief among them are the (dual) Jacobi–Trudi formula and the Lascoux–Pragacz formula, the latter being a skew analogue of the Giambelli identity. Comparatively, the skew characters of the symplectic and orthogonal groups, also known as the skew symplectic and orthogonal Schur functions, have received less attention in this direction. We establish analogues of the dual Jacobi–Trudi and Lascoux–Pragacz formulae for these characters. Our approach is entirely combinatorial, being based on lattice path descriptions of the tableaux models of Koike and Terada. Ordinary Jacobi–Trudi formulae are then derived in an algebraic manner from their duals.

1. INTRODUCTION

The classical groups, a term coined by Weyl, are the general linear groups over the real numbers, complex numbers and quaternions and certain subgroups thereof. We are concerned with the complex general linear, symplectic and orthogonal groups, which we write as $GL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$ and $O(n, \mathbb{C})$ respectively. Each of these groups carry families of irreducible representations indexed by partitions. In the case of $GL(n, \mathbb{C})$, these representations are precisely the irreducible polynomial representations whose characters are the Schur polynomials, which are symmetric polynomials in n variables. For the symplectic and orthogonal groups, the irreducible characters in question are symmetric Laurent polynomials in the variables $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$ (also known as BC_n -symmetric polynomials). These characters sometimes go by the name of symplectic and orthogonal Schur polynomials, but we will refer to them simply as the symplectic and orthogonal characters.

The Schur polynomials have several different determinantal expressions. Among them are the Jacobi–Trudi formula and its dual, the Nögelbach–Kostka identity, which express the Schur polynomial as (isobaric) determinants in the complete homogeneous or elementary symmetric functions respectively. There is also the Giambelli formula, which expresses the Schur polynomial as a determinant of Schur polynomials indexed by hook-shaped Young diagrams. These two types of determinantal expressions have analogues for skew Schur polynomials, the skew version of the Giambelli formula being due to Lascoux and Pragacz [17, 18]. Analogously, the symplectic and orthogonal characters were given Jacobi–Trudi-type expressions by Weyl [24, Theorems 7.8.E & 7.9.A], there expressed in terms of complete homogeneous or elementary symmetric functions with

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variables $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$. They also have Giambelli formulae, being first proved by Abramsky, Jahn and King [1], the structure of which is identical to the Schur case.

From a combinatorial point of view, the Schur polynomials may be defined as a weighted sum over semistandard Young tableaux or, equivalently, Gelfand–Tsetlin patterns. This extends easily to the skew case. By interpreting these tableaux as families of nonintersecting lattice paths, Gessel and Viennot [8] provided a beautiful proof of the skew Jacobi–Trudi formulae using what is now known as the Lindström–Gessel–Viennot lemma. Stembridge then applied this approach to the Giambelli and Lascoux–Pragacz formulae [22, §9].

In [5], Fulmek and Krattenthaler set out to provide similar lattice path proofs of the symplectic and orthogonal Jacobi–Trudi formulae and their duals as well as the Giambelli formulae in the straight case. For the symplectic identities they use the tableaux of King [13]. In the orthogonal case there are several different tableaux models, and of these Fulmek and Krattenthaler exploit the tableaux of Proctor, Sundaram as well as of King and Welsh [14, 21, 23]. The main tools in their proofs are the Lindström–Gessel–Viennot lemma and a modified reflection principle. They succeeded in proving the dual Jacobi–Trudi formulae for the symplectic and orthogonal characters in this way, and pass to the ordinary formulae by the dual paths of Gessel and Viennot. For the Giambelli identities, they prove the symplectic and odd orthogonal cases. However, they were unable to obtain a lattice path proof of the even orthogonal Giambelli formula. (This was due to the even orthogonal Sundaram-type tableaux they introduce not having an appropriate weight function; see the discussions in [3, §3.6] and [5, §8].)

Compared to the skew Schur polynomials, skew analogues of the symplectic and orthogonal characters have received little attention. Indeed, only very recently did Jing, Li and Wang obtain Jacobi–Trudi formulae for these characters [12, Propositions 3.2–3.3], and there only in terms of the complete symmetric functions. The goal of the present paper is to provide dual Jacobi–Trudi-type formulae for the skew symplectic and orthogonal characters. We accomplish this in a purely combinatorial way, using an approach that is somewhat in the vein of the one used by Fulmek and Krattenthaler with an extension to the skew setting. In addition, we also produce Lascoux–Pragacz-type skew analogues of the Giambelli formulae for these characters. What facilitates these more general formulae are the tableaux models for skew symplectic and orthogonal characters due to Koike and Terada [16]. These tableaux have already proved combinatorially useful in proving factorisation theorems for skew symplectic and orthogonal characters in work of Ayyer and the second named author [2]. Remarkably, our proofs are simpler than those of Fulmek and Krattenthaler, which further emphasises that the tableaux of Koike and Terada are the better tool in the combinatorial setting.

We begin in the next section by providing definitions and statements of our results. In Section 3 we introduce the tableaux of Koike and Terada as well as the corresponding families of nonintersecting lattice paths. In the following Section 4 we give proofs of the dual Jacobi–Trudi formulae. In order to relate these to the results of Jing, Li and Wang, we use the standard algebraic approach for proving the equivalence of the ordinary Jacobi–Trudi formula and its dual in Section 5. Following this, Section 6 contains the proofs of the Lascoux–Pragacz-type skew analogues of the Giambelli formulae.

2. DEFINITIONS AND MAIN RESULTS

A *partition* is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$. We call the λ_i its *parts*, $l(\mu) := k$ its *length* and $|\lambda| := \sum_{i=1}^k \lambda_i$ its *size*. The *Young diagram* of a partition λ is a collection of left-justified boxes (*cells*) with λ_i cells in the i -th row from the top. For the rest of the paper, we do not distinguish between a partition and its associated Young diagram. The *conjugate* $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ of λ is the partition where λ'_i is the number of boxes in the i -th column of the Young diagram of λ , counted from the left. For two partitions λ, μ , we say that μ is *contained* in λ , denoted by $\mu \subseteq \lambda$, if the Young diagram of μ can be obtained from the Young diagram of λ by removing boxes. In this case, we denote by λ/μ the *skew shaped Young diagram* (or *skew shape* for short) obtained by removing all boxes of the Young diagram of μ from the one of λ . The size of λ/μ is the number of boxes in λ/μ and denoted by $|\lambda/\mu|$. We will sometimes write (n^m) for the rectangular partition with m parts equal to n .

We are interested in irreducible characters of the classical Lie groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$ indexed by partitions. The definitions of these objects may be found, for instance, in [6, §24] or [21, Appendix B], and we only cover the essentials needed to state our results. The characters we are interested in are all symmetric or BC_n -symmetric polynomials with variables $\mathbf{x} := (x_1, \dots, x_n)$, the rings of which we denote by Λ_n and Λ_n^{BC} respectively. For $\mathrm{GL}(n, \mathbb{C})$, the characters of the irreducible polynomial representations are the *Schur polynomials* $s_\lambda(\mathbf{x})$ where λ runs over all partitions of length at most n . They are easily computed, for $l(\lambda) \leq n$, by

$$s_\lambda(\mathbf{x}) := \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j})}{\det_{1 \leq i, j \leq n} (x_i^{n - j})}, \quad (1)$$

setting $\lambda_j = 0$ for $j > l(\lambda)$. In fact, this convention is used throughout the whole paper. The formula is a special case of the Weyl character formula. If $l(\lambda) > n$ then $s_\lambda(\mathbf{x}) := 0$. From the definition it is clear that these are in fact symmetric polynomials of homogeneous degree $|\lambda|$. When λ is a single row or column of r boxes then the Schur polynomials reduce to the *complete homogeneous symmetric polynomials* h_r and the *elementary symmetric polynomials* e_r in \mathbf{x} respectively.

For the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$ we have irreducible characters given by the Weyl formula

$$\mathrm{sp}_\lambda(\mathbf{x}) := \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j + 1} - x_i^{-(\lambda_j + n - j + 1)})}{\det_{1 \leq i, j \leq n} (x_i^{n - j + 1} - x_i^{-(n - j + 1)})},$$

where again $l(\lambda) \leq n$. The orthogonal case is little more delicate. For a statement P we write $[P]$ for the *Iverson bracket*: $[P] = 1$ if P is true and $[P] = 0$ otherwise. Then the even orthogonal group $\mathrm{O}(2n, \mathbb{C})$ has irreducible characters

$$\mathrm{o}_\lambda(\mathbf{x}) := 2^{[\lambda_n \neq 0]} \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j} + x_i^{-(\lambda_j + n - j)})}{\det_{1 \leq i, j \leq n} (x_i^{n - j} + x_i^{-(n - j)})}.$$

If $\lambda_n = 0$ then this is also the character of the irreducible representation of the special orthogonal group $\mathrm{SO}(2n, \mathbb{C})$ corresponding to λ . However, if $\lambda_n \neq 0$ then the above splits into a sum of two irreducible characters of $\mathrm{SO}(2n, \mathbb{C})$, one indexed by λ and the other by $(\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$, which is of course not a partition. In the odd orthogonal

case $O(2n+1, \mathbb{C})$ there is no such distinction, and therefore we will label the irreducible characters of $O(2n+1, \mathbb{C})$ (or $SO(2n+1, \mathbb{C})$) by

$$\text{so}_\lambda(\mathbf{x}) := \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + n - j + 1/2} - x_i^{-(\lambda_j + n - j + 1/2)})}{\det_{1 \leq i, j \leq n} (x_i^{n - j + 1/2} - x_i^{-(n - j + 1/2)})}.$$

We will refer to these two sets of orthogonal characters as even orthogonal characters and odd orthogonal characters respectively.

Each of the above four families of characters have skew variants. In Section 3 we will explicitly define each of these families in terms of skew tableaux, however for now let us explain where they come from. We say two partitions *interlace*, written $\mu \preceq \lambda$, if $\mu \subseteq \lambda$ and

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots.$$

One of the fundamental properties of the Schur polynomials is the *branching rule* [20, p. 72]

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \preceq \lambda} x_n^{|\lambda/\mu|} s_\mu(x_1, \dots, x_{n-1}).$$

Since λ and μ interlace, the skew shape λ/μ has no two boxes in the same column. Iterating the branching rule n times naturally leads to the notion of semistandard Young tableaux. Alternatively, one could iterate only k times for $1 \leq k \leq n-1$. The coefficient of $s_\mu(x_1, \dots, x_{n-k})$ in this expansion is the skew Schur polynomial $s_{\lambda/\mu}(x_{n-k+1}, \dots, x_n)$. In other words, we have the more general branching rule

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} s_\mu(x_1, \dots, x_{n-k}) s_{\lambda/\mu}(x_{n-k+1}, \dots, x_n).$$

In representation-theoretic terms, the branching rule describes the restriction of the irreducible representation indexed by λ to the subgroup $GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$, or more generally to $GL(n-k, \mathbb{C}) \times GL(k, \mathbb{C})$.

Koike and Terada [16] carried out this same procedure for the symplectic and orthogonal groups. They use the branching rules of Zhelobenko [25] to define skew analogues of the symplectic and orthogonal characters. In their most general form these characters depend on a skew shape λ/μ and integers n, m such that $l(\mu) \leq m$ and $l(\lambda) \leq n+m$. Following our notation from above, we denote these by $\text{sp}_{\lambda/\mu}^m$, $\text{so}_{\lambda/\mu}^m$ and $\text{o}_{\lambda/\mu}^m$. These objects are, like their non-skew variants, symmetric Laurent polynomials in $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$. While we give a combinatorial definition of these characters below, we should mention that Jing, Li and Wang have given alternative definitions in terms of vertex operators [12].

The main goal of this paper is to prove, combinatorially, two types of determinantal formulae for $\text{sp}_{\lambda/\mu}^m$, $\text{so}_{\lambda/\mu}^m$ and $\text{o}_{\lambda/\mu}^m$: (i) dual Jacobi–Trudi-type formulae and (ii) Giambelli–Lascoux–Pragacz-type formulae. The Jacobi–Trudi-type formulae are then derived from their duals in an algebraic manner.

For $\mu \subseteq \lambda$, $l(\lambda) \leq n$ and $\lambda_1 \leq N$, and the Jacobi–Trudi identity and its dual are

$$s_{\lambda/\mu}(\mathbf{x}) = \det_{1 \leq i, j \leq n} (h_{\lambda_i - \mu_j - i + j}(\mathbf{x})) = \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x})), \quad (2)$$

where we remind the reader that $\mathbf{x} = (x_1, \dots, x_n)$ for a non-negative integer n that is fixed throughout the paper. Here we have stated this as an identity for the Schur

polynomials, but it also holds in the ring of symmetric functions on a countable set of variables, in which case the $s_{\lambda/\mu}$ are the *skew Schur functions*.

We set $\mathbf{x}^\pm := (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ and are now ready to state our theorems.

