A GENERALIZATION OF CONJUGATION OF INTEGER PARTITIONS

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Abstract. We exhibit, for any positive integer parameter s, an involution on the set of integer partitions of n. These involutions show the joint symmetry of the distributions of the following two statistics. The first counts the number of parts of a partition divisible by s, whereas the second counts the number of cells in the Ferrers diagram of a partition whose leg length is zero and whose arm length has remainder $s-1$ when dividing by s. In particular, for $s=1$ this involution is just conjugation. Additionally, we provide explicit expressions for the bivariate generating functions.

Our primary motivation to construct these involutions is that we know only of two other "natural" bijections on integer partitions of a given size, one of which is the Glaisher–Franklin bijection sending the set of parts divisible by s, each divided by s, to the set of parts occurring at least s times.

1. Introduction

Integer partitions are possibly one of the most important families of objects in combinatorics. However, it seems that we do not know of very many bijections on the set of integer partitions of a given size — although a large variety of bijections between sets of partitions with certain properties can be found in the literature, as witnessed by Pak in his survey [10].

Apart from conjugation of the Ferrers diagram, a well-known family of bijections is due to Glaisher and Franklin, see [10, Sec. 3.3], [7].[1] For a given positive integer s, it sends the set of parts divisible by s, each divided by s, to the set of parts occurring at least s times.

The other family of bijections we know of is due to Loehr and Warrington [9]. For each rational number x, they describe an involution that interchanges two statistics h_x^+ and h_x^-, which count the number of cells in the Ferrers diagram of a partition satisfying certain constraints on the ratio of arm and leg length. These involutions can be combined, for example, to provide a bijection sending the diagonal inversion number to the length of a partition.[2]

The purpose of this article is to present a family of involutions on the set of partitions of a given integer that interchange two statistics r_s and c_s (to be defined in the next section), where s is a positive integer. For $s=1$ we recover the operation of conjugation.

To give an outline, in the next section we recall standard notation and give definitions relevant for our considerations. In particular, there we introduce the announced statistics r_s

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[1] The case $s=2$ is www.findstat.org/Mp00312.

[2] www.findstat.org/Mp00322

and c_s. Subsequently we present our results. Theorem 1 says that there is an involution on partitions of n that interchanges the statistics r_s and c_s. The theorem is actually much finer as it leaves the sequence of the non-zero remainders after division of the parts of the partition by s invariant. Our second main result is presented in Theorem 3. It provides an explicit expression for the generating function $\sum_\lambda q^{|\lambda|}$, where the sum is over all partitions with $(r_s(\lambda), c_s(\lambda)) = (r, c)$ and a fixed sequence of non-zero remainders, with $|\lambda|$ denoting the sum of parts of λ. The symmetry in r and c is evident from the expression, see Remarks 4(1).

Sections 3 and 4 are devoted to the construction of the involution of Theorem 1. It is built up step by step. It is particularly simple if all parts of the partition are divisible by s, see Construction 1 in Section 3.1. The next case that we consider is the case of strictly increasing remainder sequences. The simple idea of Construction 1 is enhanced by the operation of "removal of remainders" (see Section 3.2) and the concept of the "remainder diagram". The result is the more general involution in Construction 2 presented in Section 3.3. In Section 4 it is argued that the general case can be reduced to the case of strictly increasing remainder sequences, see Construction 3. The resulting complete description of our involution, proving Theorem 1, is finally summarized in Construction 4.

Along the way to this involution, we derive in parallel generating function results, see Lemmas 6, 10 and 11, and in particular Theorem 12. (In fact, several ingredients to the involution are inspired by generating function calculations.) We complete the proof of Theorem 3 in Section 5 by simplifying the expression from Theorem 12. We offer actually two proofs of Theorem 3: one uses a combination of combinatorial arguments and q-series identities, the other is purely combinatorial.

The family of involutions we present here, depending on a positive integer s, was discovered by an automated search for equidistributed statistics on integer partitions in www.findstat.org such that there is no accompanying bijection in the database.[3]

2. Definitions and Results

A *partition* λ of a positive integer n is a weakly decreasing sequence of positive integers that add up to n. We write $\lambda \vdash n$ and n is also referred to as the *size* of λ, denoted by $|\lambda|$. The number of parts is the *length* of the partition, denoted by $\ell(\lambda)$. The *Ferrers diagram* of $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is the arrangement of left-justified unit boxes, called *cells*, with λ_i cells in row i. In the following, we often identify the Ferrers diagram with the partition. We use the English convention and matrix coordinates to locate cells in the Ferrers diagram. By λ' we denote the *conjugate partition* of λ, which is obtained by reflecting the Ferrers diagram about the main diagonal. The *leg length* $\operatorname{leg}(z)$ of a cell z in the partition is the number of cells in the same column strictly below the cell, while the *arm length* $\operatorname{arm}(z)$ of a cell is the number of cells in the same row strictly to the right of the cell. The Ferrers diagrams of $\lambda = (6,4,4,1)$ and of $\lambda' = (4,3,3,3,1,1)$ are shown in Figure 1.

The cells that contribute to the leg and arm lengths of the cell $(1,2)$ of the Ferrers diagram of λ are indicated in blue and red, respectively, where the cell in the i-th row and j-th column is referred to as (i,j).

[3]The case $s = 2$ is now www.findstat.org/Mp00321.

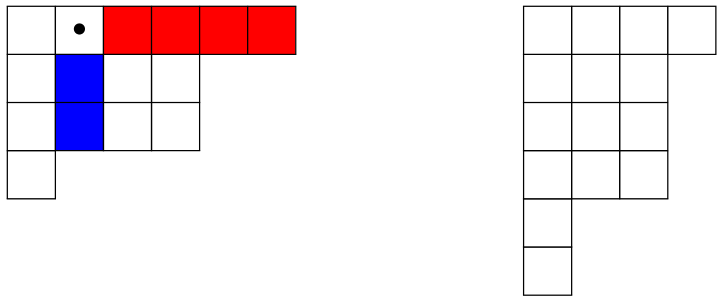

Figure 1. The Ferrers diagrams of $\lambda = (6,4,4,1)$ and $\lambda' = (4,3,3,3,1,1)$

Throughout, we fix a positive integer s. We define the following two statistics on partitions that depend on s. We let[4]

$$r_s(\lambda) = \#\text{ of parts of } \lambda \text{ divisible by } s,$$
$$c_s(\lambda) = \#\text{ of cells } z \text{ in } \lambda \text{ such that } \operatorname{leg}(z) \text{ is zero and } \operatorname{arm}(z)+1 \text{ is divisible by } s.$$

A cell that contributes to $c_s(\lambda)$ is called *s-cell*. For example, given $\lambda = (6,4,4,1)$, the 2-cells are $(1,5)$ and $(3,3)$ and we have $r_2(\lambda) = 3$ and $c_2(\lambda) = 2$.

It is worth pointing out that the statistic $c_s(\lambda)$ occurred earlier in [11, p. 23, bottom], where it is denoted by $\mathrm{BF}_{r,0}(\lambda)$, as a special case of a more general statistic $\mathrm{BF}_{\alpha,\beta}(\lambda)$.

Our main goal is to show that the polynomial

$$\sum_{\lambda \vdash n} R^{r_s(\lambda)} C^{c_s(\lambda)}$$

is symmetric in R and C by constructing an involution on partitions of n that interchanges the statistics r_s and c_s.

We will actually show a vast refinement of this statement. The *remainder sequence* of a partition λ modulo s is the sequence $\boldsymbol{\rho}_s(\lambda) = (\rho_1, \dots, \rho_m)$ of non-zero remainders of the parts of λ when dividing by s and reading λ from left to right. For example, given $\lambda = (12,9,5,4,4,3,2)$, we have $\boldsymbol{\rho}_4(\lambda) = (1,1,3,2)$. Our involution will fix the remainder sequence of the partition. As a consequence, we obtain our first main theorem.

Theorem 1. *Let s and n be positive integers, and let $\boldsymbol{\rho}$ be a vector of integers between 1 and $s-1$. Furthermore, let r and c be non-negative integers. Then the number of partitions λ of n with $\boldsymbol{\rho}_s(\lambda) = \boldsymbol{\rho}$ and $(r_s(\lambda), c_s(\lambda)) = (r,c)$ is equal to the number of partitions λ of n with $\boldsymbol{\rho}_s(\lambda) = \boldsymbol{\rho}$ and $(r_s(\lambda), c_s(\lambda)) = (c,r)$.*

Example 2. We choose $s = 3$. There are exactly 5 partitions λ of 37 with remainder sequence $(2,1,1,2,1)$ and $r_3(\lambda) = 2$ and $c_3(\lambda) = 3$, namely $(15,6,5,4,4,2,1)$, $(15,8,4,4,3,2,1)$, $(14,10,4,3,3,2,1)$, $(17,6,4,4,3,2,1)$, $(14,7,7,3,3,2,1)$. Their Ferrers diagrams are shown in Figure 2. There, the 3-cells are the black cells. Furthermore, blocks of three cells are either white or shaded in order to facilitate the identification of the row lengths that are divisible by $s = 3$.

