QFT (16/05/19)

U-turn alternating sign matrices

1. ASMs

Recall from Lawrence's talk: an ASM is an $n \times n$ matrix with entries from $\{1, 0, -1\}$ satisfying

1) the nonzero entries in each row & column alternate between 1 and -1.
2) each row/column has sum equal to 1.

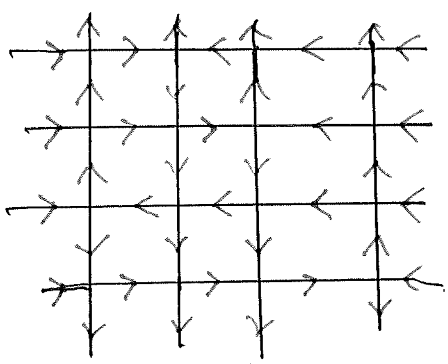
The number of ASMs is given by

(Theorem) $$A(n) = \cancel{\prod_{i=1}^{n} \frac{(3i-2)!}{(2i-1)!}} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

This formula was conjectured by Mills–Robbins–Rumsey and first proved by Zielberger, and subsequently Kuperberg.

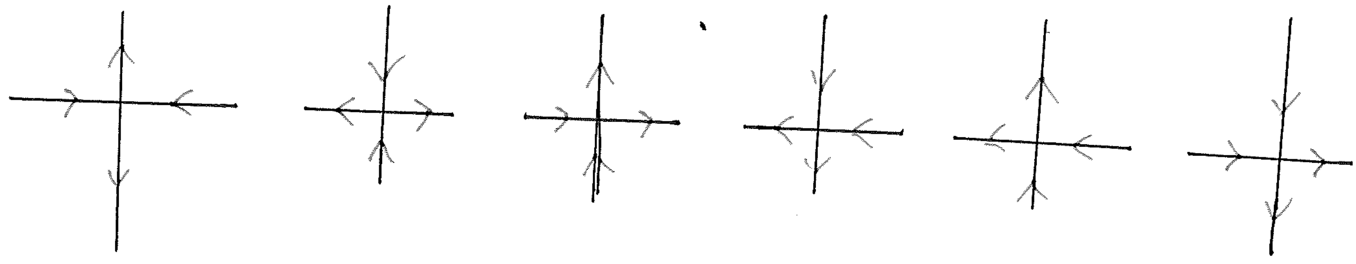
There is a bijection between ASMs and admissible states of the six-vertex model with domain-wall boundary, that is square ($n \times n$) 6-vertex models where the sides point inwards and the top and bottom point outwards:

(boring example)



$$\longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The correspondence is



$1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0$

and is bijective. Appealing to this correspondence, Kuperberg used the partition function for such a model (above) where the weight of each state is (up to a factor) equal to one. This gives an enumeration of ASMs. The partition function in question is as follows.

Theorem (Izergin – Korepin): The partition function for the six-vertex model with domain-wall boundary is

$$Z(n;X,Y) = \frac{(-1)^n \prod_{i=0}^{n-1} q^{\frac{y_i - x_i}{2}} \prod_{0\le i,j\le n} [x_i - y_j][x_i - y_j - 1]}{\prod_{0\le j<i<n} [x_i - x_j] \prod_{0\le i<j<n} [y_i - y_j]} \times \det M$$

where

$$M_{ij} = \frac{1}{[x_i - y_j][x_i - y_j - 1]}$$

and

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}.$$

Proof Uses YBE to show symmetry, then a recursion and a polynomial argument shows LHS = RHS.

Unfortunately we cannot simply plug in $X = (\frac{1}{2}, \dots, \frac{1}{2})$ and $Y = (0, \dots, 0)$, so we must purterb $Z(n, (\frac{1}{2}), (0))$ by ε and then send $\varepsilon \to 0$.

2. UASMs

A U-turn alternating sign matrix is a $2n \times n$ matrix with entries from $\{1, 0, -1\}$ satisfying:

1) each column satisfies the ASM conditions.
2) each odd row is read left-to-right and each even row is read right-to-left, such that when read odd $\to$ even satisfy the conditions of an ASM.