Theorem 2.1. *Let m, n, N be non-negative integers and λ, μ partitions such that $\mu \subseteq \lambda$, $l(\mu) \leq m$, $l(\lambda) \leq n + m$ and $\lambda_1 \leq N$. Then*

$$\text{sp}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) - e_{\lambda'_i + \mu'_j - i - j - 2m}(\mathbf{x}^\pm)), \quad (3a)$$

$$\text{so}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) + e_{\lambda'_i + \mu'_j - i - j - 2m + 1}(\mathbf{x}^\pm)), \quad (3b)$$

$$\text{o}_{\lambda/\mu}^m(\mathbf{x}) = \frac{1}{2^{[m=l(\mu)]}} \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) + e_{\lambda'_i + \mu'_j - i - j - 2m + 2}(\mathbf{x}^\pm)). \quad (3c)$$

A main contribution of our work is to provide, as we believe, enlightening combinatorial explanations of these formulae. By using an algebraic approach, we can dualise these three identities and obtain the following ordinary Jacobi–Trudi-type formulae for these characters.

Theorem 2.2. *Let m, n, N be non-negative integers and λ, μ partitions such that $\mu \subseteq \lambda$, $l(\mu) \leq m$, $l(\lambda) \leq n + m$ and $l(\lambda) \leq N$. Then*

$$\text{sp}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (h_{\lambda_i - \mu_j - i + j}(\mathbf{x}^\pm) + [j > m + 1]h_{\lambda_i - i - j + 2m + 2}(\mathbf{x}^\pm)), \quad (4a)$$

$$\text{so}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (h_{\lambda_i - \mu_j - i + j}(\mathbf{x}^\pm) + [j > m]h_{\lambda_i - i - j + 2m + 1}(\mathbf{x}^\pm)), \quad (4b)$$

$$\text{o}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (h_{\lambda_i - \mu_j - i + j}(\mathbf{x}^\pm) - [j > m]h_{\lambda_i - i - j + 2m}(\mathbf{x}^\pm)). \quad (4c)$$

For μ empty and $m = 0$, these formulae are due to Weyl [24, Theorems 7.8.E & 7.9.A]. The identities (4a) and (4c) as stated were recently obtained by Jing, Li and Wang [12, Propositions 3.2–3.3].

Our combinatorial approach also admits the derivation of Giambelli-type formulae, and, in order to formulate them, we need the *Frobenius notation* of a partition. For a partition λ , let (p, p) be the diagonal cell with maximal p which is still contained in the Young diagram of λ (p is the size of the *Durfee square*). For $1 \leq i \leq p$, let α_i be the number of cells right of (i, i) in the same row and β_i be the number of cells below (i, i) in the same column. We then write $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$. Using Frobenius notation for λ and μ , that is, $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$, the *Lascoux–Pragacz formula* reads as

$$s_{\lambda/\mu}(\mathbf{x}) = (-1)^q \det \begin{pmatrix} (s_{\alpha_i | \beta_j}(\mathbf{x}))_{1 \leq i, j \leq p} & (h_{\alpha_i - \gamma_j}(\mathbf{x}))_{1 \leq i \leq p, 1 \leq j \leq q} \\ (e_{\beta_j - \delta_i}(\mathbf{x}))_{1 \leq i \leq q, 1 \leq j \leq p} & 0 \end{pmatrix}. \quad (5)$$

This was shown by Lascoux and Pragacz in [17, 18]. The μ empty case is much older, being due to Giambelli [9], and is therefore known as the *Giambelli identity*. We provide the following analogues.

Theorem 2.3. *Let $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$ be two partitions such that $\mu \subseteq \lambda$, $l(\mu) \leq m$ and $l(\lambda) \leq n + m$. Then*

$$\mathrm{sp}_{\lambda/\mu}^m(\mathbf{x}) = (-1)^q \det \begin{pmatrix} (\mathrm{sp}_{(\alpha_i|\beta_j)}^m(\mathbf{x}))_{1 \leq i, j \leq p} & (\mathrm{sp}_{(\alpha_i)/(\gamma_j)}^m(\mathbf{x}))_{\substack{1 \leq i \leq p, \\ 1 \leq j \leq q}} \\ (\mathrm{sp}_{(1^{\beta_j+1})/(1^{\delta_i+1})}^m(\mathbf{x}))_{\substack{1 \leq i \leq q, \\ 1 \leq j \leq p}} & 0 \end{pmatrix}, \quad (6a)$$

$$\mathrm{so}_{\lambda/\mu}^m(\mathbf{x}) = (-1)^q \det \begin{pmatrix} (\mathrm{so}_{(\alpha_i|\beta_j)}^m(\mathbf{x}))_{1 \leq i, j \leq p} & (\mathrm{so}_{(\alpha_i)/(\gamma_j)}^m(\mathbf{x}))_{\substack{1 \leq i \leq p, \\ 1 \leq j \leq q}} \\ (\mathrm{so}_{(1^{\beta_j+1})/(1^{\delta_i+1})}^m(\mathbf{x}))_{\substack{1 \leq i \leq q, \\ 1 \leq j \leq p}} & 0 \end{pmatrix}, \quad (6b)$$

$$\mathrm{o}_{\lambda/\mu}^m(\mathbf{x}) = (-1)^q \det \begin{pmatrix} (\mathrm{o}_{(\alpha_i|\beta_j)}^m(\mathbf{x}))_{1 \leq i, j \leq p} & (\mathrm{o}_{(\alpha_i)/(\gamma_j)}^m(\mathbf{x}))_{\substack{1 \leq i \leq p, \\ 1 \leq j \leq q}} \\ (\mathrm{o}_{(1^{\beta_j+1})/(1^{\delta_i+1})}^m(\mathbf{x}))_{\substack{1 \leq i \leq q, \\ 1 \leq j \leq p}} & 0 \end{pmatrix}. \quad (6c)$$

As mentioned in the introduction, for μ empty these formulae reduce to those of Abramsky, Jahn and King [1]. Like their formulae, ours have the exact same structure as in the Schur case since $s_{(1^{\beta_j+1})/(1^{\delta_i+1})}(\mathbf{x}) = e_{\beta_j - \delta_i}(\mathbf{x})$ and $s_{(\alpha_i)/(\gamma_j)}(\mathbf{x}) = h_{\alpha_i - \gamma_j}(\mathbf{x})$.

Weyl's Jacobi–Trudi formulae for the symplectic and orthogonal characters are used by Koike and Terada to define the *universal characters* for these groups [15]. This is achieved by “forgetting” the variables \mathbf{x}^\pm in either (3) and (4) and then treating the determinants as polynomials in the e_r or h_r respectively, and thus as elements of the ring of symmetric functions at an arbitrary alphabet. By specialising the arbitrary alphabet to \mathbf{x}^\pm the actual characters are recovered. Many identities between the general linear characters (Schur functions) and the orthogonal and symplectic characters are derived by Koike and Terada using only the universal characters. It would be interesting to investigate whether using (3) and (4) to define skew analogues of the universal characters leads to similar “universal” proofs of identities such as, for example, branching rules.

Note that Hamel [10] has given determinantal formulae for skew analogues of the symplectic and odd orthogonal characters using outside decompositions as introduced in [11]. However, the skew tableaux she defines are different to those of Koike and Terada, and so are the associated skew characters.

3. TABLEAUX AND LATTICE PATHS

In this section, we introduce the underlying combinatorial models for $s_{\lambda/\mu}$, $\mathrm{sp}_{\lambda/\mu}^m$, $\mathrm{so}_{\lambda/\mu}^m$ and $\mathrm{o}_{\lambda/\mu}^m$ in terms of tableaux. There are several possibilities for the models underlying $\mathrm{so}_{\lambda/\mu}^m$ and $\mathrm{o}_{\lambda/\mu}^m$, and we choose those defined by Koike and Terada in [16]. In the non-skew case, all of the various combinatorial models and their equivalences may be found in [21] (also see [4]). We then also introduce the corresponding families of non-intersecting lattice paths.

3.1. Semistandard Young tableaux. Let $\mu \subseteq \lambda$ be two partitions. A *semistandard Young tableau* of shape λ/μ is a filling of the cells of the Young diagram λ/μ with positive integers such that the entries increase weakly along rows and strictly down columns; see Figure 1 for an example. We denote by $\mathrm{SSYT}_{\lambda/\mu}^n$ the set of semistandard Young tableaux of shape λ/μ and maximal entry n . The weight \mathbf{x}^T of a semistandard

Young tableau T is defined as the monomial

$$\mathbf{x}^T = x_1^{\# \text{ of } 1\text{'s in } T} \dots x_n^{\# \text{ of } n\text{'s in } T}.$$

The *skew Schur polynomial* $s_{\lambda/\mu}(\mathbf{x})$ corresponding to the shape λ/μ is defined as the multivariate generating function of semistandard Young tableaux of shape λ/μ , i.e.,

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \text{SSYT}_{\lambda/\mu}^n} \mathbf{x}^T.$$

Our path models depend on arbitrary integers $N \geq \lambda_1$ and $n \geq l(\lambda)$. A *family of Schur paths* associated with the shape λ/μ is a family of N non-intersecting lattice paths with starting points $S_i = (\mu'_i - i + 1, 2l(\mu) - \mu'_i + i - 1)$, end points $E_j = (\lambda'_j - j + 1, n - \lambda'_j + j - 1 + 2l(\mu))$ for $1 \leq i, j \leq N$, where also here we set $\lambda'_j = 0$ and $\mu'_i = 0$ for $j > l(\lambda')$ and $i > l(\mu')$, and with step set $\{(1, 0), (0, 1)\}$. The weight of a family of paths is the product of the weights of its steps, where the i -th step of a path has weight x_i if it is horizontal and 1 otherwise. Phrased differently, the weight of a horizontal step starting at (a, b) has weight $x_{a+b-2l(\mu)+1}$ (this is called the *e-labelling*).

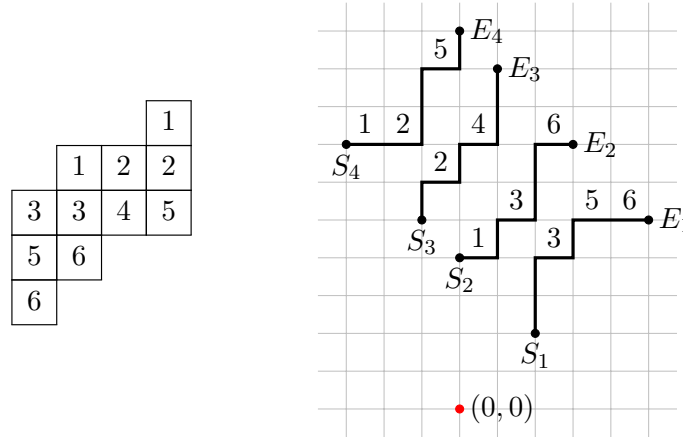


FIGURE 1. A semistandard Young tableau of shape $(4, 4, 4, 2, 1)/(3, 1)$ (left) and the corresponding family of Schur paths (right) for $N = 4$ and $n = 6$.

The bijection between semistandard Young tableaux T of shape λ/μ and families of Schur paths associated with the shape λ/μ is as follows. The path starting at S_i corresponds to the i -th column of T and the j -th step in the path is a horizontal step if and only if j appears as a filling in the i -th column. See Figure 1 for an example.

3.2. Skew (n, m) -symplectic semistandard Young tableaux. Let $\mu \subseteq \lambda$ be two partitions and n, m integers with $l(\mu) \leq m$ and $l(\lambda) \leq n + m$. A *skew (n, m) -symplectic semistandard Young tableau* (or *skew (n, m) -symplectic tableau*) of skew shape λ/μ is a semistandard Young tableau of skew shape λ/μ which is filled by

$$\bar{1} < 1 < \bar{2} < 2 < \dots < \bar{n} < n,$$

and satisfies the

- *m-symplectic condition*: the entries in row $m + i$ are at least \bar{i} .

We denote by $\text{SPT}_{\lambda/\mu}^{(n,m)}$ the set of skew (n, m) -symplectic tableaux of shape λ/μ . For a skew (n, m) -symplectic tableau T , the weight \mathbf{x}^T is defined as

$$\mathbf{x}^T = \prod_{i=1}^n x_i^{(\# \text{ of } i\text{'s in } T) - (\# \text{ of } \bar{i}\text{'s in } T)}.$$

The *skew m-symplectic character* $\text{sp}_{\lambda/\mu}^m(\mathbf{x})$ is defined as the multivariate generating function of skew (n, m) -symplectic tableaux of shape λ/μ :

$$\text{sp}_{\lambda/\mu}^m(\mathbf{x}) = \sum_{T \in \text{SPT}_{\lambda/\mu}^{(n,m)}} \mathbf{x}^T.$$

For our path model, we again fix an integer $N \geq \lambda_1$. A *family of (n, m) -symplectic paths* associated with the shape λ/μ a family of N non-intersecting lattice paths with starting points $S_i = (\mu'_i - i + 1, 2m - \mu'_i + i - 1)$ end points $E_j = (\lambda'_j - j + 1, 2n + 2m - \lambda'_j + j - 1)$ for $1 \leq i, j \leq N$. The step set of these paths is $\{(1, 0), (0, 1)\}$, and additionally each path must stay weakly above the line $y = x - 1$. The weight of a vertical step is 1 and a horizontal step has weight x_i if it is the $(2i)$ -th step of a path and weight x_i^{-1} if it is the $(2i - 1)$ -st step of a path. Equivalently, the weight of a horizontal step starting at (a, b) is $x_{(a+b)/2-m+1}^{-1}$ if $a + b$ is even and $x_{(a+b+1)/2-m}$ if $a + b$ is odd.