[4] We use the letter "r" in r_s and the letter "c" in c_s since, clearly, the first statistics is associated with the rows of the Ferrers diagram of the partition, and since we think of the latter statistics to be associated with the columns of the Ferrers diagram.

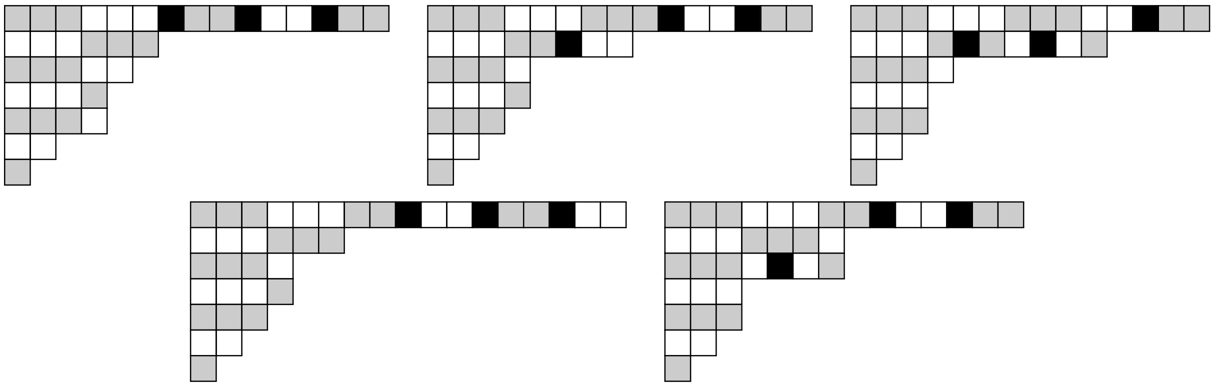

FIGURE 2. The partitions λ of 37 with remainder sequence $(2,1,1,2,1)$, $r_3(\lambda) = 2$ and $c_3(\lambda) = 3$

On the other hand, there are exactly 5 partitions λ of 37 with remainder sequence $(2,1,1,2,1)$ and $r_3(\lambda) = 3$ and $c_3(\lambda) = 2$, namely $(11,10,4,3,3,3,2,1)$, $(12,8,4,4,3,3,2,1)$, $(12,6,5,4,4,3,2,1)$, $(11,7,7,3,3,3,2,1)$, $(14,6,4,4,3,3,2,1)$. Their Ferrers diagrams are shown in Figure 3. The shadings in the figure have the same meaning as in Figure 2.

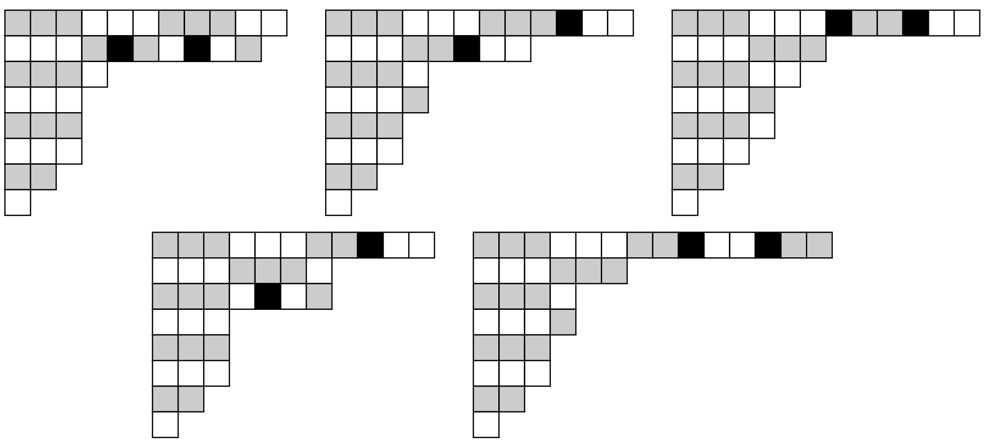

FIGURE 3. The partitions λ of 37 with remainder sequence $(2,1,1,2,1)$, $r_3(\lambda) = 3$ and $c_3(\lambda) = 2$

Apart from a bijective proof we also present a proof by computation. Both proofs imply the following result. In order to state it, recall that the *q-binomial coefficient* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

with

$$[n]_q! = \prod_{i=1}^{n} (1 + q + \cdots + q^{i-1}) = \prod_{i=1}^{n} \frac{1-q^i}{1-q}.$$

We extend the notion of size to finite sequences so that for $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$ we have $|\boldsymbol{\rho}| = \rho_1 + \cdots + \rho_m$. We say that $\boldsymbol{\rho}$ has a *weak descent* at position j if $\rho_j \geq \rho_{j+1}$. Finally, the *weak* *major index* of $\boldsymbol{\rho}$ is the sum of the positions of its weak descents, that is

$$\operatorname{wmaj}(\boldsymbol{\rho}) = \sum_{j:\rho_j \geq \rho_{j+1}} j.$$

This is a special case of the so called "graphical major indices" introduced by Foata and Zeilberger [6] and further investigated by Clarke and Foata [1, 3, 4, 2] as well as Foata and one of the authors [5]. Using the language of the articles by Clarke and Foata, the "weak major index" is the major index defined solely on "large" letters.

Our announced generating function result is the following.

Theorem 3. *Let s be a positive integer, $\boldsymbol{\rho}$ be a vector of integers between 1 and $s-1$ of length m, and r, c be non-negative integers. The generating function with respect to the weight $q^{|\lambda|}$ of partitions λ with $\boldsymbol{\rho}_s(\lambda) = \boldsymbol{\rho}$ and $(r_s(\lambda), c_s(\lambda)) = (r, c)$ is*

$$q^{|\boldsymbol{\rho}|} Q^{-\operatorname{wmaj}(\boldsymbol{\rho}) + \binom{m}{2} + r + c} \left(\begin{bmatrix} r+m-1 \\ m-1 \end{bmatrix}_Q \begin{bmatrix} r+c+m-2 \\ c \end{bmatrix}_Q + Q^{m-1} \begin{bmatrix} r+m \\ m \end{bmatrix}_Q \begin{bmatrix} r+c+m-2 \\ c-1 \end{bmatrix}_Q \right),$$

where $Q = q^s$.

Remarks 4. (1) An alternative way to write the above expression is as

$$q^{|\boldsymbol{\rho}|} Q^{-\operatorname{wmaj}(\boldsymbol{\rho}) + \binom{m}{2} + r + c} \left(\frac{[r+c+m-1]_Q!}{[r]_Q!\,[c]_Q!\,[m-1]_Q!} + Q^{m-1} \frac{[r+c+m-2]_Q!}{[r-1]_Q!\,[c-1]_Q!\,[m]_Q!} \right),$$

from which the symmetry in r and c expressed in Theorem 1 is obvious.

(2) A surprising feature of the formula is that the dependence on the remainder sequence $\boldsymbol{\rho}$ is only in the exponent of q in front of the expression. This "almost-independence" from $\boldsymbol{\rho}$ is explained by the bijection of Construction 3.

Example 5. If we choose $s = 3$, $m = 5$, $\boldsymbol{\rho} = (2, 1, 1, 2, 1)$, and $(r, c) = (2, 3)$ (respectively $(r, c) = (3, 2)$) in the formula of Theorem 3, then we obtain

$$\begin{aligned} &q^7 q^{3\cdot(-7+10+2+3)} \left(\begin{bmatrix} 6 \\ 4 \end{bmatrix}_{q^3} \begin{bmatrix} 8 \\ 3 \end{bmatrix}_{q^3} + q^{3\cdot 4} \begin{bmatrix} 7 \\ 5 \end{bmatrix}_{q^3} \begin{bmatrix} 8 \\ 2 \end{bmatrix}_{q^3} \right) \\ &\qquad = q^{31} \left(1 + 2q^3 + 5q^6 + 9q^9 + 17q^{12} + \cdots + 16q^{66} + 9q^{69} + 5q^{72} + 2q^{75} + q^{78} \right). \end{aligned}$$

In particular, the coefficient of q^{37} in this polynomial equals 5, corresponding to the five partitions for each of $(r, c) = (2, 3)$ and $(r, c) = (3, 2)$ in Example 2.

3. Some special cases

The purpose of this section is to define the involution of Theorem 1 in two simpler cases: first for the case where all parts of the partitions are divisible by s (see Construction 1), and then for the more general case where the non-zero remainders that the parts leave after division by s are in increasing order (see Construction 2). Moreover, we provide the necessary auxiliary results that imply that the constructed mappings are indeed involutions and have the desired properties in relation to the statistics r_s and c_s. Finally, working towards the proof of Theorem 3, we also provide corresponding generating function results, the upshot being Lemma 10.