For example:

$$\left.\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} \text{U-turns}$$

Our main result:

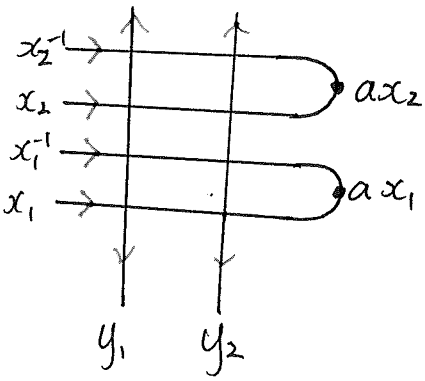
Theorem (Kuperberg) (Robbins): The number of UASMs is

$$A_U(2n) = 2^n (-3)^{n^2} \prod_{i=1}^{2n+1} \prod_{j=1}^{n} \frac{1 + 6j - 3i}{2j + i - 1}.$$

Further, we have

$$A_U(2n) = A_U(2n-2) \cdot \frac{\binom{6n-2}{2n}}{\binom{4n-2}{2n}}. \quad \text{(Don't state).}$$

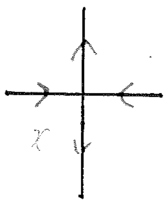
Mimicking ASMs, we will introduce a modified six-vertex model with "U-boundary".



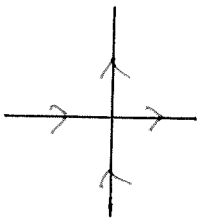
Following Kuperberg we use the notation

$$\sigma(x) = x - x^{-1}, \quad \alpha(x) = \sigma(ax)\sigma(ax^{-1})$$

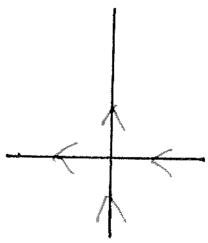
The weights will be given as



$\sigma(a^2)$

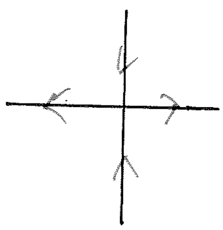


$\sigma(ax^{-1})$

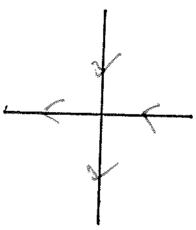


$\sigma(ax)$

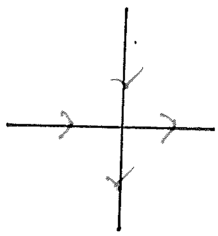
(R-matrix)



$\sigma(a^2)$

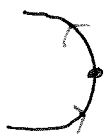


$\sigma(ax^{-1})$



$\sigma(ax)$

and for a U-turn



$\sigma(bx)$

$\sigma(bx^{-1})$

(K-matrix)

Theorem (Kuperberg): The partition function for the six-vertex model with right U-boundary is

$$Z_U(n;\vec{x},\vec{y})$$

$$= \frac{(\sigma(a^2))^n \prod_{i=1}^{n} \sigma(b y_i^{-1})\sigma(a^2 x_i^2) \prod_{i,j=1}^{n} \alpha(x_i y_j^{-1})\alpha(x_i y_j)}{\prod_{1\le i<j\le n} \sigma(x_i^{-1}x_j)\sigma(y_j y_i^{-1}) \prod_{1\le i\le j\le n} \sigma(x_i^{-1}x_j^{-1})\sigma(y_i y_j)} \det M_U \quad (*)$$

where

$$(M_U)_{ij} = \frac{1}{\alpha(x_i y_j^{-1})} - \frac{1}{\alpha(x_i y_j)}.$$

We prove the theorem with a succession of lemmas. The main idea is that:

1) We show $Z_U(n;\vec{x},\vec{y})$ is symmetric in $\vec{y}$ and in $\vec{x}$. Further we show $Z_U(n;\vec{x},\vec{y})$ picks up a factor of

$$\frac{\sigma(a^2(x_i^{-1})^2)}{\sigma(a^2 x_i^2)}$$

when $x_i \leftrightarrow x_i^{-1}$ for fixed i.