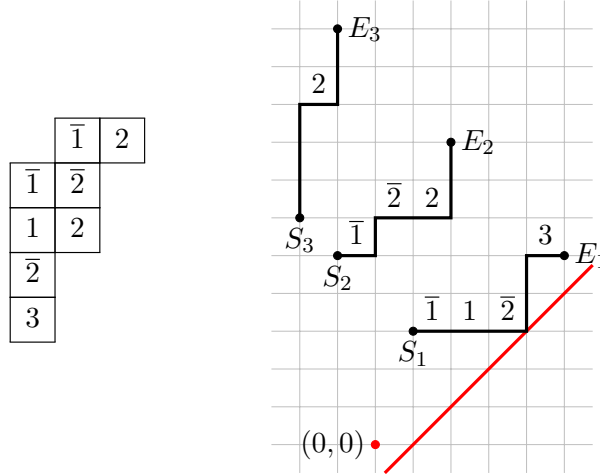


FIGURE 2. A skew $(3, 2)$ -symplectic tableau of shape $(3, 2, 2, 1, 1)/(1)$ (left) and its associated family of $(3, 2)$ -symplectic paths (right).

There is a bijection between skew (n, m) -symplectic tableaux and families of (n, m) -symplectic paths that is very similar to the one in the Schur case. Namely, given a skew (n, m) -symplectic tableau T , the path starting at S_i of the corresponding family of symplectic paths is obtained by associating the j -th step of the path with the j -th element in the sequence $\bar{1}, 1, \bar{2}, 2, \dots$ and letting a step being horizontal if and only if the corresponding element in the sequence appears in the i -th column. An example is given in Figure 2.

3.3. Skew (n, m) -odd orthogonal semistandard Young tableaux. Let λ, μ be partitions such that $\mu \subseteq \lambda$, $l(\mu) \leq m$ and $l(\lambda) \leq n + m$. A *skew (n, m) -odd orthogonal semistandard Young tableau* (or *skew (n, m) -odd orthogonal tableau*) associated with the shape λ/μ is a skew semistandard Young tableau of shape λ/μ which is filled by

$$\widehat{1} < \overline{1} < 1 < \cdots < \widehat{n} < \overline{n} < n$$

and satisfies

- the *modified m -symplectic condition*: the entries in row $m + i$ are at least \widehat{i} , and
- the *m -odd orthogonal condition*: the symbol \widehat{i} can only appear in the first column of the $(m + i)$ -th row.

Note that, unlike in [16], we use \widehat{i} instead of \natural_i . Denote by $\text{SOT}_{\lambda/\mu}^{(n,m)}$ the set of skew (n, m) -odd orthogonal tableaux of shape λ/μ . We define the weight \mathbf{x}^T of a tableau $T \in \text{SOT}_{\lambda/\mu}^m$ as

$$\mathbf{x}^T = \prod_{i=1}^n x_i^{(\# \text{ of } i\text{'s in } T) - (\# \text{ of } \bar{i}\text{'s in } T)}.$$

The *skew m -odd orthogonal character* $\text{so}_{\lambda/\mu}^m(\mathbf{x})$ is the multivariate generating function of skew (n, m) -odd orthogonal tableaux of shape λ/μ :

$$\text{so}_{\lambda/\mu}^m(\mathbf{x}) = \sum_{T \in \text{SOT}_{\lambda/\mu}^{(n,m)}} \mathbf{x}^T.$$

A *family of (n, m) -odd orthogonal paths* associated with the shape λ/μ is a family of non-intersecting (n, m) -symplectic paths for which we extend the step set by a diagonal step $(1, 1)$ which can only occur when the starting point is of the form (i, i) . The weight of a family of (n, m) -odd orthogonal paths is the product of the weights of all steps, where horizontal and vertical steps have the same weight as in the symplectic case and diagonal steps have weight 1.

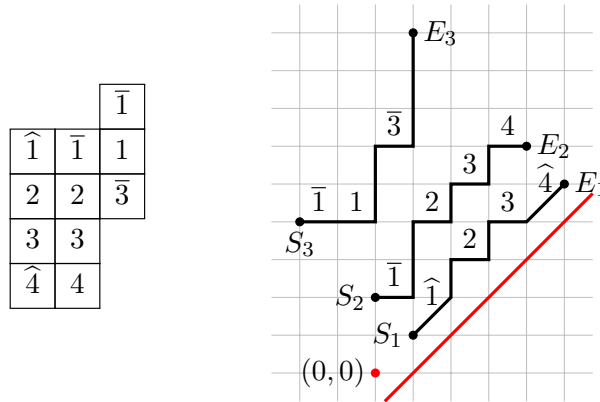


FIGURE 3. A skew $(4, 1)$ -odd orthogonal tableau of shape $(3, 3, 3, 2, 2)/(2)$ (left) and its associated family of $(4, 1)$ -odd orthogonal paths (right).

We obtain a weight preserving bijection from skew (n, m) -odd orthogonal tableaux of shape λ/μ to families of (n, m) -odd orthogonal paths associated with the shape λ/μ as follows. We follow the same procedure as in the symplectic setting by interpreting first each \widehat{i} entry as \bar{i} . Note that each horizontal step corresponding to an \widehat{i} entry ends at the line $y = x - 1$ and therefore has to be followed by a vertical step. Now replace each of these horizontal steps coming from an \widehat{i} entry and its following vertical step by a diagonal step. See Figure 3 for an example.

3.4. Skew (n, m) -even orthogonal semistandard Young tableaux. As before, let $\mu \subseteq \lambda$ be partitions such that $l(\mu) \leq m$ and $l(\lambda) \leq n+m$. A *skew (n, m) -even orthogonal semistandard Young tableau* (or *skew (n, m) -even orthogonal tableau*) of shape λ/μ is a semistandard Young tableau of shape λ/μ which is filled by the symbols

$$\check{1} < \widehat{1} < \bar{1} < 1 < \cdots < \check{n} < \widehat{n} < \bar{n} < n$$

and satisfies

- the *modified m -symplectic condition*: the entries in row $m+i$ are at least \widehat{i} ,
- the *modified m -odd orthogonal condition*: the symbol \widehat{i} can only appear in the first column of the $(m+i)$ -th row and it appears if and only if the entry above is \check{i} , and
- the *m -even orthogonal condition*: if \bar{i} appears in the first column of the $(m+i)$ -th row and i also appears in the same row, then there is an \bar{i} immediately above this i entry.

Note that, unlike in [16], we use the symbols \check{i} and \widehat{i} instead of $\#_i \flat_i$; this corresponds to interchanging the roles of \check{i} and \widehat{i} if compared to [2].

We denote the set of skew (n, m) -even orthogonal tableaux of shape λ/μ by $\text{OT}_{\lambda/\mu}^{(n, m)}$. The weight \mathbf{x}^T of a skew (n, m) -even orthogonal tableau T is defined as

$$\mathbf{x}^T = \prod_{i=1}^n x_i^{(\# \text{ of } i \text{ entries in } T) - (\# \text{ of } \bar{i} \text{ entries in } T)}.$$

The *skew m -even orthogonal character* $\text{o}_{\lambda/\mu}^m(\mathbf{x})$ is the multivariate generating function of skew (n, m) -even orthogonal tableaux of shape λ/μ , so

$$\text{o}_{\lambda/\mu}^m(\mathbf{x}) = \sum_{T \in \text{OT}_{\lambda/\mu}^{(n, m)}} \mathbf{x}^T.$$

We say that a family of lattice paths is *strongly non-intersecting* if there is no intersection between any pair of paths when considering them as subsets of \mathbb{R}^2 . (For our application of the Lindström–Gessel–Viennot Lemma 4.2, we will also need the notion of *weakly non-intersecting* in Section 4.2, where we only forbid intersections at lattice points that are endpoints of steps in both paths.)

A *family of (n, m) -even orthogonal paths* associated with the shape λ/μ is a family of strongly non-intersecting symplectic paths for which we extend the step set by a horizontal step $(2, 0)$, called an *o-horizontal step*, which can only occur starting at a point of the form $(i-2, i)$, and where the family of paths does not have any trapped position as defined below. We draw o-horizontal steps as arcs to avoid confusion.

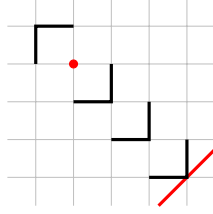


FIGURE 4. The local configuration around a trapped position which is marked as a red dot.

For positive integers i, d , we call the position $(m+i-d, m+i+d-1)$ *trapped* if the following is satisfied:

- the lattice point $(m+i-d, m+i+d-1)$ is not contained in any path,
- for each $d' \in \{0, \dots, d-1\}$, there is a lattice path which passes through $(m+i-d', m+i+d'-1)$ by a horizontal step followed by a vertical step, and
- there is a path passing through the point $(m+i-d-1, m+i+d)$ by a vertical step followed by a horizontal step.

See Figure 4 for an example with $d = 3$.

We now describe the weight preserving bijection from (n, m) -even orthogonal tableaux of shape λ/μ to families of (n, m) -even orthogonal paths associated with the same shape that are strongly non-intersecting. First we interpret each \bar{i} as \bar{i} and each \hat{i} as i and apply the map from (n, m) -symplectic tableaux to families of (n, m) -symplectic paths. Then we replace the pairs of horizontal steps associated with \bar{i}, \hat{i} coming from the modified m -odd orthogonal condition by an o-horizontal step. For an example see Figure 5. In order to see that this map is a bijection it suffices to check that a tableau contradicts the m -even orthogonal condition if and only if the corresponding family of paths has a trapped position, which is done next.

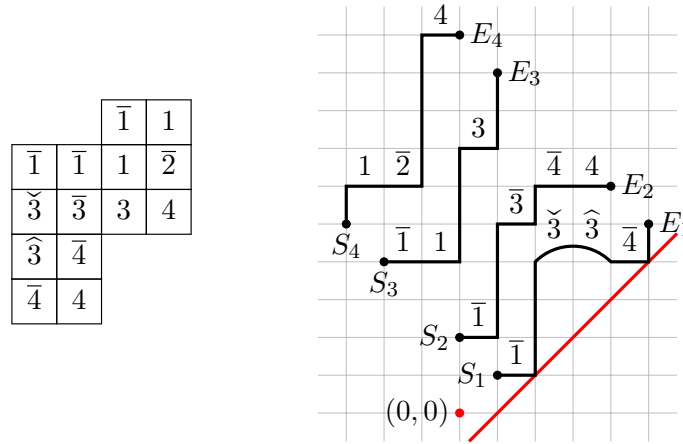


FIGURE 5. A skew $(4,1)$ -even orthogonal tableau of shape $(4,4,4,2,2)/(2)$ (left) and its associated family of $(4,1)$ -even orthogonal paths (right).

Assume that T is a tableau contradicting the m -even orthogonal-condition, i.e., there exists an integer i such that the $(m+i)$ -th row starts with an entry \bar{i} and also contains an entry i whose entry above is not \bar{i} . It is immediate that the top neighbour of the first i in this row can not be \bar{i} since this would contradict the condition that rows are weakly increasing. Denote by d the number of \bar{i} entries in the $(m+i)$ -th row. When looking at the corresponding family of paths, the $(d'+1)$ -st entry \bar{i} corresponds to a horizontal step ending at the point $(m+i-d', m+i+d'-1)$. Since the bottommost of these paths touches the line $y = x - 1$ at $(m+i, m+i-1)$ and all paths are strongly non-intersecting, each of these horizontal steps are followed by a vertical step. The first entry i in the $(m+i)$ -th row corresponds to a horizontal step starting at $(m+i-d-1, m+i+d)$. Since the top neighbour of this i is not \bar{i} , the step before has to be a vertical step. This implies that the position $(m+i-d, m+i+d-1)$ is trapped.

4. COMBINATORIAL PROOFS OF THE DUAL JACOBI–TRUDI FORMULAE

The proofs of the dual Jacobi–Trudi formulae for all of the skew characters under consideration — skew symplectic, skew odd orthogonal and skew even orthogonal — follow a similar scheme: we interpret the respective tableaux columnwise as non-intersecting lattice paths as seen in Section 3. Thus, each column corresponds to a lattice path whose generating function can be written in terms of elementary symmetric functions in the alphabet $\mathbf{x}^\pm = (x_1^{-1}, x_1, \dots, x_n^{-1}, x_n)$. Applying the Lindström–Gessel–Viennot Lemma 4.2 then yields a determinantal formula.

The proofs increase in complexity and a brief summary is as follows.