3.1. **Bijective proof for the case of the empty remainder sequences.** In the special case where the remainder sequence of $\lambda = (\lambda_1, \dots, \lambda_\ell)$ modulo s is empty, each row of the Ferrers diagram can be partitioned into segments of length s. We shrink each of these segments to one cell, i.e., we consider the partition $(\frac{\lambda_1}{s}, \frac{\lambda_2}{s}, \dots, \frac{\lambda_l}{s})$. Then $r_s(\lambda)$ is the number of rows and $c_s(\lambda)$ is the number of columns of the shrunk partition. In this case, conjugation of the shrunk diagram and subsequent expansion of each cell again into a row segment of length s give the involution.

This involution is also the basis for the general case. To describe it formally, we introduce some notation. On the one hand, let

$$\lambda{\downarrow_s} = (\lfloor \lambda_1/s \rfloor, \dots, \lfloor \lambda_\ell/s \rfloor)\,.$$

We call $\lambda{\downarrow_s}$ the *s-reduction* of λ. On the other hand, let

$$\lambda{\uparrow_s} = (s \cdot \lambda_1, \dots, s \cdot \lambda_\ell)$$

and call $\lambda{\uparrow_s}$ the *s-blow-up* of λ.

We have $\lambda{\uparrow_s}{\downarrow_s} = \lambda$ for every partition λ. We also have $\lambda{\downarrow_s}{\uparrow_s} = \lambda$ if and only if λ has empty remainder sequence modulo s. The involution on partitions with empty remainder sequence can now be stated as follows.

Construction 1 (Empty remainder sequence). *Let λ be a partition with empty remainder sequence. We define the mapping*

$$\lambda \mapsto [\lambda{\downarrow_s}]'{\uparrow_s}\,.$$

Our reasoning above demonstrates that Construction 1 is an involution on partitions with empty remainder sequence modulo s that interchanges the statistics r_s and c_s.

The discovery of the general involution, proving Theorem 1, was inspired by generating function considerations that led to a proof of Theorem 3, as indicated throughout the presentation. The preceding construction thus corresponds to the statement of the following lemma. We use the standard notation for Q-shifted factorials,

$$(a;Q)_n := \prod_{i=0}^{n-1} (1 - aQ^i).$$

Lemma 6. *The generating function with respect to the weight $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ of partitions λ with empty remainder sequence is given by*

$$1 + \sum_{k \geq 1} R^k \frac{CQ^k}{(CQ;Q)_k},$$

where $Q = q^s$.

Proof. By shrinking row segments of length s as above, it suffices to compute the generating function of all partitions λ with respect to the weight

$$R^{\#\text{ of rows of }\lambda} C^{\#\text{ of columns of }\lambda} q^{|\lambda|}$$

and then replace q by $Q = q^s$, which takes care of expanding the cells into row segments again.

Now, the "1" in the claimed expression takes care of the empty partition, while

$$R^k \frac{Cq^k}{\prod_{i=1}^{k} (1 - Cq^i)}$$

is the generating function of partitions of length k as is easily seen by considering the conjugate partition λ' of λ and applying elementary combinatorial reasoning. $\square$

3.2. **A crucial operation: removal of the final non-zero remainder.** In the following, we also need to keep track of the positions of parts with a non-zero remainder modulo s in a partition λ. We define the *row position sequence* $\boldsymbol{\gamma}_s(\lambda) = (\gamma_1, \dots, \gamma_m)$ to be the sequence of indices $1 \le \gamma_1 < \cdots < \gamma_m$ such that λ_{γ_j} has non-zero remainder after division by s. Let $\Delta_s\lambda$ be the partition we obtain by deleting the last ρ_m cells in the γ_m-th row of the Ferrers diagram of λ.

Lemma 7. *Let λ be a partition with remainder sequence $\boldsymbol{\rho}_s(\lambda) = (\rho_1, \dots, \rho_m)$ and row position sequence $\boldsymbol{\gamma}_s(\lambda) = (\gamma_1, \dots, \gamma_m)$, $m \ge 1$. Then*

$$c_s(\Delta_s\lambda) = \begin{cases} c_s(\lambda), & \text{if } m = 1 \text{ and } \gamma_1 = 1, \text{ or } \gamma_{m-1} = \gamma_m - 1 \text{ and } \rho_{m-1} \ge \rho_m, \\ c_s(\lambda) + 1, & \text{otherwise.} \end{cases}$$

Proof. First note that the case where $m = \gamma_1 = 1$ is obvious. From now on we tacitly assume that we are not in this case.

Next we observe that the deletion only has an effect on the s-cells in rows $\gamma_m - 1$ and γ_m. However, since the lengths of the rows below row γ_m are all divisible by s, the number of s-cells in row γ_m does not change. If $\gamma_{m-1} < \gamma_m - 1$ then there is one more s-cell in row $\gamma_m - 1$ of $\Delta_s\lambda$ than in the same row of λ. This is still true if $\gamma_{m-1} = \gamma_m - 1$ and $\rho_{m-1} < \rho_m$. However, if $\gamma_{m-1} = \gamma_m - 1$ and $\rho_{m-1} \ge \rho_m$ the number of s-cells does not change. $\square$

Setting $s = 3$, we have $c_s(\Delta_s\lambda) = c_s(\lambda) + 1$ for the partition on the left in Figure 4, and $c_s(\Delta_s\lambda) = c_s(\lambda)$ for the partition on the right. The non-zero remainders are indicated in green.

FIGURE 4. Example partitions for Lemma 7

3.3. **Bijective proof for the case of strictly increasing remainder sequences.** After having understood empty remainder sequences, the next easiest task is to accommodate strictly increasing remainder sequences. The reason is that, in this case, the statistic c_s increases by 1 when successively removing the final non-zero remainders, i.e. $c_s(\Delta_s^i\lambda) = c_s(\Delta_s^{i-1}\lambda) + 1$ for $i = 1, \dots, m$, except for the case when there is just one non-zero remainder left and it is the remainder of the first part of the partition.

Let $\boldsymbol{\gamma}_s(\lambda) = (\gamma_1, \dots, \gamma_m)$ be the row position sequence of λ. The *column position sequence* $\boldsymbol{\gamma}'_s(\lambda) = (\gamma'_1, \dots, \gamma'_m)$ is the sequence $(\lceil \lambda_{\gamma_1}/s \rceil, \dots, \lceil \lambda_{\gamma_m}/s \rceil)$. Informally, these are the column indices corresponding to the removed remainders in $\lambda{\downarrow_s}$. To give an example, let $s = 4$ and let λ be the partition $(4s+1, 4s, 3s+2, 3s, 2s, s+3)$. Its Ferrers diagram is shown in Figure 5 on the left, while its s-reduction is shown on the right (the bullets should be ignored at this

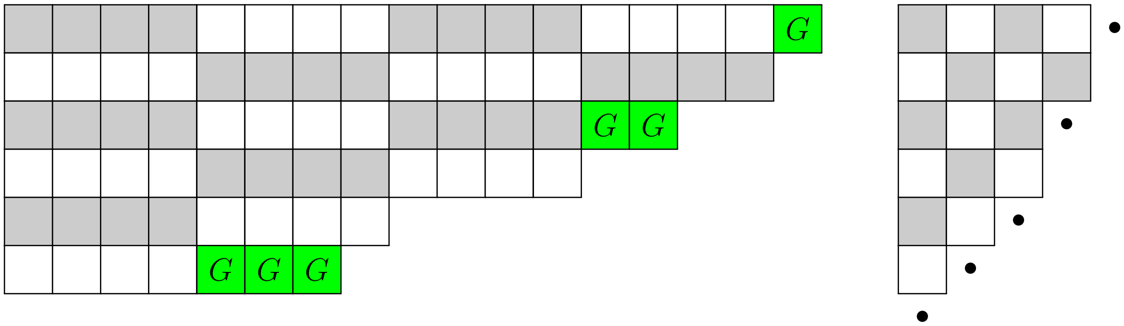

FIGURE 5. A partition of 74 and its 4-reduction

point). In this example, we have $\boldsymbol{\gamma}_s(\lambda) = (1,3,6)$ and $\boldsymbol{\gamma}'_s(\lambda) = (5,4,2)$, and the remainder sequence is $\boldsymbol{\rho}_s(\lambda) = (1,2,3)$ (corresponding to the green cells in Figure 5).

Given a partition λ with strictly increasing remainder sequence, the *green cells* $\mathrm{green}_s(\lambda)$ are defined as the cells $(\gamma_1,\gamma'_1),\dots,(\gamma_m,\gamma'_m)$. In our running example of Figure 5, these are $(1,5)$, $(3,4)$, and $(6,2)$.

Recall that an *outer corner* of a Ferrers diagram λ is a cell z not contained in the diagram such that the union $\lambda \cup z$ is a Ferrers diagram. For example, the outer corners of the Ferrers diagram on the right of Figure 5 are indicated by black dots. Next we show that all green cells are outer corners of the s-reduction.