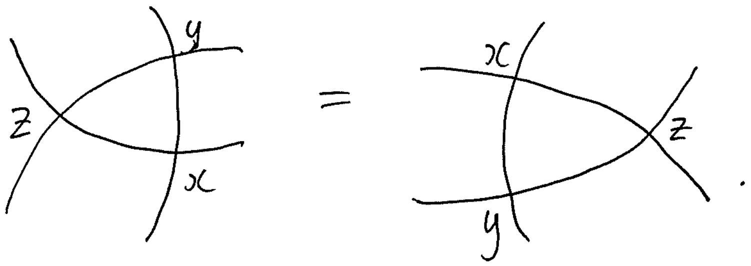
2) We prove a recurrence relation for $Z_U(n;\vec{x},\vec{y})$ of the form

$$Z_U(n;\vec{x}_n,\vec{y}_n)\Big|_{x_1=ay_1} = \text{Constant} \times Z_U(n-1;\vec{x}_{n-1},\vec{y}_{n-1}).$$

3) Both sides of $\circledast$ are Laurent polynomials in $\vec{x}$ and in $\vec{y}$ with width $2n-1$

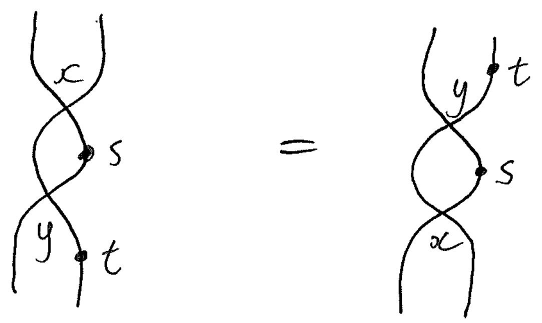
3 Proof of the expression for the partition function

<u>Lemma 1</u> (YBE) If $xyz = a^{-1}$, then

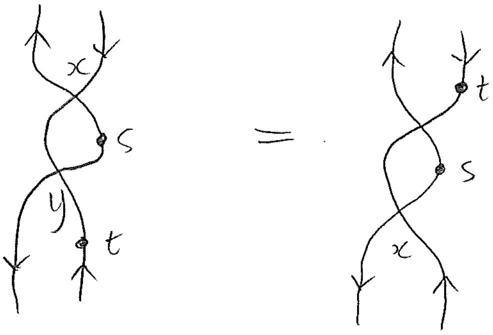


<u>Proof</u> Earlier in the semester. $\square$

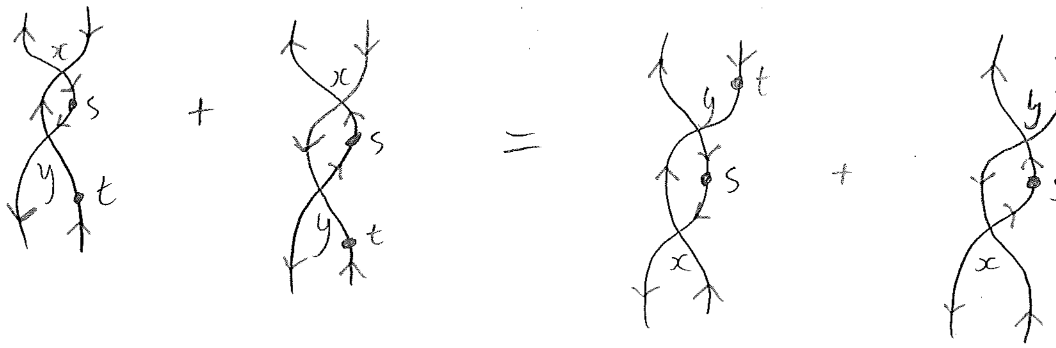
<u>Lemma 2</u> (Reflection equation): If $st = ay$ and $st^{-1} = ax$ then



<u>Proof</u> Both sides are zero unless two boundaries point in and two point out. Moreover reflecting in a horizontal line through $\bullet s$ and reversing arrows. Out of the six remaining (nonzero) cases, this symmetry proves four of them. The final two equations are equivalent. One of these is



which reads

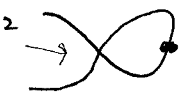
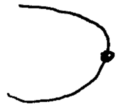


or

$$\sigma(bt)\sigma(a^2)\left(\sigma(ay)\sigma(bs^{-1}) + \sigma(ax)\sigma(bs)\right)$$
$$= \sigma(bt^{-1})\sigma(a^2)\left(\sigma(ax)\sigma(b\bar{s}) + \sigma(ay)\sigma(bs)\right).$$

Using the conditions shows this to be true, completing the proof. $\square$

Lemma 3 (Fish equation): For any a, x

$a^{-1}x^2 \rightarrow$  ax $= \sigma(a^2x^2)$  ax^{-1}.