- For the *skew symplectic case*, we have to compute the generating function of lattice paths consisting of unit horizontal and vertical steps in the positive direction which do not cross the line $y = x - 1$. This is achieved by using a modified reflection principle (Lemma 4.1) by Fulmek and Krattenthaler [5], which provides a difference of two elementary symmetric functions as the generating function, see Lemma 4.3.
- In the case of *skew odd orthogonal characters*, we need to compute the generating function of lattice paths which may additionally have diagonal steps along the line $y = x$. For each fixed number of such diagonal steps in the lattice path, we obtain a difference of elementary symmetric functions in Lemma 4.4. Adding these differences together allow telescopic cancelling, which finally yields a sum of two elementary symmetric functions in Corollary 4.5.
- In the case of *skew even orthogonal characters*, we allow horizontal double steps ending on the line $y = x$ instead of diagonal steps. By similar means (Lemma 4.6), we also obtain a sum of two elementary symmetric functions as the generating function for these lattice paths in Corollary 4.7. However, applying the Lindström–Gessel–Viennot Lemma 4.2 in this case also results in families of lattice paths that do not correspond to a skew (n, m) -even orthogonal tableau. We provide a sign-reversing involution between those and families of lattice paths with trapped positions at the end of the section.

4.1. Modified reflection principle. One of the main tools in proving the dual Jacobi–Trudi formulae is a *modified reflection principle* which we present next.

A lattice point $(x, y) \in \mathbb{Z}^2$ is said to be *even* if $x + y$ is even, otherwise it is said to be *odd*. Suppose we have a lattice path starting in $P = (a, b)$ that consists of unit horizontal and vertical steps in the positive direction. In addition, we assume that P is an even point. We assign weights to the steps as follows. Vertical steps have weight 1, whereas the weights of horizontal steps are given by a modified e -labelling: the step from (i, j) to $(i + 1, j)$ has weight $x_{(i+j-a-b)/2+1}^{-1}$ if (i, j) is even and $x_{(i+j-a-b+1)/2}$ if (i, j) is odd. Now the weight of the path is the product of the weights of its steps.

Furthermore, consider a line $y = x + d$ such that d is even and P lies above that diagonal line, that is, $b > a + d$. The modified reflection principle will enable us to derive combinatorially a formula for the generating function of such lattice paths that start at P and that have no intersection with the line $y = x + d$. This is done by computing the generating function of those paths that have an intersection with $y = x + d$ and then subtracting it from the generating function of all lattice paths.

Lemma 4.1. *Let $a, b, c, d, f \in \mathbb{Z}$ such that $a + b$ and d are even, $b > a + d$ and $f > c + d$. There is a weight-preserving bijection between lattice paths with unit horizontal and vertical steps that start at the point $P = (a, b)$ and end at (c, f) , and that have an intersection with the line $y = x + d$, and lattice paths with unit horizontal and vertical steps that start at the reflected point $P' = (b - d, a + d)$ of P along $y = x + d$ and also end at (c, f) .*

Proof. The proof is illustrated in Figure 6. Suppose our path touches the line for the

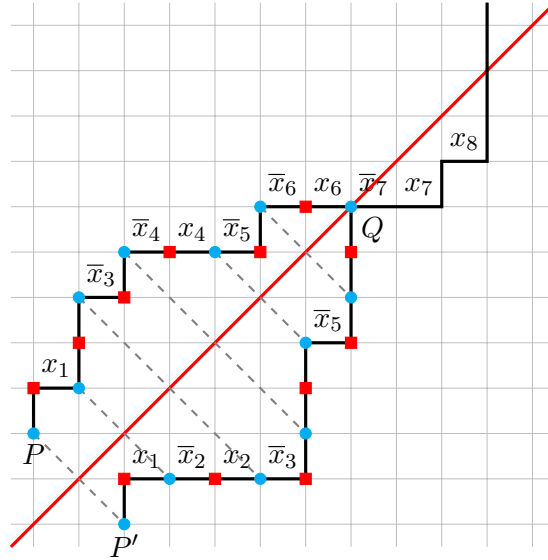


FIGURE 6. The modified reflection principle where $\bar{x}_i := x_i^{-1}$.

first time at the point Q , when traversing it starting at P . The modified reflection of the path from P to Q along the line $y = x + d$ works as follows: P is reflected in the usual way, so it is mapped to $P' = (b - d, a + d)$. The same applies to all other even points and to odd points at which the path does not turn; these are odd points that lie in between either two vertical or two horizontal steps. The remaining points are odd

points in which the path turns. We reflect these points in such a way that the directions of the turns are maintained. Concretely, if an odd point (x, y) comes with a left turn (a horizontal step followed by a vertical step), it is mapped to $(y - d + 1, x + d - 1)$; if it comes with a right turn (a vertical step followed by a horizontal step), it is mapped to $(y - d - 1, x + d + 1)$.

Note that the modification of the usual reflection ensures that the mapping is weight-preserving. \square

4.2. The Lindström–Gessel–Viennot Lemma. Another important tool is the well-known Lindström–Gessel–Viennot Lemma [7, 8, 19]. We state it in a form that is convenient for us. We need the following two notions.

- We say that a family of lattice paths is *weakly non-intersecting* if no pair of paths has an intersection at a lattice point that is the endpoint of steps in both paths.
- We say that a family of lattice paths is *strongly non-intersecting* if no pair of paths intersects when considering them as subsets of \mathbb{R}^2 .

One may interpret the first notion of non-intersecting when considering the paths as graphs, and the second notion when considering their natural embeddings in \mathbb{R}^2 . The background for these notions is that the Gessel–Viennot sign-reversing involution can only be applied if there is a pair of paths that intersect at a lattice point that is the endpoint of steps in both paths, which is precisely the case when the family is *not* weakly non-intersecting. Also note that the two notions of non-intersecting are equivalent for families of Schur paths, of m -symplectic paths, and of m -odd orthogonal paths, however, they differ for families of m -even orthogonal paths due to the steps of length 2. Thus we may simply say non-intersecting in the first two cases.

Lemma 4.2 (The Lindström–Gessel–Viennot Lemma). *We consider lattice paths on the lattice \mathbb{Z}^2 which is equipped with edge weights and where the allowed steps are given by a finite subset of \mathbb{Z}^2 such that the step set does not allow self-intersections of paths. Let S_1, \dots, S_n and E_1, \dots, E_n be lattice points and let $\mathcal{P}(S_i \rightarrow E_j)$ denote the generating function of lattice paths from S_i and E_j with respect to the step set and the edge weights. Then*

$$\det_{1 \leq i, j \leq n} (\mathcal{P}(S_i \rightarrow E_j))$$

is the signed generating function of families of n lattice paths such that the following is satisfied.

- *The S_1, S_2, \dots, S_n are the starting points and the E_1, E_2, \dots, E_n are the ending points.*
- *The paths are weakly non-intersecting.*
- *The sign is $\text{sgn } \sigma$ where σ is the permutation of $\{1, 2, \dots, n\}$ such that S_i is connected to $E_{\sigma(i)}$ for $i = 1, 2, \dots, n$.*

4.3. Skew symplectic characters. For the skew symplectic character $\text{sp}_{\lambda/\mu}^m$, we have to compute the generating function of families of non-intersecting lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leq i, j \leq N$ with $\lambda_1 \leq N$, $l(\mu) \leq m$ and $l(\lambda) \leq n + m$ that consist of unit horizontal and vertical steps in the positive direction and that do not cross the line $y = x - 1$, see Section 3.2. In

order to apply Lemma 4.2, we first compute the generating function of paths between pairs of starting and ending points.

Lemma 4.3. *Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and the points (a, b) and $(c, 2n + a + b - c)$ lie strictly above the line $y = x - 1$. Then, with the path set-up of Lemma 4.1, the generating function of paths from (a, b) to $(c, 2n + a + b - c)$ is given by*

$$e_{c-a}(\mathbf{x}^\pm) - e_{c-b-2}(\mathbf{x}^\pm),$$

where $\mathbf{x}^\pm = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$.

Proof. We use the modified reflection principle from Lemma 4.1 with $(a, b, c, d, f) \mapsto (a, b, c, -2, 2n + a + b - c)$ for this proof. The generating function of all lattice paths from (a, b) to $(c, 2n + a + b - c)$ with the given step set regardless of whether they cross the line $y = x - 1$ is $e_{c-a}(\mathbf{x}^\pm)$.

Next, we derive the generating function of all lattice paths from (a, b) to $(c, 2n + a + b - c)$ that do cross that line. Those paths touch at least once the line $y = x - 2$. We take the initial part of a path from (a, b) to that first intersection point and reflect it in a weight-preserving way according to the modified reflection principle. This yields paths from $(b + 2, a - 2)$ to $(c, 2n + a + b - c)$. The generating function of these paths is $e_{c-b-2}(\mathbf{x}^\pm)$, which we need to subtract from $e_{c-a}(\mathbf{x}^\pm)$ to obtain the result. \square

In view of the above, the generating function of lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ which do not cross the line $y = x - 1$ is given by

$$e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) - e_{\lambda'_i + \mu'_j - i - j - 2m}(\mathbf{x}^\pm).$$

Note the requirement that $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ lies above the line $y = x - 1$ is equivalent to $l(\lambda) \leq n + m$. Applying the Lindström–Gessel–Viennot Lemma 4.2 finally yields

$$\text{sp}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) - e_{\lambda'_i + \mu'_j - i - j - 2m}(\mathbf{x}^\pm)),$$

and this concludes the combinatorial proof of Theorem 2.1 (3a).

4.4. Skew odd orthogonal characters. For the skew odd orthogonal character $\text{so}_{\lambda/\mu}^m$, we have to compute as before the generating function of families of non-intersecting lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leq i, j \leq N$ with $\lambda_1 \leq N$, $l(\mu) \leq m$ and $l(\lambda) \leq n + m$ that do not cross the line $y = x - 1$; see Section 3.3. In this case, however, we have a bigger step set. We allow unit horizontal and vertical steps in the positive direction and, in addition, diagonal steps $(1, 1)$ on the line $y = x$ which have weight 1. In order to compute this generating function, we first derive the generating function of individual paths with a given number k of diagonal steps and then we sum over all non-negative integers k .

Lemma 4.4. *Let k be a positive integer and let $\{(1, 0), (0, 1), (1, 1)\}$ be the step set such that diagonal steps are only allowed along the line $y = x$. Also let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and the points (a, b) and $(c, 2n + a + b - c)$ lie strictly above the line $y = x - 1$. Then the generating function of lattice paths from (a, b) to $(c, 2n + a + b - c)$ that do not cross the line $y = x - 1$ and that have exactly k diagonal steps is*

$$e_{c-b-k}(\mathbf{x}^\pm) - e_{c-b-k-2}(\mathbf{x}^\pm),$$

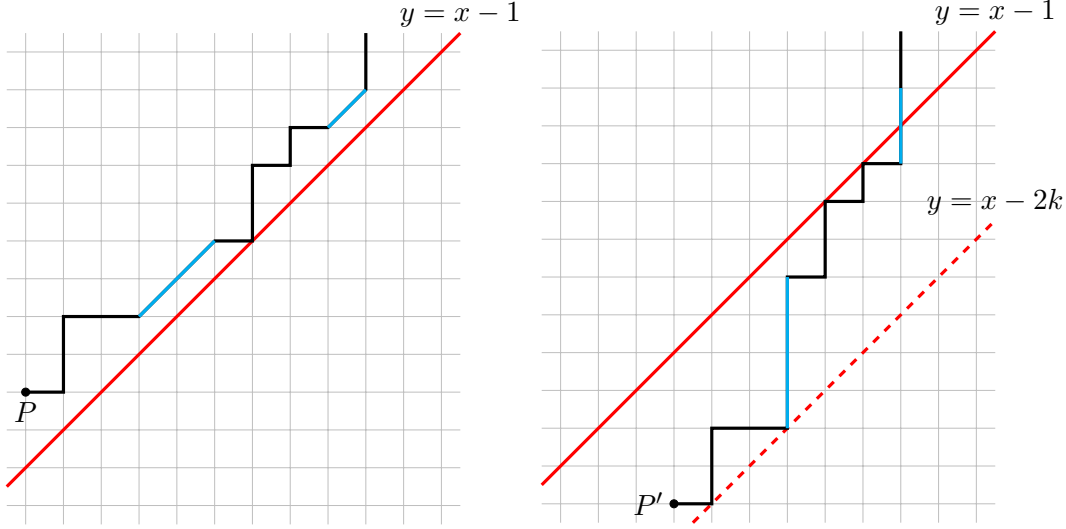


FIGURE 7. Situation in Lemma 4.4 when replacing diagonal steps by vertical steps of length 2.

where $\mathbf{x}^\pm = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$.

Proof. First, we map a path with k diagonal steps to a path which only consists of unit horizontal and vertical steps as follows. Replace every diagonal step by two vertical steps. In the process, we keep the terminal part of the path and shift the initial part accordingly. This is illustrated in Figure 7. We obtain a lattice path from $(a+k, b-k)$ to $(c, 2n+a+b-c)$ that does not cross the line $y = x - 2k - 1$ but intersects the line $y = x - 2k$.