Proposition 8. *Let λ be a partition with strictly increasing remainder sequence modulo s. Then the cells in $\mathrm{green}_s(\lambda)$ are outer corners of $\lambda{\downarrow_s}$.*

Proof. We have $\lceil \lambda_i/s \rceil = \lfloor \lambda_i/s \rfloor + 1$ if and only if λ_i is not divisible by s, so the cells in $\mathrm{green}_s(\lambda)$ are indeed just outside of $\lambda{\downarrow_s}$. Since the remainder sequence is strictly increasing, the cells in $\mathrm{green}_s(\lambda)$ have distinct column indices. $\square$

The *remainder diagram* $\nu_s^+(\lambda)$ is obtained from the Ferrers diagram of $\lambda{\downarrow_s}$ by adding the green cells, as coloured cells.[5] We call $\lambda{\downarrow_s}$ the *interior* of $\nu_s^+(\lambda)$. Figure 6 displays the remainder diagram $\nu_s^+(\lambda)$ of the partition λ from Figure 5. There, the green cells are marked in green, while the remaining — non-coloured — cells form the interior of $\nu_s^+(\lambda)$.

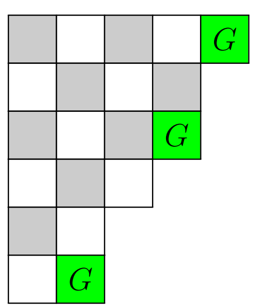

FIGURE 6. The remainder diagram of the partition of Figure 5

Next we show that the statistics r_s and c_s are determined by the remainder diagram.

[5]The concept of the "remainder diagram" has some similarities with parts of the Littlewood-like decomposition of partitions in [11, p. 12], although there does not seem to be a direct overlap.

Lemma 9. *Let λ be a partition with strictly increasing remainder sequence modulo s and remainder diagram $\nu_s^+(\lambda)$. Then*

$$r_s(\lambda) = \#\text{ of rows of } \nu_s^+(\lambda) - |\operatorname{green}_s(\lambda)|,$$
$$c_s(\lambda) = \#\text{ of columns of } \nu_s^+(\lambda) - |\operatorname{green}_s(\lambda)|.$$

Proof. The first equation holds because the green cells correspond to the parts of λ which are not divisible by s.

For a partition λ with empty remainder sequence, we have $|\operatorname{green}_s(\lambda)| = 0$ and $\nu_s^+(\lambda) = \lambda{\downarrow_s}$, and $c_s(\lambda)$ equals the number of columns of $\lambda{\downarrow_s}$. If $m = 1$ and $\gamma_1 = 1$, then $c_s(\lambda)$ also equals the number of columns of $\lambda{\downarrow_s}$. However, $\nu_s^+(\lambda)$ has precisely one more column than $\lambda{\downarrow_s}$. Otherwise, by Lemma 7, each green cell of $\nu_s^+(\lambda)$ reduces the number of cells counted by $c_s(\lambda)$ by one. ☐

The conjugate of a remainder diagram is obtained in the same way as the conjugate of a Ferrers diagram, by reflecting about the main diagonal. Thus, the green cells are at positions $(\gamma_1', \gamma_1), \dots, (\gamma_m', \gamma_m)$ of the conjugate remainder diagram.

Conjugating $\nu_s^+(\lambda)$, then expanding the cells of the interior again into row segments of s cells and putting the remainders back into the green cells, in increasing order from top to bottom, we obtain the involution that swaps the two statistics in this special case.

To write down the bijection formally we need one further definition. Let ν^+ be a partition with m coloured cells that are at the end of their respective rows, and let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)$ be a vector of integers between 1 and $s-1$. Then we define $\nu^+ \leftarrow_s \boldsymbol{\rho}$ to be obtained from the s-blow-up of the interior of ν^+ (that is, of the uncoloured cells) by adding ρ_i cells to the rows corresponding to the coloured cells, in order.

Construction 2 (Strictly increasing remainder sequence)**.** *Let λ be a partition with strictly increasing remainder sequence $\boldsymbol{\rho} = \boldsymbol{\rho}_s(\lambda)$. We define the mapping*

$$\lambda \mapsto \left([\nu_s^+(\lambda)]' \leftarrow_s \boldsymbol{\rho} \right).$$

Our reasoning above demonstrates that Construction 2 is an involution on partitions with strictly increasing remainder sequence modulo s that interchanges r_s and c_s.

We now extend Lemma 6 to the case of strictly increasing remainder sequences.

Lemma 10. *Let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)$ be a vector of integers between 1 and $s-1$ with strictly increasing coordinates. The generating function of partitions λ with $\boldsymbol{\rho}_s(\lambda) = \boldsymbol{\rho}$ with respect to the weight $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ is given by*

$$q^{|\boldsymbol{\rho}|} \sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{|\boldsymbol{\gamma}|-m} \left(R^{\gamma_m - m} + \sum_{k \ge 1} \frac{CQ^k}{(CQ;Q)_k} R^{\max(\gamma_m - m, k-m)} \right),$$

where, as before, $Q = q^s$.

Proof. Let λ be a partition with $\boldsymbol{\rho}_s(\lambda) = \boldsymbol{\rho}$ and $\boldsymbol{\gamma}_s(\lambda) = \boldsymbol{\gamma}$. We modify λ as follows: we delete the last ρ_m cells in row γ_m, i.e., we apply Δ_s to λ, and then delete s cells in each row strictly above row γ_m. By Lemma 7, this does not change the statistic c_s. These deletions are taken into account by the terms q^{ρ_m} and $Q^{\gamma_m - 1}$ in the generating function. We continue in this manner: we delete the last cells ρ_j in row γ_j and delete s cells in each row strictly above row γ_j for $j = m-1, m-2, \dots, 1$. This does not change the statistic c_s and, in total, the deletions are taken into account by the terms $q^{|\boldsymbol{\rho}|}$ and $Q^{|\boldsymbol{\gamma}|-m}$.

We are left with a partition with empty remainder sequence. Suppose k is the length of this partition. The case $k = 0$ is taken care of by the term $R^{\gamma_m - m}$. In the case $k \geq 1$, as can be seen in the proof of Lemma 6, the generating function of such partitions with respect to the weight $C^{c_s(\lambda)} q^{|\lambda|}$ is $\frac{CQ^k}{\prod_{i=1}^k (1 - CQ^i)}$. The number of parts divisible by s in the original partition is $k - m$ if $\gamma_m \leq k$ and $\gamma_m - m$ if $\gamma_m > k$. The assertion follows. $\square$

4. The general case

In this section we provide an algorithm, presented in Construction 3, that affords a reduction of the general case to the case of strictly increasing remainder sequences, the case that we had just discussed in Section 3.3. This leads in particular to the completion of the proof of Theorem 1, with the involution summarized in Construction 4. As already in the previous section, also here we derive in parallel the corresponding generating function results, culminating in Theorem 12, which constitutes the basis for the proof of Theorem 3 in Section 5.

Since it provides the inspiration for the constructions to follow, we start from the generating function side. We show next how the observation from Section 3.2 can be used to generalize Lemma 10 in a straightforward manner to the general case. In order to express the generating function, it is useful to define a 01-sequence $\mathbf{d}(\boldsymbol{\rho}, \boldsymbol{\gamma}) = (d_1, \ldots, d_m)$ of length m, which depends on a vector $\boldsymbol{\rho}$ of length m of integers between 1 and $s-1$ and a strictly increasing sequence of positive integers $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ as follows; later on, γ_j will again be the row of the remainder ρ_j: we set $d_j = 1$ unless $j > 1$, $\rho_{j-1} \geq \rho_j$ and $\gamma_j = \gamma_{j-1} + 1$, in which case we set $d_j = 0$. The motivation for this definition comes from the operation provided in Section 3.2. Note that $d_1 = 1$.

Lemma 11. *Let $\boldsymbol{\rho}$ be a vector of integers between 1 and $s-1$ of length m. The generating function with respect to the weight $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ of partitions λ with remainder sequence $\boldsymbol{\rho}$ is given by*

$$q^{|\boldsymbol{\rho}|} \sum_{1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{\mathbf{d}(\boldsymbol{\rho}, \boldsymbol{\gamma}) \cdot (\boldsymbol{\gamma} - \mathbf{1})} \left(R^{\gamma_m - m} + \sum_{k \geq 1} \frac{CQ^k}{(CQ; Q)_k} \, R^{\max(\gamma_m - m, k - m)} \right),$$

where $Q = q^s$ and $\mathbf{d}(\boldsymbol{\rho}, \boldsymbol{\gamma}) \cdot (\boldsymbol{\gamma} - \mathbf{1})$ denotes the standard inner product of $\mathbf{d}(\boldsymbol{\rho}, \boldsymbol{\gamma})$ and $(\boldsymbol{\gamma} - \mathbf{1}) = (\gamma_1 - 1, \ldots, \gamma_m - 1)$.