We observe that this mapping is a weight-preserving bijection. For the inverse mapping, we consider a lattice path from $(a+k, b-k)$ to $(c, 2n+a+b-c)$ not crossing the line $y = x - 2k - 1$ but intersecting the line $y = x - 2k$. Since the path goes to $(c, 2n+a+b-c)$, there must be a point where the path touches the line $y = x - 2k$ and continues with at least two vertical steps. We take the rightmost of such points, replace the first two vertical steps by a diagonal step and shift the initial part of the path by $(-1, 1)$. We thus obtain a path from $(a+k-1, b-k+1)$ to $(c, 2n+a+b-c)$ not crossing the line $y = x - 2k + 1$ but intersecting the line $y = x - 2k + 2$. As before, there is a point where the path intersects with the line $y = x - 2k + 2$ and continues with at least two vertical step. We iteratively repeat the step of replacing two vertical steps by a diagonal step at the rightmost occurrence and shifting the initial part of the path by $(-1, 1)$ until we have got a path that starts at (a, b) and has k diagonal steps.

With this weight-preserving bijection in mind, the statement in the theorem follows by applying twice the modified reflection principle from Lemma 4.1 as follows.

To compute the generating function of paths from $(a+k, b-k)$ to $(c, 2n+a+b-c)$ that intersect the line $y = x - 2k$, we reflect $(a+k, b-k)$ along $y = x - 2k$, which yields $(b+k, a-k)$. Hence, the generating function is $e_{c-b-k}(\mathbf{x}^\pm)$.

From the latter generating function, we have to subtract the generating function of paths that cross the line $y = x - 2k - 1$; note that such paths always have to intersect

the line $y = x - 2k$ because of the end point $(c, 2n + a + b - c)$. Paths that cross the line $y = x - 2k - 1$ definitely intersect the line $y = x - 2k - 2$. Reflecting $(a + k, b - k)$ along $y = x - 2k - 2$ gives $(b + k + 2, a - k - 2)$, which implies that the generating function of paths crossing the line $y = x - 2k - 1$ is $e_{c-b-k-2}(\mathbf{x}^\pm)$. This concludes the proof of the lemma. \square

The generating function of paths from (a, b) to $(c, 2n + a + b - c)$ without diagonal steps is $e_{c-a}(\mathbf{x}^\pm) - e_{c-b-2}(\mathbf{x}^\pm)$, which follows from Lemma 4.3. Next we sum the generating functions for paths with exactly k diagonal steps for all non-negative k . This sum turns out to be a telescoping sum which reduces to a sum of two terms after cancellation:

$$(e_{c-a}(\mathbf{x}^\pm) - e_{c-b-2}(\mathbf{x}^\pm)) + \sum_{k \geq 1} (e_{c-b-k}(\mathbf{x}^\pm) - e_{c-b-k-2}(\mathbf{x}^\pm)) = e_{c-a}(\mathbf{x}^\pm) + e_{c-b-1}(\mathbf{x}^\pm).$$

This leaves us with the following.

Corollary 4.5. *Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and the points (a, b) and $(c, 2n + a + b - c)$ lie strictly above the line $y = x - 1$. Then the generating function of lattice paths from (a, b) to $(c, 2n + a + b - c)$ with step set $\{(1, 0), (0, 1), (1, 1)\}$ that do not cross the line $y = x - 1$ and where diagonal steps are only allowed on the line $y = x$ is*

$$e_{c-a}(\mathbf{x}^\pm) + e_{c-b-1}(\mathbf{x}^\pm),$$

where $\mathbf{x}^\pm = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$.

By setting $a = \mu'_j - j + 1$, $b = 2m - \mu'_j + j - 1$ and $c = \lambda'_i - i + 1$ in Corollary 4.5 and applying the Lindström–Gessel–Viennot Lemma (Lemma 4.2), we finally obtain

$$\text{so}_{\lambda/\mu}^m(\mathbf{x}) = \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) + e_{\lambda'_i + \mu'_j - i - j - 2m + 1}(\mathbf{x}^\pm)),$$

and this concludes the combinatorial proof of Theorem 2.1 (3b).

4.5. Skew even orthogonal characters. Recall that in the case of skew even orthogonal characters, we have to compute the generating function of strongly non-intersecting lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leq i, j \leq N$ with $\lambda_1 \leq N$, $l(\mu) \leq m$ and $l(\lambda) \leq n + m$ that do not cross the line $y = x - 1$ and where a certain pattern that corresponds to the m -even orthogonal condition is not allowed; see Section 3.4. The step set is $\{(1, 0), (0, 1), (2, 0)\}$, where the horizontal steps $(2, 0)$, called o-horizontal steps, are only allowed to start at points on the line $y = x + 2$ (and thus end on the line $y = x$). The o-horizontal steps are equipped with the weight 1 and the unit steps have weights according to the standard e -labelling.

We compute the generating function of such paths in Corollary 4.7 in a similar manner to the previous section. However, we also need to deal with the avoidance of the patterns that take care of the m -even orthogonal condition, that is, with trapped positions. It turns out that this combines nicely with the application of the Lindström–Gessel–Viennot Lemma 4.2. Namely, when applying the lemma to the generating functions of paths from Corollary 4.7, we may have pairs of paths that are weakly non-intersecting but not strongly non-intersecting. With our step set such intersections may only occur when o-horizontal steps intersect with two unit vertical steps. Therefore, we present a sign-reversing involution that shows that families of lattice paths with such intersections

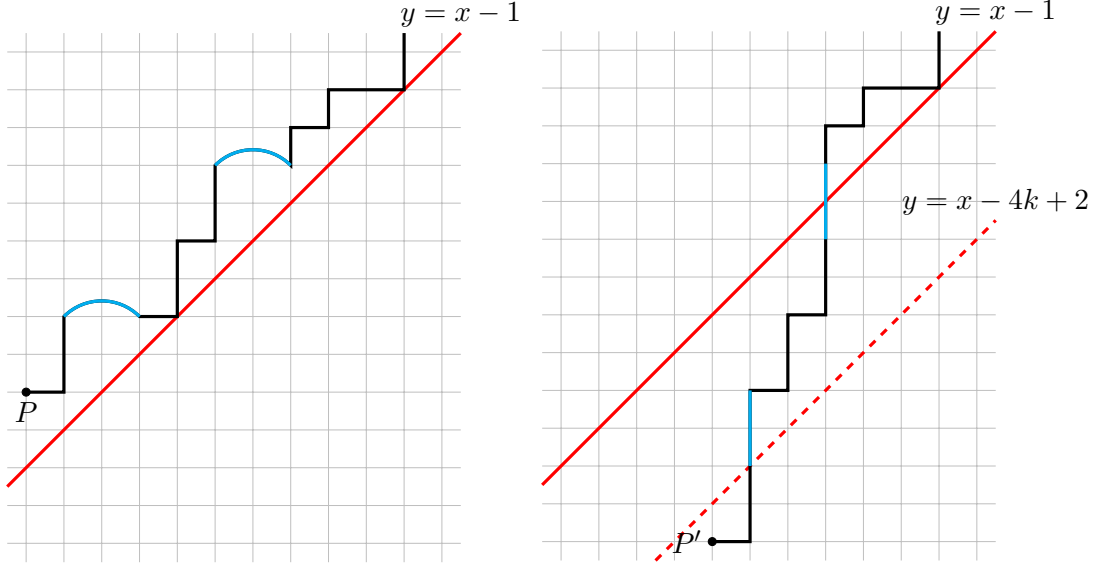


FIGURE 8. Situation in Lemma 4.6 when replacing o-horizontal steps by vertical steps of length 2.

and with trapped positions indeed cancel after applying the Lindström–Gessel–Viennot Lemma 4.2.

Lemma 4.6. *Let k be a positive integer and let $\{(1, 0), (0, 1), (2, 0)\}$ be the step set such that o-horizontal steps are only allowed to end on the line $y = x$. Also let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and (a, b) and $(c, 2n + a + b - c)$ lie strictly above the line $y = x - 1$. Then the generating function of lattice paths from (a, b) to $(c, 2n + a + b - c)$ that do not cross the line $y = x - 1$ and that have exactly k o-horizontal steps is*

$$\begin{cases} e_{c-b-2k+2}(\mathbf{x}^\pm) - e_{c-b-2k-2}(\mathbf{x}^\pm) & \text{if } b - a \geq 2, \\ e_{c-a-2k}(\mathbf{x}^\pm) - e_{c-b-2k-2}(\mathbf{x}^\pm) & \text{otherwise,} \end{cases}$$

where $\mathbf{x}^\pm = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$.

Proof. We proceed in a way similar to the proof of Lemma 4.4. First, we replace every o-horizontal step by a double vertical step whilst keeping the terminal part of the path but shifting the initial part accordingly. See Figure 8 for an illustration. This results in a path from $(a + 2k, b - 2k)$ to $(c, 2n + a + b - c)$ that intersects the line $y = x - 4k + 2$ but that does not cross the line $y = x - 4k - 1$.

This mapping is again a weight-preserving bijection. We continue by using the modified reflection principle in Lemma 4.1.

In order to compute the generating function of paths from $(a + 2k, b - 2k)$ to $(c, 2n + a + b - c)$ that intersect the line $y = x - 4k + 2$, we have to distinguish two cases. If $(a + 2k, b - 2k)$ already lies below $y = x - 4k + 2$, that is, if $b - a = 0$, then the generating function is simply $e_{c-a-2k}(\mathbf{x}^\pm)$. Otherwise, we reflect $(a + 2k, b - 2k)$ along $y = x - 4k + 2$ and obtain $(b + 2k - 2, a - 2k + 2)$. Thus, the generating function of these paths is $e_{c-b-2k+2}(\mathbf{x}^\pm)$.

Lattice paths from $(a+2k, b-2k)$ to $(c, 2n+a+b-c)$ which cross the line $y = x-4k-1$ touch also the line $y = x-4k-2$. Reflecting $(a+2k, b-2k)$ along $y = x-4k-2$ gives $(b+2k+2, a-2k-2)$, and thus the generating function of these paths is $e_{c-b-2k-2}(\mathbf{x}^\pm)$. This completes the proof. \square

Using Lemma 4.3, we know that the generating function of lattice paths from (a, b) to $(c, 2n+a+b-c)$ that do not go below the line $y = x-1$ and have no o-horizontal steps is $e_{c-a}(\mathbf{x}^\pm) - e_{c-b-2}(\mathbf{x}^\pm)$. By summing over all possible numbers of o-horizontal steps, we finally obtain in the case $b-a \geq 2$:

$$(e_{c-a}(\mathbf{x}^\pm) - e_{c-b-2}(\mathbf{x}^\pm)) + \sum_{k \geq 1} (e_{c-b-2k+2}(\mathbf{x}^\pm) - e_{c-b-2k-2}(\mathbf{x}^\pm)) = e_{c-a}(\mathbf{x}^\pm) + e_{c-b}(\mathbf{x}^\pm).$$

On the other hand, if $b-a=0$, we obtain

$$(e_{c-a}(\mathbf{x}^\pm) - e_{c-a-2}(\mathbf{x}^\pm)) + \sum_{k \geq 1} (e_{c-a-2k}(\mathbf{x}^\pm) - e_{c-a-2k-2}(\mathbf{x}^\pm)) = e_{c-a}(\mathbf{x}^\pm).$$

Corollary 4.7. *Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a+b$ is even and the points (a, b) and $(c, 2n+a+b-c)$ lie strictly above the line $y = x-1$. Then the generating function of lattice paths from (a, b) to $(c, 2n+a+b-c)$ with step set $\{(1, 0), (0, 1), (2, 0)\}$ that do not cross the line $y = x-1$ and where o-horizontal steps are only allowed to end on the line $y = x$ is*

$$\begin{cases} e_{c-a}(\mathbf{x}^\pm) & \text{if } b-a=0, \\ e_{c-a}(\mathbf{x}^\pm) + e_{c-b}(\mathbf{x}^\pm) & \text{if } b-a \geq 2, \end{cases}$$

where $\mathbf{x} = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$.

We now apply the Lindström–Gessel–Viennot Lemma 4.2 to all lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leq i, j \leq N$ with $\lambda_1 \leq N$, $l(\mu) \leq m$, $l(\lambda) \leq n+m$ and step set $\{(1, 0), (0, 1), (2, 0)\}$ such that the paths weakly stay above the line $y = x-1$ and o-horizontal steps are only allowed to end on the line $y = x$. By Corollary 4.7 this gives

$$\frac{1}{2^{[m=l(\mu)]}} \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) + e_{\lambda'_i + \mu'_j - i - j - 2m + 2}(\mathbf{x}^\pm)). \quad (7)$$

Note that the prefactor $\frac{1}{2}$ takes care of the case $m = l(\mu)$ because then the first column of the underlying matrix in the previous determinant is $(2e_{\lambda'_1 - \mu'_1}(\mathbf{x}^\pm), 2e_{\lambda'_2 - \mu'_1 - 1}(\mathbf{x}^\pm), \dots, 2e_{\lambda'_N - \mu'_1 - N + 1}(\mathbf{x}^\pm))^\top$.