Proof. The proof follows essentially the steps from the proof of Lemma 10, except for the following detail: when we delete the last ρ_j cells in row γ_j then we delete s cells in each row strictly above row γ_j if and only if $d_j = 1$. If $d_j = 0$, we do not delete cells above row γ_j. This is because the observation in Lemma 7 on removing non-zero remainders says that the statistic c_s does not change when deleting the last ρ_j cells in row γ_j if and only if $d_j = 0$. $\square$

It turns out that the generating function in Lemma 11 can be simplified.

Theorem 12. *Let $\boldsymbol{\rho}$ be a vector of integers between 1 and $s-1$ of length m. The generating function with respect to the weight $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ of partitions λ with remainder sequence $\boldsymbol{\rho}$ is*

$$q^{|\boldsymbol{\rho}|} Q^{-\operatorname{wmaj}(\boldsymbol{\rho})} \sum_{i \geq m} Q^{\binom{m}{2} + i - m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q \left(R^{i-m} + \sum_{k \geq 1} \frac{CQ^k}{(CQ; Q)_k} R^{\max(i-m, k-m)} \right)$$

where $Q = q^s$.

The theorem follows from Lemma 11, the observation that, for fixed γ_m, we have

$$\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{|\boldsymbol{\gamma}|-m} = Q^{\binom{m}{2}+\gamma_m-m} \begin{bmatrix} \gamma_m - 1 \\ m-1 \end{bmatrix}_Q, \tag{4.1}$$

and from Lemma 13 below. Equation (4.1) holds since $\left[{n+m \atop m}\right]_q$ is the generating function $\sum_\lambda q^{|\lambda|}$ of partitions λ of length at most m and parts no greater than n, and since

$$\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{|\boldsymbol{\gamma}|-m} = Q^{\gamma_m - m + 1 + 2 + \cdots + m-1} \sum_{0 \le \gamma_1^- \le \gamma_2^- \le \cdots \le \gamma_{m-1}^- \le \gamma_m - m} Q^{\gamma_1^- + \cdots + \gamma_{m-1}^-},$$

by the transformation $\gamma_k^- = \gamma_k - k$.

Lemma 13. *Let $\boldsymbol{\rho}$ be a vector of integers between 1 and $s-1$ of length m. Then, for fixed γ_m, we have*

$$\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{\mathbf{d}(\boldsymbol{\rho},\boldsymbol{\gamma})\cdot(\boldsymbol{\gamma}-\mathbf{1})} = Q^{-\operatorname{wmaj}(\boldsymbol{\rho})} \sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{|\boldsymbol{\gamma}|-m},$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$.

Proof. We need the following generalization of the weak major index: for k with $1 \le k < m$, we define

$$\operatorname{wmaj}_k(\boldsymbol{\rho}) = \sum_{\substack{j:\rho_j \ge \rho_{j+1} \\ j \le k}} j.$$

This simply is the weak major index of the tuple $\boldsymbol{\rho}$ cut off after the $(k+1)$-st entry. Note that $\operatorname{wmaj}_{m-1} = \operatorname{wmaj}$ for sequences of length m.

The proof is by induction on m. For the start of the induction we note that for $m = 1$ the statement is obvious.

By the induction hypothesis, we may assume

$$\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{\mathbf{d}(\boldsymbol{\rho},\boldsymbol{\gamma})\cdot(\boldsymbol{\gamma}-\mathbf{1})} = Q^{-\operatorname{wmaj}_{m-2}(\boldsymbol{\rho})} \sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{\sum_{j=1}^{m-1}(\gamma_j-1)+d_m(\gamma_m-1)}.$$

If $\rho_{m-1} < \rho_m$, then $d_m = 1$ and $\operatorname{wmaj}(\boldsymbol{\rho}) = \operatorname{wmaj}_{m-2}(\boldsymbol{\rho})$, and the assertion follows in this case. If, on the other hand, we have $\rho_{m-1} \ge \rho_m$, then $\operatorname{wmaj}(\boldsymbol{\rho}) = \operatorname{wmaj}_{m-2}(\boldsymbol{\rho}) + m - 1$, and, by the definition of d_m, we have

$$\begin{aligned} &\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} < \gamma_m} Q^{\sum_{j=1}^{m-1}(\gamma_j-1)+d_m(\gamma_m-1)} \\ &\qquad = \sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} < \gamma_m - 1} Q^{|\boldsymbol{\gamma}|-m} + \sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{m-2} < \gamma_m - 1} Q^{\sum_{j=1}^{m-2}(\gamma_j-1)+\gamma_m-2}. \end{aligned} \tag{4.2}$$

We need to show that this is equal to

$$Q^{-m+1} \sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_m} Q^{|\boldsymbol{\gamma}|-m}.$$

We provide a combinatorial proof. First note that the first term in the second line of (4.2) can be transformed as follows:

$$\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} < \gamma_m - 1} Q^{|\boldsymbol{\gamma}|-m} = Q^{-m+1} \sum_{2 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} < \gamma_m} Q^{|\boldsymbol{\gamma}|-m}.$$

Here we have used the transformation $\gamma_i \to \gamma_i - 1$ for $i \in \{1, 2, \dots, m-1\}$. The second term in the second line of (4.2) is

$$\sum_{1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{m-2} < \gamma_m - 1} Q^{\sum_{j=1}^{m-2}(\gamma_j - 1) + \gamma_m - 2} = Q^{-m+1} \sum_{1 = \gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} < \gamma_m} Q^{|\gamma| - m},$$

where we have used the transformation $\gamma_i \to \gamma_{i+1} - 1$ for $i \in \{1, 2, \dots, m-2\}$ and have set $\gamma_1 = 1$. This completes the proof. ☐

We will now use the combinatorial proof of the previous lemma to provide the missing piece of our bijection. More concretely, the combinatorial argument allows us to reduce everything to the essence of Construction 2.

We extend the notion of the remainder diagram to the case of arbitrary remainder sequences as follows. To explain it, consider the example for $s = 3$ in Figure 7.

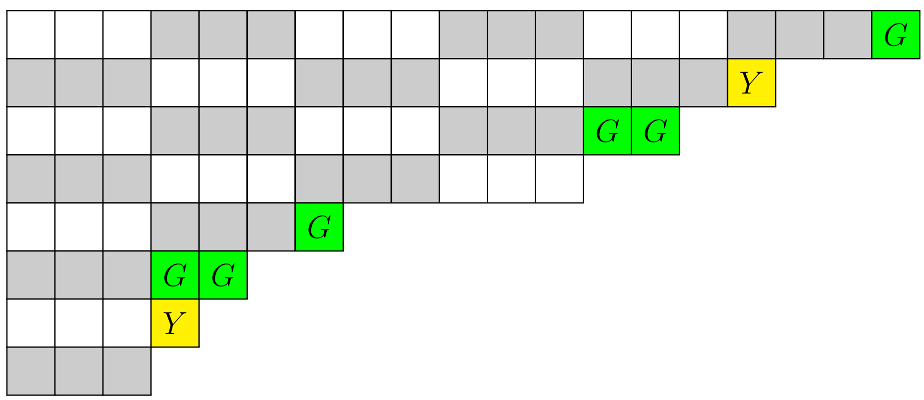

FIGURE 7. Green and yellow remainders in a partition

Consider the i-th remainder from the bottom (!), $i \geq 1$. This remainder is marked green if $c_s(\Delta_s^i \lambda) = c_s(\Delta_s^{i-1} \lambda) + 1$, and it is marked yellow if $c_s(\Delta_s^i \lambda) = c_s(\Delta_s^{i-1} \lambda)$ (cf. Lemma 7). The only exception from this rule is a non-zero remainder in the top row, which is always marked green; see Figure 7.

For a partition λ let, as before, $\mathrm{green}_s(\lambda)$ be the set of green cells that correspond to the green remainders, and let $\mathrm{yellow}_s(\lambda)$ be the set of yellow cells that correspond to the yellow remainders. Yellow cells are also located outside of the s-reduction; in their row, they are adjacent to the final cell of the s-reduction, however, they need not be outer corners of the s-reduction. In the following, we sometimes refer to the green and the yellow cells as the coloured cells.

The Ferrers diagram of $\nu = \lambda{\downarrow_s}$ together with the yellow and green cells is the *(extended) remainder diagram $\nu_s^+(\lambda)$ for λ*. For the example above, the remainder diagram is shown in Figure 8.

More generally, an *(extended) remainder diagram* ν^+ is a partition ν together with a collection of green cells $\mathrm{green}(\nu^+)$ and a collection of yellow cells $\mathrm{yellow}(\nu^+)$, none of them in ν, provided the following three conditions are met:

- Green cells are outer corners of ν.
- Yellow cells are located at the end of a (possibly empty) row of ν.