To complete the proof, we want to show that (7) indeed represents a formula for $\mathbf{o}_{\lambda/\mu}^m(\mathbf{x})$. This is done by a sign-reversing involution. First, we recall that the Lindström–Gessel–Viennot Lemma 4.2 (7) yields a signed enumeration of weakly non-intersecting lattice paths where o-horizontal steps might intersect with vertical steps. Figure 9 shows an example of lattice paths with such intersections. On the other hand, there might still be trapped positions. The sign-reversing involution will transform an intersection into a trapped position, or vice versa.

The intersections between o-horizontal steps and a pair of unit vertical steps only occur along the line $y = x+1$. Consider such an intersection and assume it is at the point $(d, d+1)$. That means we have an o-horizontal step $(d-1, d+1) \rightarrow (d+1, d+1)$

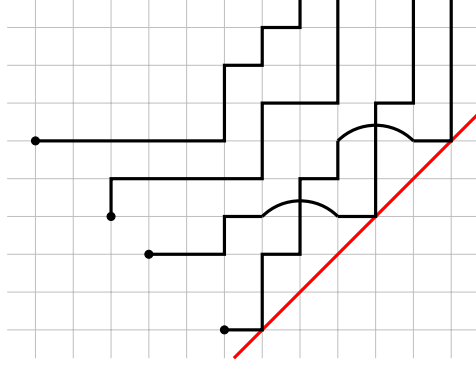


FIGURE 9. Weakly non-intersecting lattice paths that are enumerated by the Lindström–Gessel–Viennot Lemma.

in one path, and two unit vertical steps $(d, d) \rightarrow (d, d+1) \rightarrow (d, d+2)$ in the other path. We perform the following local changes to the family of lattice paths along the line $y + x = 2d + 1$ as indicated in Figure 10: First we resolve the crossing by replacing it by the following two turns $(d, d) \rightarrow (d+1, d) \rightarrow (d+1, d+1)$ and $(d-1, d+1) \rightarrow (d, d+1) \rightarrow (d, d+2)$, which will then appear in different paths. Note that this changes the sign of the permutation, however, it does not change the weight. Then we look for the unoccupied point (x_0, y_0) on the line $y + x = 2d + 1$ that is above the line $y = x - 1$ and has minimal $y_0 - x_0$. This point is unique and between that point and the line $y = x - 1$ there is a series of left turns along the line $y + x = 2d + 1$. We take the left turn $(x_0, y_0 - 1) \rightarrow (x_0 + 1, y_0 - 1) \rightarrow (x_0 + 1, y_0)$ and replace it by $(x_0, y_0 - 1) \rightarrow (x_0, y_0) \rightarrow (x_0 + 1, y_0)$.

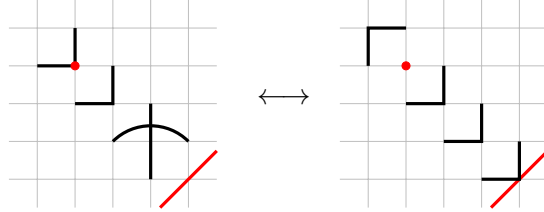


FIGURE 10. Local changes between lattice paths with an intersection of an o-horizontal step with two vertical steps (left) and with a trapped position (right).

Note that this local transformation only changes the sign of the weight. In addition, we see that the resulting constellations such as on the right of Figure 10 are exactly the trapped positions that cannot occur in the lattice path interpretation of skew (n, m) -even orthogonal tableaux due to the m -even orthogonal condition. This observation leads us to the announced sign-reversing involution.

Consider the families of lattice paths whose signed enumeration is given by (7). If it contains a crossing of an o-horizontal step with two vertical steps or if it contains a trapped position then choose a canonical occurrence as follows. Both the crossings and the trapped position are located on lines $y + x = 2d + 1$ for integers d . We consider the

crossing or the trapped position for which the d is minimal. Then we perform the local changes as exemplified in Figure 10. This mapping changes the sign of the weight and is readily invertible. We are left with families of lattice paths that have neither crossings nor trapped positions, which are exactly those that correspond to skew (n, m) -even orthogonal tableaux. This finally proves that

$$o_{\lambda/\mu}^m(\mathbf{x}) = \frac{1}{2^{[m=l(\mu)]}} \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^\pm) + e_{\lambda'_i + \mu'_j - i - j - 2m + 2}(\mathbf{x}^\pm)),$$

and concludes the combinatorial proof of Theorem 2.1 (3c).

5. PROOFS OF THE JACOBI–TRUDI FORMULAE

The purpose of this section is to derive ordinary Jacobi–Trudi-type formulae for the characters $\text{sp}_{\lambda/\mu}^m$, $\text{so}_{\lambda/\mu}^m$ and $o_{\lambda/\mu}^m$. We achieve this by an algebraic approach using complementary cofactors, which is one of the standard ways to prove the equivalence of the Jacobi–Trudi formula for Schur functions and its dual. These computations all work in the ring of symmetric functions on a countable alphabet $\mathbf{X} = (x_1, x_2, x_3, \dots)$, and so in this section we work in such generality. The formulae for the actual characters are obtained by substituting $\mathbf{x}^\pm = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ for \mathbf{X} .

Given an $N \times N$ matrix A and a pair of sequences σ, τ of the same length, we let A_τ^σ denote the matrix obtained by extracting rows and columns with indices from σ and τ respectively in the given order. The basic lemma we need is the following [6, Lemma A.42].

Lemma 5.1. *Let A and B be $N \times N$ mutually inverse matrices and let (σ, σ') and (τ, τ') be pairs of complementary subsets of $\{1, \dots, N\}$ such that $|\sigma| = |\tau|$. Then*

$$\det(A_\tau^\sigma) = \varepsilon \det(A) \det(B_{\sigma'}^{\tau'}),$$

where ε is the product of the signs of the permutations formed by the words $\sigma\sigma'$ and $\tau\tau'$.

Recall that the matrices

$$(e_{i-j}(\mathbf{X}))_{1 \leq i, j \leq r} \quad \text{and} \quad ((-1)^{i-j} h_{i-j}(\mathbf{X}))_{1 \leq i, j \leq r},$$

are lower-unitriangular, i.e., lower-triangular with diagonal entries all 1, and where as before $\mathbf{X} = (x_1, x_2, x_3, \dots)$. Moreover, thanks to the relationship [20, p. 21]

$$\sum_{k=0}^r (-1)^k e_{r-k}(\mathbf{X}) h_k(\mathbf{X}) = \delta_{r,0} \tag{8}$$

where $\delta_{a,b}$ is the usual Kronecker delta, they are mutually inverse. We actually require the following slightly more general pair of matrices.

Lemma 5.2. *For $m, N \in \mathbb{N}$, $k \in \mathbb{Z}$ and a parameter t the matrices*

$$\mathcal{E}(N, m, k; t) := (e_{i-j}(\mathbf{X}) + [j < m + \lceil k/2 \rceil] t e_{i+j-2m-k}(\mathbf{X}))_{1 \leq i, j \leq N}$$

and

$$\mathcal{H}(N, m, k; t) := ((-1)^{i-j} (h_{i-j}(\mathbf{X}) - [i > m + \lfloor k/2 \rfloor] (-1)^k t h_{2m-i-j+k}(\mathbf{X})))_{1 \leq i, j \leq N}$$

are lower-unitriangular and mutually inverse.

Proof. The (i, j) -th entry in the product of the two matrices is

$$\begin{aligned} \sum_{\ell=0}^N (-1)^{\ell-j} e_{i-\ell}(\mathbf{X}) h_{\ell-j}(\mathbf{X}) + \sum_{\ell=0}^{m+\lceil k/2 \rceil - 1} (-1)^{\ell-j} t e_{i+\ell-2m-k}(\mathbf{X}) h_{\ell-j}(\mathbf{X}) \\ - \sum_{\ell=m+\lfloor k/2 \rfloor + 1}^N (-1)^{\ell-j+k} t e_{i-\ell}(\mathbf{X}) h_{2m-\ell-j+k}(\mathbf{X}), \end{aligned}$$

where we assume that $i \geq j$ since the lower-triangularity is clear from the definition. The first sum in this expression simplifies to

$$\sum_{\ell=0}^N (-1)^{\ell-j} e_{i-\ell}(\mathbf{X}) h_{\ell-j}(\mathbf{X}) = \sum_{\ell=0}^{i-j} (-1)^{\ell} e_{i-j-\ell}(\mathbf{X}) h_{\ell}(\mathbf{X}) = \delta_{i-j,0},$$

by (8). For the remaining sums, we obtain by substituting $\ell \mapsto m - \ell + \lceil k/2 \rceil - 1$ in the first sum and $\ell \mapsto m + \ell + \lfloor k/2 \rfloor + 1$ in the second sum

$$\begin{aligned} \sum_{\ell=0}^{m+\lceil k/2 \rceil - 1} (-1)^{\ell-j} t e_{i+\ell-2m-k}(\mathbf{X}) h_{\ell-j}(\mathbf{X}) - \sum_{\ell=m+\lfloor k/2 \rfloor + 1}^N (-1)^{\ell-j+k} t e_{i-\ell}(\mathbf{X}) h_{2m-\ell-j+k}(\mathbf{X}) \\ = \sum_{\ell=0}^{m+\lceil k/2 \rceil - 1} (-1)^{\ell-j+\lceil k/2 \rceil + m + 1} t e_{i-\ell-m-\lfloor k/2 \rfloor - 1}(\mathbf{X}) h_{m+\lceil k/2 \rceil - \ell - j - 1}(\mathbf{X}) \\ - \sum_{\ell=0}^{N-m-\lfloor k/2 \rfloor - 1} (-1)^{\ell-j+\lceil k/2 \rceil + m + 1} t e_{i-\ell-m-\lfloor k/2 \rfloor - 1}(\mathbf{X}) h_{m+\lceil k/2 \rceil - \ell - j - 1}(\mathbf{X}) \\ = 0, \end{aligned}$$

where the last equality follows since the summands are equal and vanish unless the index satisfies $\ell \leq \min\{m + \lceil k/2 \rceil - 1, N - m - \lfloor k/2 \rfloor - 1\}$. \square

Recall from [20, p. 3] that for a partition $\lambda \subseteq (N^M)$ the sets

$$\{\lambda_i + M - i + 1 : 1 \leq i \leq M\} \quad \text{and} \quad \{M + j - \lambda'_j : 1 \leq j \leq N\} \quad (9)$$

form a disjoint union of $\{1, \dots, N + M\}$. This implies that for partitions $\mu \subseteq \lambda \subseteq (N^M)$ the sequences

$$\begin{aligned} (\sigma, \sigma') &:= (\lambda'_1 + N, \dots, \lambda'_N + 1, N - \lambda_1 + 1, \dots, N + M - \lambda_M) \\ (\tau, \tau') &:= (\mu'_1 + N, \dots, \mu'_N + 1, N - \mu_1 + 1, \dots, N + M - \mu_M). \end{aligned}$$

form permutations of $\{1, \dots, N + M\}$. Applying Lemma 5.1 with these choices of (σ, σ') and (τ, τ') , $A = \mathcal{E}(N + M, m, k; t)$ and $B = \mathcal{H}(N + M, m, k; t)$ we obtain that the determinants

$$\det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{X}) + [N + \mu'_j - j + 1 < m + \lceil k/2 \rceil] t e_{\lambda'_i + \mu'_j - i - j + 2(N - m + 1) - k}(\mathbf{X})) \quad (10a)$$

and

$$\det_{1 \leq i, j \leq M} (h_{\lambda_i - \mu_j - i + j}(\mathbf{X}) - [N - \mu_j + j > m + \lfloor k/2 \rfloor] (-1)^k t h_{\lambda_i + \mu_j - i - j + 2(m - N) + k}(\mathbf{X})), \quad (10b)$$

are equal. Note that in this case the product of the signs of the permutations is $\varepsilon = (-1)^{|\lambda|+|\mu|}$, which is a consequence of the proof that (9) form a disjoint union of $\{1, \dots, N + M\}$. If one sets $t = 0$ in (10) then

$$\det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{X})) = \det_{1 \leq i, j \leq M} (h_{\lambda_i - \mu_j - i + j}(\mathbf{X})),$$

showing that the Jacobi–Trudi formula and its dual are equal. The dual forms of our Jacobi–Trudi formulae for the skew characters $\mathrm{sp}_{\lambda/\mu}^m$, $\mathrm{so}_{\lambda/\mu}^m$ and $\mathrm{o}_{\lambda/\mu}^m$ are similarly contained in (10). We can now prove Theorem 2.2.

Proof of Theorem 2.2. Fix partitions $\mu \subseteq \lambda$ such that $l(\mu) \leq m$, $l(\lambda) \leq n + m$ and $\lambda \subseteq (N^M)$. For the symplectic case we set $(m, k, t) \mapsto (N + m, 2, -1)$ in (10), which gives

$$\begin{aligned} \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{X}) - [\mu'_j - j < m] e_{\lambda'_i + \mu'_j - i - j - 2m}(\mathbf{X})) \\ = \det_{1 \leq i, j \leq M} (h_{\lambda_i - \mu_j - i + j}(\mathbf{X}) + [j - \mu_j > m + 1] h_{\lambda_i + \mu_j - i - j + 2m + 2}(\mathbf{X})). \end{aligned}$$

Since $\mu'_j \leq m$ for $1 \leq j \leq N$ we always have two terms in each entry of the determinant on the left. After replacing \mathbf{X} by $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ it is equal to $\mathrm{sp}_{\lambda/\mu}^m$ by (3a). Further, $m + \mu_j - j + 1 < 0$ is only true for $j > m + 1$, so the determinant on the right-hand side becomes (4a) with $N \mapsto M$ and the same variable substitution.