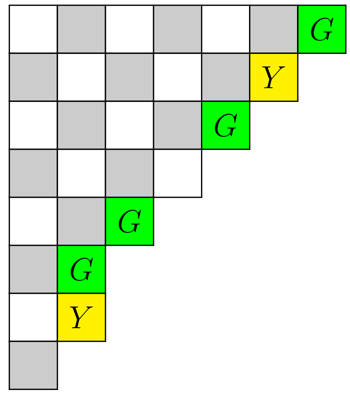

FIGURE 8. The (extended) remainder diagram for the partition in Figure 7

- The cell at the end of the row preceding a row with a yellow cell is always a coloured cell of ν (cf. the first case in Lemma 7). In particular, a coloured cell in the top row must be green.

A remainder diagram with coloured cells in rows $\boldsymbol{\gamma}$ is *compatible* with a vector $\boldsymbol{\rho}$ of integers between 1 and $s-1$ provided that for any weak descent $\rho_{k-1} \geq \rho_k$ of $\boldsymbol{\rho}$ the coloured cell in row γ_k is yellow if and only if $\gamma_{k-1} = \gamma_k - 1$.

We can now express the statistics r_s and c_s in terms of the (extended) remainder diagram, thus generalizing Lemma 9.

Lemma 14. *Let λ be a partition with remainder diagram $\nu_s^+(\lambda)$. Then*

$$\begin{aligned} r_s(\lambda) &= \#\ \textit{of rows of}\ \nu_s^+(\lambda) - |\operatorname{green}_s(\lambda)| - |\operatorname{yellow}_s(\lambda)|, \\ c_s(\lambda) &= \#\ \textit{of columns of}\ \nu_s^+(\lambda) - |\operatorname{green}_s(\lambda)|. \end{aligned}$$

Proof. The proof of Lemma 9 carries over verbatim. □

Next we describe a bijection between remainder diagrams compatible with a given remainder sequence and remainder diagrams without yellow cells. To state it precisely, inspired by Lemma 14 we define for any remainder diagram ν^+ the two statistics

$$\begin{aligned} r(\nu^+) &= \#\ \text{of rows of}\ \nu^+ - |\operatorname{green}(\nu^+)| - |\operatorname{yellow}(\nu^+)| \quad \text{and} \\ c(\nu^+) &= \#\ \text{of columns of}\ \nu^+ - |\operatorname{green}(\nu^+)|. \end{aligned}$$

The following construction is a translation of the combinatorial proof of Lemma 13.

Construction 3 (REDUCTION TO REMAINDER DIAGRAMS WITHOUT YELLOW CELLS)**.** *Let λ be a partition with remainder sequence $\boldsymbol{\rho} = \boldsymbol{\rho}_s(\lambda)$ and remainder diagram $\nu_s^+(\lambda)$.*

(1) *Initialization: We let $k := 1$, $\nu^+ := \nu_s^+(\lambda)$, and $\nu := \lambda{\downarrow_s}$.*
(2) *If k equals the length of $\boldsymbol{\rho}$ then go to (4). If not and if there is a weak descent of $\boldsymbol{\rho}$ at k, i.e., if $\rho_k \geq \rho_{k+1}$, then go to (3). Otherwise increase k by 1 and repeat (2) with this new value of k.*
(3) *By construction, all coloured cells strictly above row γ_{k+1} in the diagram ν^+ are already green and thus outer corners of ν.*
   (3A) *If the coloured cell in row γ_{k+1} is green, then the next green cell above is not in row $\gamma_{k+1}-1$, i.e., $\gamma_k < \gamma_{k+1}-1$ (cf. the definition of the (extended) remainder*

diagram and Lemma 7). We add the outer corners of ν in rows $\gamma_1, \dots, \gamma_k$ to ν and add for each of them a green cell to ν^+ in the row below.

(3B) *If the coloured cell in row γ_{k+1} is yellow, then we delete this coloured cell from ν^+. In this case, the next coloured cell above is in row $\gamma_{k+1} - 1$, i.e., $\gamma_k = \gamma_{k+1} - 1$, and all coloured cells above row γ_{k+1} are outer corners. We add the (coloured) outer corners in rows $\gamma_1, \dots, \gamma_k$ to ν and add for each of them a green cell to ν^+ in the row below*[6]*. Finally, we add a green cell to ν^+ in the first row.*

Increase k by 1 and go to (2).

(4) *The output of the algorithm is the remainder diagram ν^+ with interior ν.*

We illustrate this construction with the help of the example in Figure 7 with remainder diagram in Figure 8. In this case, the row position sequence is $(1, 2, 3, 5, 6, 7)$ and the remainder sequence is $(1, 1, 2, 1, 2, 1)$. Hence we have weak descents of the remainder sequence at $k = 1$, 3, and 5. The sequence of pairs (ν^+, ν) we obtain when applying the algorithm of Construction 3 is shown in Figure 9. There, the white and shaded cells form the partitions ν, while the complete diagrams — including the green and yellow cells — form the partitions ν^+. Note that the final remainder diagram is not compatible with the original remainder sequence.

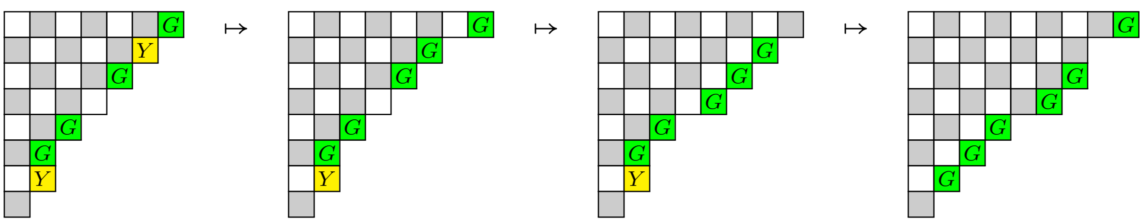

FIGURE 9. Application of Construction 3

The following lemma confirms that Construction 3 has all the required properties such that it indeed achieves the desired reduction to the case of remainder diagrams without yellow cells.

Lemma 15. *Given a partition λ with remainder sequence $\boldsymbol{\rho} = \boldsymbol{\rho}_s(\lambda)$ and remainder diagram $\nu^+(\lambda)$, the algorithm of Construction 3 constructs a remainder diagram ν^+ without yellow cells, whose interior is by $\operatorname{wmaj}(\boldsymbol{\rho})$ cells larger than $\lambda\!\downarrow_s$ and such that $r(\nu^+) = r_s(\lambda)$ and $c(\nu^+) = c_s(\lambda)$.*

Proof. To see that the interior of ν has increased by $\operatorname{wmaj}(\boldsymbol{\rho})$, note that in both (3A) and (3B) we add k cells to the interior of the remainder diagram, which is precisely the contribution of the weak descent at position k to the weak major index of $\boldsymbol{\rho}$.

To see that $r(\nu^+) = r_s(\lambda)$ and $c(\nu^+) = c_s(\lambda)$, note that the total number of cells which are either green or yellow and also the number of rows do not change. In a step (3A), the number of columns does not change either. In a step (3B), the number of columns increases by one, whereas the number of yellow cells decreases by one.

Each step of the construction is invertible, since we can determine from the image in which of steps (3A) or (3B) we were: there is a green cell in the first row of the image if and only if the coloured cell in row γ_{k+1} in the preimage is yellow. $\square$

[6]Note that this has the effect that the former yellow cell in row γ_{k+1} is replaced by a green cell.

In the example in Figure 9, applying the inverse of Construction 3 to the conjugate of the last diagram with respect to the remainder sequence $(1, 1, 2, 1, 2, 1)$, we obtain the sequence of diagrams in Figure 10.

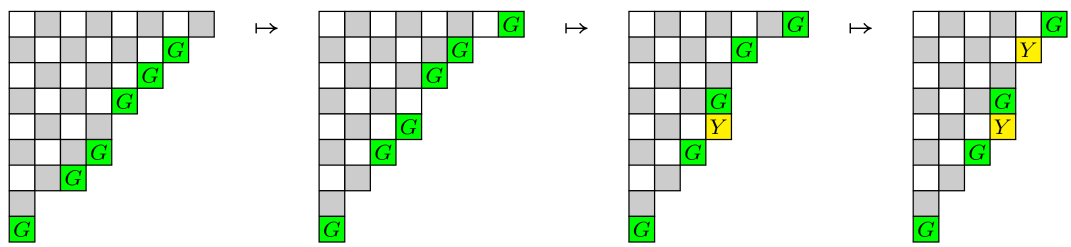

FIGURE 10. Application of the inverse of Construction 3

We can now put together the preceding constructions to obtain a bijection for the general case.

Construction 4. *Let λ be a partition and $\boldsymbol{\rho} = \boldsymbol{\rho}_s(\lambda)$ its remainder sequence.*

- *We apply Construction 3 to $\nu_s^+(\lambda)$ with respect to $\boldsymbol{\rho}$ to obtain a remainder diagram μ^+ without yellow cells.*
- *We apply the inverse of Construction 3 to $[\mu^+]'$ with respect to $\boldsymbol{\rho}$ to obtain a remainder diagram κ^+.*
- *We transform κ^+ into a partition by applying the s-blow-up to the interior of the diagram and replacing the coloured cells by the remainders of the original partition to obtain $\kappa^+ \leftarrow_s \boldsymbol{\rho}$ (compare with Construction 2).*

Note that the resulting partition is again compatible with $\boldsymbol{\rho}$ by construction.

Example 16. We apply Construction 4 to the partition in Figure 7. Its remainder diagram is displayed in Figure 8. The application of Construction 3 to the remainder diagram is performed in Figure 9. Let μ^+ denote the resulting remainder diagram. The application of the inverse of Construction 3 to $[\mu^+]'$ is performed in Figure 10. After applying the s-blow-up to the interior of the diagram and putting the remainders back, we obtain the partition in Figure 11.

Remark 17. The partitions of 37 in Example 2 appear in the order as "dictated" by the involution in Construction 4. To be precise, if one applies Construction 4 to the partitions in Figure 2 then the output partitions are the ones in Figure 3, in the given order.

5. Proof of Theorem 3

By Theorem 1, the generating functions in Lemmas 6, 10 and 11 and Theorem 12 are all symmetric in R and C, however this is not visible from the formulas. We transform the formula in Theorem 12 and extract the coefficient of $R^r C^c$ to obtain a form where the symmetry in R and C is obvious. The result is stated in Theorem 3 (with the symmetric rewriting of the formula given in Remarks 4(1)), and the content of this section is its proof. We start with a proof by computation and provide a combinatorial proof afterwards.

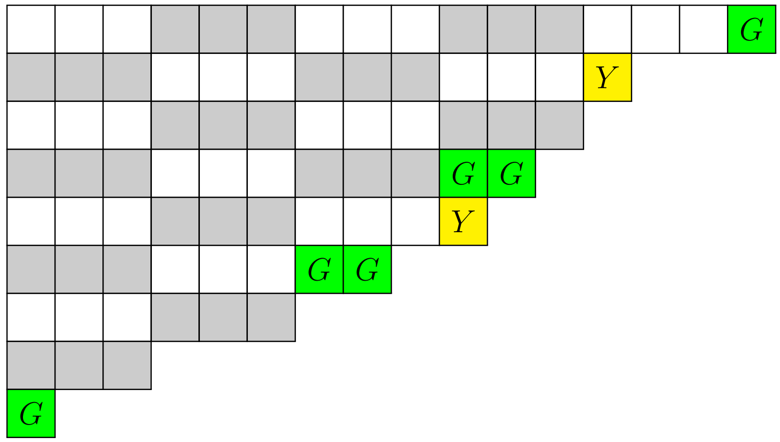

FIGURE 11. Partition obtained after applying the bijection summarized in Construction 4 to the partition in Figure 7

First proof. First note that we can extend the sum over i in the formula in Theorem 12 over all $i \geq 0$ since $\left[{i-1 \atop m-1} \right]_Q = 0$ if $0 \leq i < m$. We neglect the prefactor $q^{|\boldsymbol{\rho}|} Q^{-\operatorname{wmaj}(\boldsymbol{\rho})+\binom{m}{2}}$ in the formula since it is independent of R and C, and we start by decomposing the sum over k in the formula in Theorem 12 to get rid of the maximum as

$$\sum_{i\geq 0} Q^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q \left(R^{i-m} + \sum_{k\geq 1} \frac{CQ^k}{(CQ;Q)_k} R^{\max(i-m,k-m)} \right)$$
$$= \sum_{i\geq 0} (RQ)^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q + \sum_{k\geq 1} \frac{CQ^k}{(CQ;Q)_k} \sum_{i>k} (RQ)^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q$$
$$+ \sum_{k\geq 1} \frac{CQ^k R^{k-m}}{(CQ;Q)_k} \sum_{i=0}^{k} Q^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q. \quad (5.1)$$

We rewrite the first term as

$$\sum_{i\geq m} (RQ)^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q = \sum_{i\geq m} (RQ)^{i-m} \frac{(Q^m;Q)_{i-m}}{(Q;Q)_{i-m}} = \sum_{i\geq 0} (RQ)^i \frac{(Q^m;Q)_i}{(Q;Q)_i}.$$

By the Q-binomial theorem (cf. [8, Eq. (1.3.2); Appendix (II.3)]) the last sum evaluates to

$$\frac{(RQ^{m+1};Q)_\infty}{(RQ;Q)_\infty} = \frac{1}{(RQ;Q)_m}.$$

Thus, we arrive at the expression

$$\frac{1}{(RQ;Q)_m} + \sum_{k\geq 1} \frac{CQ^k}{\prod_{i=1}^{k}(1-CQ^i)} \sum_{i>k} (RQ)^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q$$
$$+ \sum_{k\geq 1} \frac{CQ^k R^{k-m}}{\prod_{i=1}^{k}(1-CQ^i)} \sum_{i=0}^{k} Q^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q. \quad (5.2)$$

Next we extract the coefficient of $R^r C^c$. In order to do so, we will make use of the simple expansion

$$\frac{1}{(z;Q)_k} = \sum_{l\ge 0} \begin{bmatrix} l+k-1 \\ l \end{bmatrix}_Q z^l.$$

We start with the case $c=0$. Making use of the expansion above, we see that the coefficient of R^r in (5.2) is

$$Q^r \begin{bmatrix} r+m-1 \\ r \end{bmatrix}_Q . \tag{5.3}$$

If we let $c\ge 1$, making again use of the expansion above, we see that the coefficient of $R^r C^c$ in (5.2) equals

$$Q^r \begin{bmatrix} r+m-1 \\ m-1 \end{bmatrix}_Q \sum_{k=1}^{r+m-1} Q^{k+c-1} \begin{bmatrix} c+k-2 \\ c-1 \end{bmatrix}_Q + Q^{r+m+c-1} \begin{bmatrix} r+c+m-2 \\ c-1 \end{bmatrix}_Q \sum_{i=m}^{r+m} Q^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q . \tag{5.4}$$

Using the simple summation

$$\sum_{k=1}^{N} Q^{k-1} \begin{bmatrix} M+k-1 \\ M \end{bmatrix}_Q = \begin{bmatrix} M+N \\ M+1 \end{bmatrix}_Q ,$$

the first expression in (5.4) can be evaluated to

$$Q^r \begin{bmatrix} r+m-1 \\ m-1 \end{bmatrix}_Q \sum_{k=1}^{r+m-1} Q^{k+c-1} \begin{bmatrix} c+k-2 \\ c-1 \end{bmatrix}_Q = Q^{r+c} \begin{bmatrix} r+m-1 \\ m-1 \end{bmatrix}_Q \begin{bmatrix} r+c+m-2 \\ c \end{bmatrix}_Q .$$

In the second expression in (5.4) we reverse the order of summation in the sum over i and then rewrite the resulting sum in basic hypergeometric notation (cf. [8]). In this manner we obtain

$$\begin{aligned}\sum_{i=m}^{r+m} Q^{i-m} \begin{bmatrix} i-1 \\ m-1 \end{bmatrix}_Q = \frac{1}{(Q;Q)_{m-1}} \sum_{j=0}^{r} Q^j (Q^{j+1};Q)_{m-1} &= \frac{1}{(Q;Q)_{m-1}} \sum_{j=0}^{r} Q^{r-j} (Q^{r-j+1};Q)_{m-1} \\ &= \frac{Q^r (Q^{r+1};Q)_{m-1}}{(Q;Q)_{m-1}} {}_2\phi_1\!\left[\begin{matrix} Q, Q^{-r} \\ Q^{1-m-r} \end{matrix}; Q, Q^{-m}\right].\end{aligned}$$

This ${}_2\phi_1$-series can be evaluated by means of the Q-Chu–Vandermonde summation (cf. [8, Eq. (1.5.2); Appendix (II.7)]), which reads

$${}_2\phi_1\!\left[\begin{matrix} a, Q^{-n} \\ c \end{matrix}; Q, \frac{cQ^n}{a}\right] = \frac{(c/a;Q)_n}{(c;Q)_n},$$

provided n is a non-negative integer. After some simplification, we obtain

$$\frac{Q^r (Q^{r+1};Q)_{m-1}}{(Q;Q)_{m-1}} {}_2\phi_1\!\left[\begin{matrix} Q, Q^{-r} \\ Q^{1-m-r} \end{matrix}; Q, Q^{-m}\right] = \begin{bmatrix} r+m \\ m \end{bmatrix}_Q .$$

Putting everything back together, we see that (5.4) equals

$$Q^{r+c} \begin{bmatrix} r+m-1 \\ m-1 \end{bmatrix}_Q \begin{bmatrix} r+c+m-2 \\ c \end{bmatrix}_Q + Q^{r+c+m-1} \begin{bmatrix} r+c+m-2 \\ c-1 \end{bmatrix}_Q \begin{bmatrix} r+m \\ m \end{bmatrix}_Q ,$$

which, aside from the neglected prefactor $q^{|\boldsymbol{\rho}|} Q^{-\operatorname{wmaj}(\boldsymbol{\rho})+\binom{m}{2}}$, is exactly the expression in Theorem 3. $\square$