Instead taking $(m, k, t) \mapsto (N + m, 1, 1)$ in (10) produces

$$\begin{aligned} \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{X}) + [\mu'_j - j < m] e_{\lambda'_i + \mu'_j - i - j - 2m + 1}(\mathbf{X})) \\ = \det_{1 \leq i, j \leq M} (h_{\lambda_i - \mu_j - i + j}(\mathbf{X}) + [j - \mu_j > m] h_{\lambda_i + \mu_j - i - j + 2m + 1}(\mathbf{X})). \end{aligned}$$

By the same arguments as above the left-hand side is equal to $\mathrm{so}_{\lambda/\mu}^m$ and the right-hand side to (4b) with $N \mapsto M$, after appropriate substitution of variables.

Finally, choosing $(m, k, t) \mapsto (N + m, 0, 1)$ in (10) yields

$$\begin{aligned} \det_{1 \leq i, j \leq N} (e_{\lambda'_i - \mu'_j - i + j}(\mathbf{X}) + [\mu'_j - j < m - 1] e_{\lambda'_i + \mu'_j - i - j - 2m + 2}(\mathbf{X})) \\ = \det_{1 \leq i, j \leq M} (h_{\lambda_i - \mu_j - i + j}(\mathbf{X}) - [j - \mu_j > m] h_{\lambda_i + \mu_j - i - j + 2m}(\mathbf{X})). \end{aligned}$$

The right-hand clearly gives (4c) with $N \mapsto M$. If $m = l(\mu) = \mu'_1$ then the $(1, j)$ -th entry of the determinant on the left-hand side is equal to $e_{\lambda'_i - \mu'_j - i + j}(\mathbf{X})$ which is half of the $(1, j)$ -th entry of (3c). Hence by factoring out $\frac{1}{2}$ from the first column, the left-hand side of the above equation is equal to (3c). \square

In [5], Fulmek and Krattenthaler derive the Jacobi–Trudi formulae for sp_λ and o_λ using Gessel and Viennot’s technique of dual paths, which is, in essence, a combinatorial realisation of the approach of this section.

6. COMBINATORIAL PROOFS OF THE GIAMBELLI FORMULAE

Fulmek and Krattenthaler [5] provided combinatorial proofs of the Giambelli identities for ordinary symplectic and odd orthogonal characters based on Stembridge’s proof of the Giambelli identity for ordinary Schur functions [22]. We adapt their ideas in

order to prove combinatorially the respective skew Giambelli identities analogous to formula (5). By means of a sign-reversing involution similar to the one in Section 4.5, we are also able to provide a combinatorial proof of the Giambelli identity for skew even orthogonal characters.

In Section 3, we obtained the lattice path models (families of (n, m) -symplectic/odd orthogonal/even orthogonal paths) that we used to prove the dual Jacobi–Trudi formulae by reading off the corresponding tableaux columnwise. Ordinary Giambelli-type identities express characters of arbitrary shapes as determinants whose entries are characters of hook shapes. That observation suggests to read off tableaux hookwise.

Consider the Young diagram of shape λ/μ for two partitions $\mu \subseteq \lambda$. The *hook* at position (i, j) consists of the cells

- $(i, j) \cup \{(i, k) : j + 1 \leq k \leq \lambda_i\} \cup \{(k, j) : i + 1 \leq k \leq \lambda'_j\}$ if $(i, j) \in \lambda/\mu$ and
- $\{(i, k) : \mu_i + 1 \leq k \leq \lambda_i\} \cup \{(k, j) : \mu'_j + 1 \leq k \leq \lambda'_j\}$ if $(i, j) \notin \lambda/\mu$.

In other words, the hook at a position (i, j) is the union of the cell (i, j) (the *corner* of the hook) together with the cells to the right of it in the i -th row (the *arm* of the hook) and with the cells below of it in the j -th column (the *leg* of the hook) provided that (i, j) is part of the Young diagram of shape λ/μ . If (i, j) is not an element of the skew shape λ/μ , that is, $(i, j) \notin \lambda/\mu$, then the hook is not connected and we call it *broken*. Figure 11 illustrates the two different types of hooks.

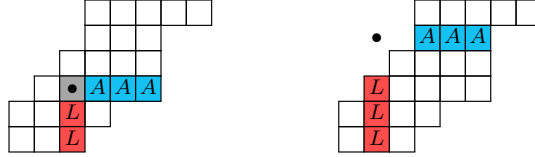


FIGURE 11. A Young diagram of shape $(8, 6, 6, 6, 4, 3)/(3, 3, 2, 1)$ with the hook at position $(4, 3)$ (left) and with the broken hook at position $(2, 2)$ (right) marked. The arms of the hooks are shaded blue with inscribed *A*'s and the legs shaded red with inscribed *L*'s.

Next, we introduce the set-up common to all three considered models for the lattice paths that we obtain by reading off tableaux hookwise; for the specific details of each model see the following subsections. Let $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$ be two partitions $\mu \subseteq \lambda$ with $l(\mu) \leq m$ and $l(\lambda) \leq n + m$. Define $A_i := (-\alpha_i, 2n + 2m - 1)$ and $B_i := (\beta_i + 1, 2n + 2m - \beta_i - 1)$ for $1 \leq i \leq p$ and $C_i := (-\gamma_i, 2m)$ and $D_i := (\delta_i + 1, 2m - \delta_i - 1)$ for $1 \leq i \leq q$.

We will associate lattice paths corresponding to the *principal hooks* of the skew tableaux: the i -th principal hook is the hook at the diagonal position (i, i) . For $q + 1 \leq i \leq p$, the paths from A_i to B_i correspond to the i -th principal hook read from right to left and from top to bottom. For $1 \leq i \leq q$, the i -th principal hook is broken. The paths from A_i to C_i correspond to the arm of the i -th principal hook read from right to left, whereas the paths from D_i to B_i correspond to the leg of the i -th principal hook read from top to bottom.

The lattice paths fulfil the following properties:

- In the region $\{(x, y) \in \mathbb{Z}^2 : x \leq 0\}$, lattice paths stay weakly above the line $x = 2m$ and horizontal unit steps $(i, 2m+j) \rightarrow (i+1, 2m+j)$ get the weight $x_{(j+2)/2}^{-1}$ if j is even and $x_{(j+1)/2}$ if j is odd.
- In the region $\{(x, y) \in \mathbb{Z}^2 : x \geq 0\}$, lattice paths stay weakly above the lines $y = -x + 2m$ and $y = x - 1$ and horizontal unit steps $(i, 2m+j) \rightarrow (i+1, 2m+j)$ are assigned the weight $x_{(i+j+2)/2}^{-1}$ if $i+j$ is even and $x_{(i+j+1)/2}$ if $i+j$ is odd.

All other steps have weight 1, and the weight of a family of paths is the product of the weights of all its steps. Note that this set-up constitutes a combined e - and h -labelling: We have an e -labelling in the region $x > 0$ for the legs of the hooks and an h -labelling in the region $x \leq 0$ for the arms and the corners of the hooks. The exact lattice path model for each of the skew characters is specified in the respective section.

A crucial observation will be the following: For each of the different lattice path models, the families of $p+q$ lattice paths from $\{A_1, \dots, A_p, D_1, \dots, D_q\}$ to $\{B_1, \dots, B_p, C_1, \dots, C_q\}$ are strongly non-intersecting if and only if we have paths from A_i to C_i as well as from D_i to B_i for $1 \leq i \leq q$ and from A_i to B_i for $q+1 \leq i \leq p$. By applying the Lindström–Gessel–Viennot Lemma 4.2 and — in the case of skew even orthogonal characters — by a sign-reversing involution, we will show that

$$(-1)^q \det \begin{pmatrix} (\mathcal{P}(A_i \rightarrow B_j))_{1 \leq i, j \leq p} & (\mathcal{P}(A_i \rightarrow C_j))_{\substack{1 \leq i \leq p, \\ 1 \leq j \leq q}} \\ (\mathcal{P}(D_i \rightarrow B_j))_{\substack{1 \leq i \leq q, \\ 1 \leq j \leq p}} & (\mathcal{P}(D_i \rightarrow C_j))_{1 \leq i, j \leq q} \end{pmatrix} \quad (11)$$

yields the respective Giambelli-type formulae, where $(-1)^q$ is the sign of the permutation

$$\begin{pmatrix} 1 & \cdots & q & q+1 & \cdots & p & p+1 & \cdots & p+q \\ p+1 & \cdots & p+q & q+1 & \cdots & p & 1 & \cdots & q \end{pmatrix}.$$

The sign can be readily computed since the permutation is equal to

$$(1 \ p+1)(2 \ p+2) \cdots (q \ p+q)$$

in cycle notation, which is a product of exactly q transpositions.

In the following three subsections, we prove Theorem 2.3.

6.1. Proof of the skew symplectic Giambelli identity (6a). A skew (n, m) -symplectic tableau is encoded by non-intersecting lattice paths from A_i to C_i and from D_i to B_i for $1 \leq i \leq q$ as well as from A_i to B_i for $q+1 \leq i \leq p$ with step set $\{(1, 0), (0, -1)\}$ in the region $x \leq 0$ and step set $\{(1, 0), (0, 1)\}$ in the region $x \geq 0$ that stay all weakly above the line $y = x - 1$; see Figure 12 for an example. Given a skew shape λ/μ , the generating function $\text{sp}_{\lambda/\mu}^m(\mathbf{x})$ is thus the sum over all families of $p+q$ strongly non-intersecting lattice paths with starting points $\{A_1, \dots, A_p, D_1, \dots, D_q\}$ and endpoints $\{B_1, \dots, B_p, C_1, \dots, C_q\}$ such that A_i is connected to C_i and D_i to B_i for $1 \leq i \leq q$ as well as having A_i connected to B_i for $q+1 \leq i \leq p$. Applying the Lindström–Gessel–Viennot Lemma 4.2 implies that $\text{sp}_{\lambda/\mu}^m(\mathbf{x})$ is given by (11). There are clearly no lattice paths from any D_i to C_j , hence $\mathcal{P}(D_i \rightarrow C_j) = 0$. Moreover, the paths from A_i to C_j correspond to complete homogeneous symmetric functions, that is, $\mathcal{P}(A_i \rightarrow C_j) = h_{\alpha_i - \gamma_j}(\mathbf{x}^\pm) = \text{sp}_{(\alpha_i)/(\gamma_j)}^m(\mathbf{x})$. Regarding the paths from D_i to B_j , we have to keep in mind that no lattice paths are allowed to touch or cross the line $y = x - 2$. Hence, Lemma 4.3 implies that $\mathcal{P}(D_i \rightarrow B_j) = e_{\beta_j - \delta_i}(\mathbf{x}^\pm) - e_{\beta_j + \delta_i - 2m}(\mathbf{x}^\pm)$.

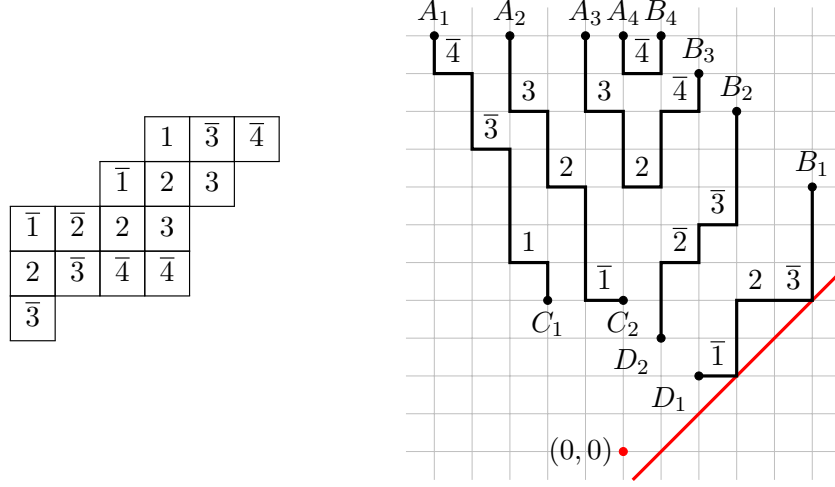


FIGURE 12. A skew $(4,2)$ -symplectic tableau of shape $(5, 3, 1, 0 | 4, 2, 1, 0) / (2, 0 | 1, 0)$ (left) and its associated family of non-intersecting lattice paths (right).

Finally, the paths from A_i to B_j correspond to skew m -symplectic characters indexed by hooks, so $\mathcal{P}(A_i \rightarrow B_j) = \text{sp}_{(\alpha_i | \beta_j)}^m$.