Second proof. By Construction 3, it suffices to consider the case $\operatorname{wmaj}(\boldsymbol{\rho}) = 0$. We need to show that the generating function of partitions λ with strictly increasing remainder sequence $\boldsymbol{\rho}$ modulo s of length m and $(r_s(\lambda), c_s(\lambda)) = (r, c)$ is

$$q^{|\boldsymbol{\rho}|}\Bigg(Q^{0+1+\cdots+m-2} Q^{r+c+m-1} \begin{bmatrix} r+m-1 \\ m-1 \end{bmatrix}_Q \begin{bmatrix} r+c+m-2 \\ c \end{bmatrix}_Q \\ + Q^{0+1+\cdots+m-1} Q^{r+m+c-1} \begin{bmatrix} r+c+m-2 \\ c-1 \end{bmatrix}_Q \begin{bmatrix} r+m \\ m \end{bmatrix}_Q \Bigg). \tag{5.5}$$

It is useful to think in terms of remainder diagrams. We need to show that the generating function with respect to the weight $Q^{\#\text{ of interior cells}}$ of remainder diagrams without yellow cells, where the number of rows is $r+m$ and the number of columns is $c+m$ (including rows and columns of green cells), is equal to the previous expression when neglecting $q^{|\boldsymbol{\rho}|}$.

We distinguish between two cases.

Case 1: the bottom row of the remainder diagram contains an interior cell. We claim that this case is covered by the second summand in (5.5). To see this, consider $\lambda{\downarrow_s}$, and let us decompose it as follows. First we cut out the columns of the green cells in $\lambda{\downarrow_s}$. This gives us m columns of distinct lengths where the largest column has at most $r+m-1$ boxes, the smallest one being allowed to be empty, since $\lambda{\downarrow_s}$ has $r+m$ rows and no green cell is below the last row of $\lambda{\downarrow_s}$.

We illustrate this with the example in Figure 5, with the remainder diagram given in Figure 6. Note that we have $m = 3, c = 2, r = 3$ in this case. In Figure 12, the m columns of distinct lengths appear as the first shape on the right-hand side. The corresponding generating function is $Q^{0+1+\cdots+m-1} \begin{bmatrix} r+m \\ m \end{bmatrix}_Q$, where here and in the following the colours of the expressions hint at the corresponding parts of the Ferrers diagrams in the figures.

Next we cut off the outer frame of the remaining partition, that is, all boxes of the first row and first column. This gives us $r+m-1+c$ boxes; compare with the second shape on the right-hand side of Figure 12. These boxes are taken into account by $Q^{r+c+m-1}$. What remains is a partition with at most $c-1$ columns of size at most $r+m-1$, as in the final shape in Figure 12. The corresponding generating function is given by the remaining factor $\begin{bmatrix} r+c+m-2 \\ c-1 \end{bmatrix}_Q$.

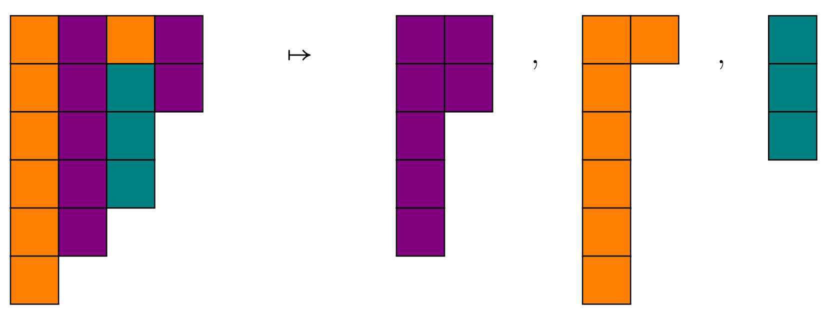

Figure 12. The decomposition of the interior of the remainder diagram of Figure 5 described in Case 1 of the second proof of Theorem 3

Case 2: the bottom row of the remainder diagram consists only of a green cell. There are $m-1$ green cells that are in the last row of the s-reduced diagram or above.

We claim that this case is covered by the first summand in (5.5). The argument is analogous to the one above. Again we consider the s-reduced diagram and decompose it as follows. We start by cutting out the $m-1$ columns of the green cells different from the bottommost green cell in $\lambda{\downarrow_s}$. This gives us $m-1$ columns of different lengths, where the largest has length at most $r+m-2$, since λ has $r+m-1$ rows and we only consider the green cells different from the bottommost green cell.

We illustrate this with the example in Figure 13, with the remainder diagram given in Figure 14. Note that we have $m=2, c=3, r=5$ in this case. In Figure 15, these $m-1$ columns of different lengths appear as the first shape on the right-hand side. Here the generating function is $Q^{0+1+\cdots+m-2}\left[{r+m-1 \atop m-1}\right]_Q$. From the remaining partition we cut off the outer frame of $r+m-2+(c+1)$ boxes; compare with the second shape on the right-hand side of Figure 15. These boxes are taken into account by the factor $Q^{r+c+m-1}$. What remains is a partition with at most c columns of length at most $r+m-2$, as in the final shape in Figure 15. The corresponding generating function is given by the remaining factor of $\left[{r+c+m-2 \atop c}\right]_Q$.

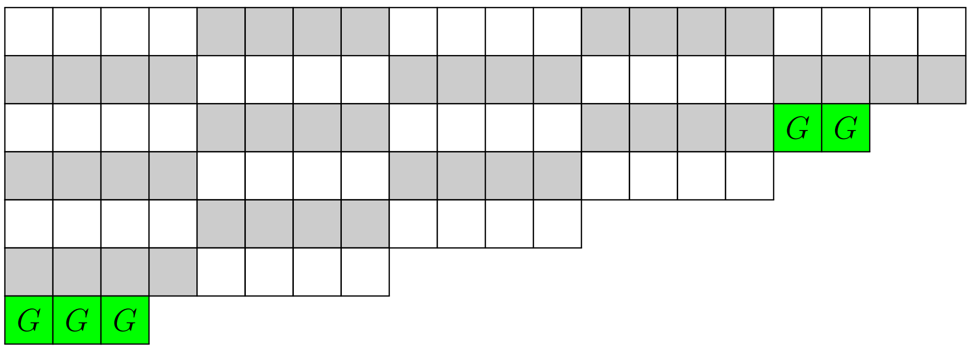

Figure 13. The example used to illustrate Case 2 of the second proof of Theorem 3

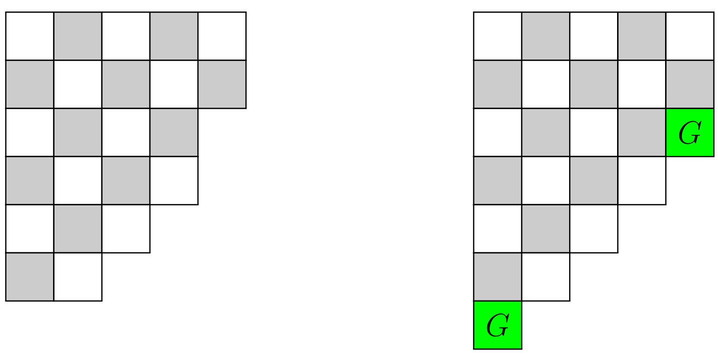

Figure 14. The remainder diagram of the partition in Figure 13 (right) and its interior (left)

This completes the proof of the theorem. $\square$

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Four of the authors wish to express their gratitude to Deutsche Bahn for delaying one of their trains on the way back from the ??th Séminaire Lotharingien de Combinatoire, resulting

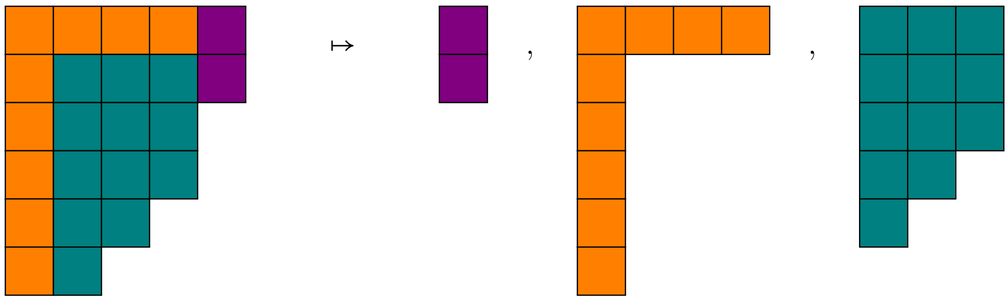

FIGURE 15. The composition of the interior of the remainder diagram of Figure 13 described in Case 2 of the second proof of Theorem 3

in a nightly stopover in a decent hotel in München — paid by Deutsche Bahn — during which this work was initiated.

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