6.2. Proof of the skew odd orthogonal Giambelli identity (6b). In the case of skew m -odd orthogonal characters $\text{so}_{\lambda/\mu}^m$, we consider similar lattice paths as in the previous section with the only addition that we also allow diagonal steps $(1,1)$ along the line $y = x$. These steps correspond to the entries \hat{i} in the $(m+i)$ -th row of the first column of the skew (n,m) -odd orthogonal tableau as before and are equipped with the weight 1. See Figure 13 for an example. As seen before, it suffices to compute the determinantal expression (11) in our setting. The expressions for $\mathcal{P}(D_i \rightarrow C_j)$ and $\mathcal{P}(A_i \rightarrow C_j)$ are the same as in the symplectic case, so $\mathcal{P}(D_i \rightarrow C_j) = 0$ and $\mathcal{P}(A_i \rightarrow C_j) = h_{\alpha_i - \gamma_j}(\mathbf{x}^\pm) = \text{so}_{(\alpha_i)/(\gamma_j)}^m(\mathbf{x})$. The entry $\mathcal{P}(D_i \rightarrow B_j) = e_{\beta_j - \delta_i}(\mathbf{x}^\pm) + e_{\beta_j + \delta_i - 2m + 1}(\mathbf{x}^\pm) = \text{so}_{(1^{\beta_j+1})/(1^{\delta_i+1})}^m(\mathbf{x})$ is a simple consequence of Corollary 4.5, and $\mathcal{P}(A_i \rightarrow B_j)$ corresponds to the skew m -odd orthogonal character $\text{so}_{(\alpha_i | \beta_j)}^m(\mathbf{x})$.

6.3. Proof of the skew even orthogonal Giambelli identity (6c). Now, we consider lattice paths in our general set-up with step set $\{(1,0), (0,-1)\}$ in the region $x \leq 0$ and step set $\{(1,0), (0,1), (2,0)\}$ in the region $x \geq 0$ such that all lattice paths stay weakly above the line $y = x - 1$ and o-horizontal steps are only allowed if they end on the line $y = x$. As before, we interpret the entries of a skew (n,m) -even orthogonal tableau of shape λ/μ hookwise as lattice paths. Yet again, consecutive entries \hat{i} and \hat{i} in the first column are interpreted as an o-horizontal step with weight 1 drawn as an arch. As a result, we obtain a family of $p+q$ strongly non-intersecting lattice paths, where $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$. See Figure 14 for an example. However, not every family of non-intersecting lattice paths in our set-up corresponds to a skew (n,m) -even orthogonal tableau. In fact, the m -even orthogonal

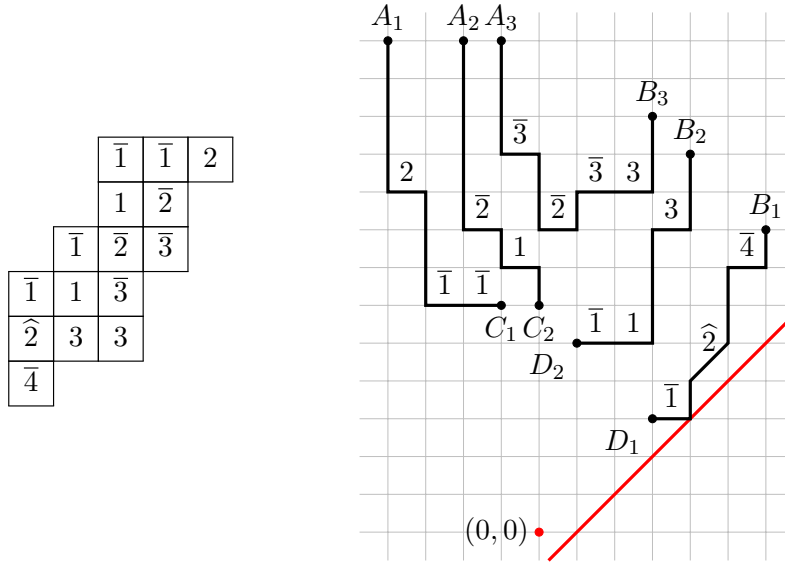


FIGURE 13. A skew $(4, 3)$ -odd orthogonal tableau of shape $(4, 2, 1 | 5, 3, 2) / (1, 0 | 2, 0)$ (left) and its associated family of non-intersecting lattice paths (right).

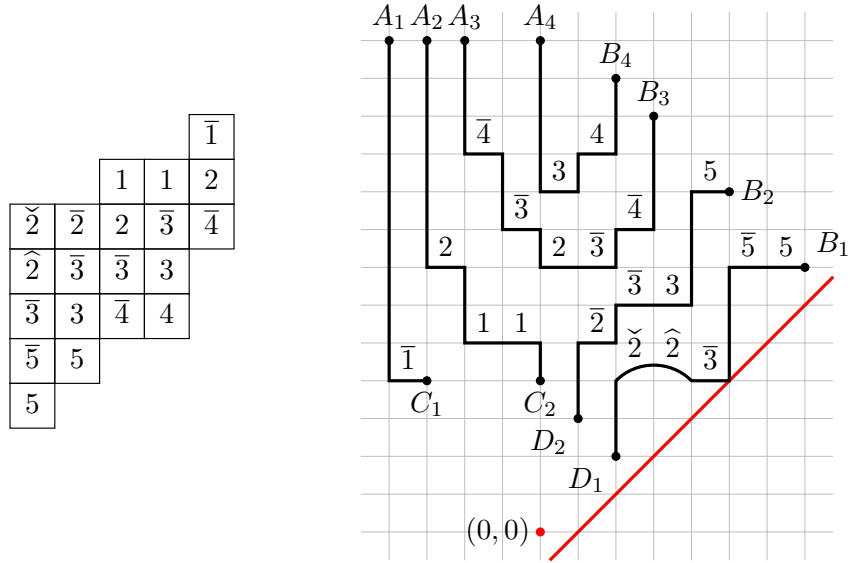


FIGURE 14. A skew $(5, 2)$ -even orthogonal tableau of shape $(4, 3, 2, 0 | 6, 4, 2, 1) / (3, 0 | 1, 0)$ (left) and its associated family of non-intersecting lattice paths (right).

condition implies that the configurations illustrated in Figure 15 cannot occur. To be more precise, assume we have a skew (n, m) -even orthogonal tableau with entry \bar{i} in the first column of the $(m + i)$ -th row and i also appears in the j -th column of that row.

Then the entry in the same column one row above is supposed to be \bar{i} . If that entry at position $(m+i-1, j)$ is not \bar{i} , then we obtain one of the three cases in Figure 15; namely (a) if $j < m+i$, (b) if $j = m+i$ and (c) if $j > m+i$. In particular, the vacancies indicated by the red points imply that the families of lattice paths are not obtained by skew (n, m) -even orthogonal tableaux. Note that case (a) is equivalent to the case of the trapped position in Section 3.4. The reason for the other cases is that the trapped position moves from a region with e -labelling to a region with h -labelling.

The corners of the nested left turns in Figure 15 are each lying on the line $y + x = 2m + 2i - 1$; we say that $2m + 2i - 1$ is the distance between the origin and the trapped positions.

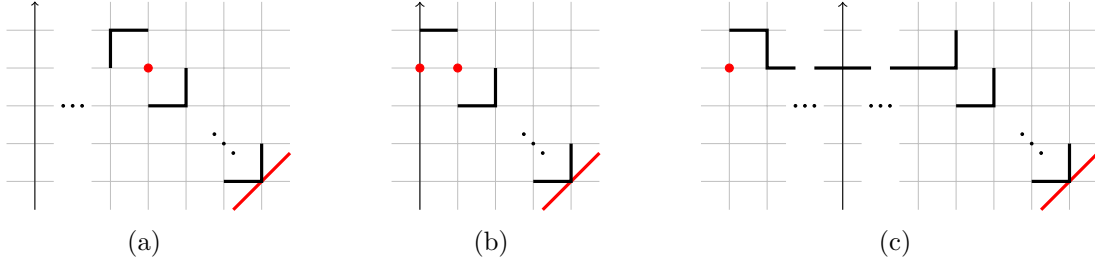


FIGURE 15. Local configurations of trapped positions at $(m+i-1, j)$ if (a) $j < m+i$, (b) $j = m+1$ or (c) $j > m+1$. The large points indicate vacancies.

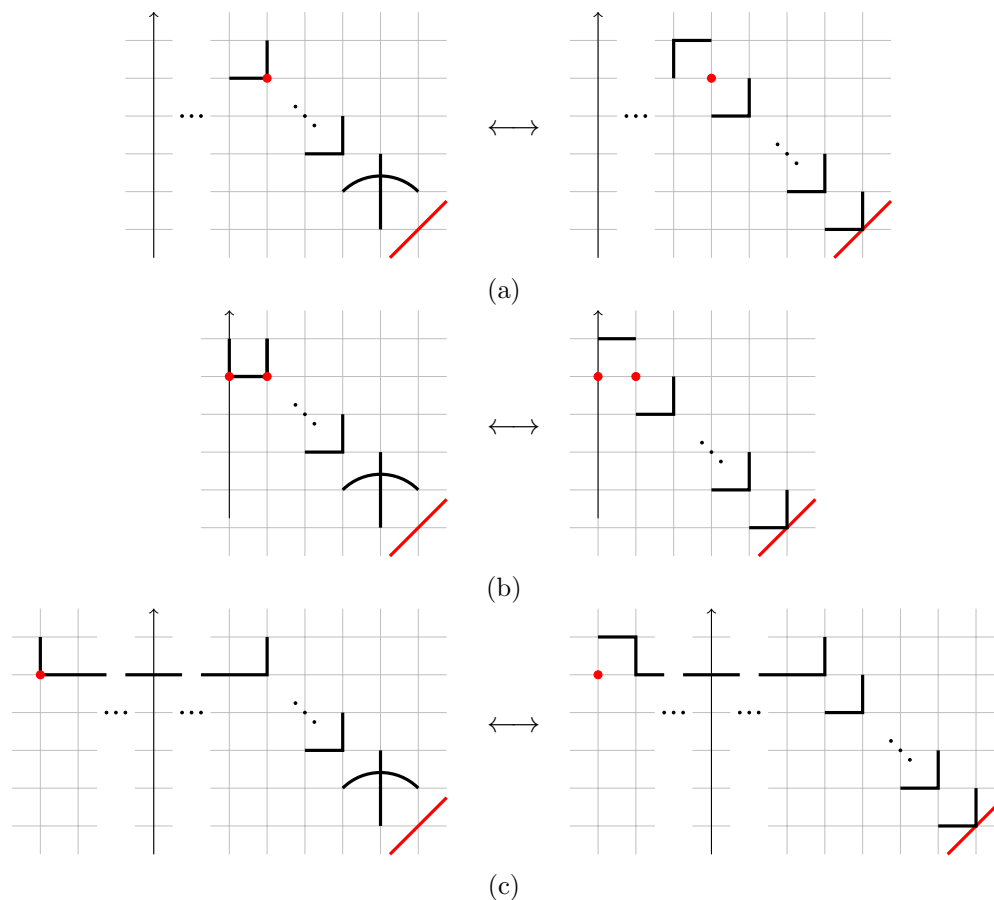
As in the cases before, we want to apply the Lindström–Gessel–Viennot Lemma 4.2. The evaluation of the determinant (11) in the current set-up yields

$$\frac{(-1)^q}{2^{[m=l(\mu) \wedge m \neq 0]}} \det \begin{pmatrix} (o_{(\alpha_i|\beta_j)}^m(\mathbf{x}))_{1 \leq i, j \leq p} & (h_{\alpha_i - \gamma_j}(\mathbf{x}^\pm))_{\substack{1 \leq i \leq p, \\ 1 \leq j \leq q}} \\ (e_{\beta_j - \delta_i}(\mathbf{x}^\pm) + e_{\beta_j + \delta_i - 2m+2}(\mathbf{x}^\pm))_{\substack{1 \leq i \leq q, \\ 1 \leq j \leq p}} & 0 \end{pmatrix},$$

which, by rewriting the entries in terms of the even orthogonal characters, is equal to (6c). However, a priori this is not the generating function of skew (n, m) -even orthogonal tableaux since it also enumerates the trapped positions in Figure 15 as well as families of weakly but not strongly non-intersecting lattice paths with intersections of o-horizontal steps with vertical steps along the line $y = x + 1$. We provide a sign-reversing involution under which these families of lattice paths cancel out, ultimately showing that the above is indeed equal to $o_{\lambda/\mu}^m(\mathbf{x})$.

Consider the families of lattice paths enumerated by (6c) as described above. Assume a given family of lattice paths has intersections involving o-horizontal steps or trapped positions that we have specified above. If an intersection lies on the line $y + x = d$, then we say that d is the distance between the origin and that intersection. We take the unique trapped position or intersection with the smallest distance to the origin and perform the corresponding local changes shown in Figure 16; the rest of the paths are left unchanged. Thus, the weight of the family of lattice paths is exactly changed by a factor of -1 .

This sign-reversing involution cancels all families of lattice paths that contain one of the trapped positions or an intersection of an o-horizontal step with two vertical



steps. We are left with exactly those families of strongly non-intersecting lattice paths that correspond to skew (n, m) -even orthogonal tableaux. This completes the proof of Theorem 2.3.

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