Proof Computation. $\square$

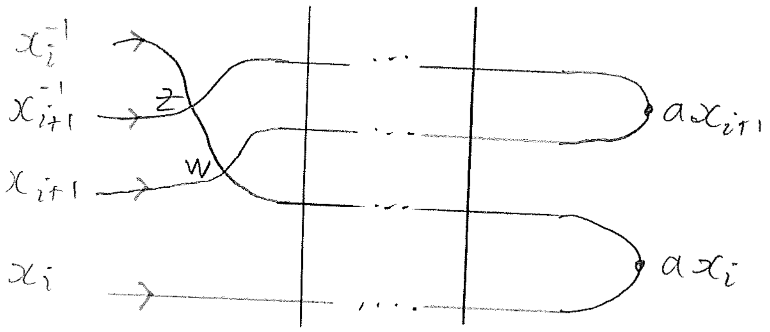
Lemma 4 The partition function $Z_u(n; \vec{x}, \vec{y})$ is symmetric in $\vec{x}, \vec{y}$, and gains a factor of

$$\frac{\sigma(a^2(x_i^{-1})^2)}{\sigma(a^2 x_i^2)}$$

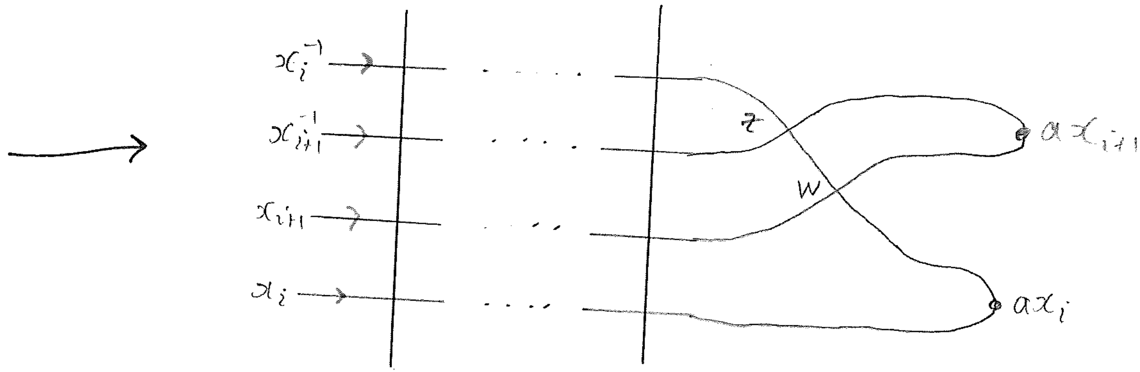
when $x_i \longmapsto x_i^{-1}$ for fixed i.

Proof Symmetry in $\vec{y}$ follows from the YBE.

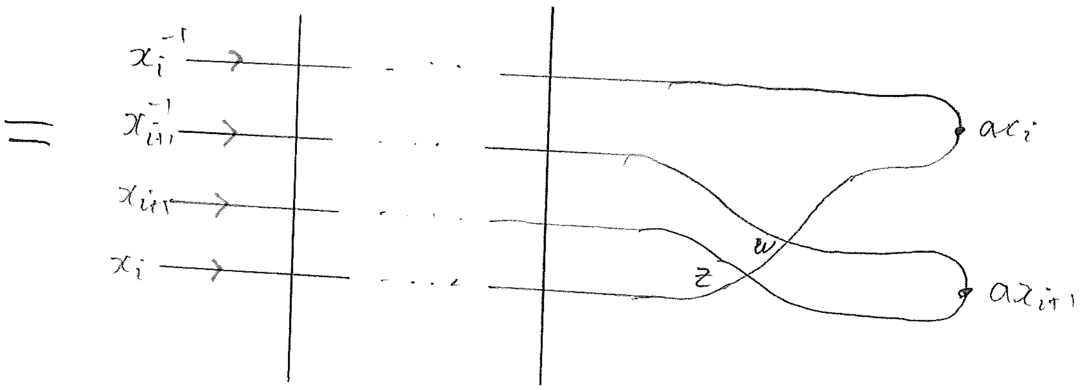
To show symmetry in $\vec{x}$, for any $i \le n-1$ we cross x_i^{-1} above the rows x_{i+1} and x_{i+1}^{-1} with spectral parameters $z = a^{-1} x_i^{-1} x_{i+1}$ and $w = \bar{a}^{1} x_i^{-1} x_{i+1}^{-1}$ so that



Then by YBE



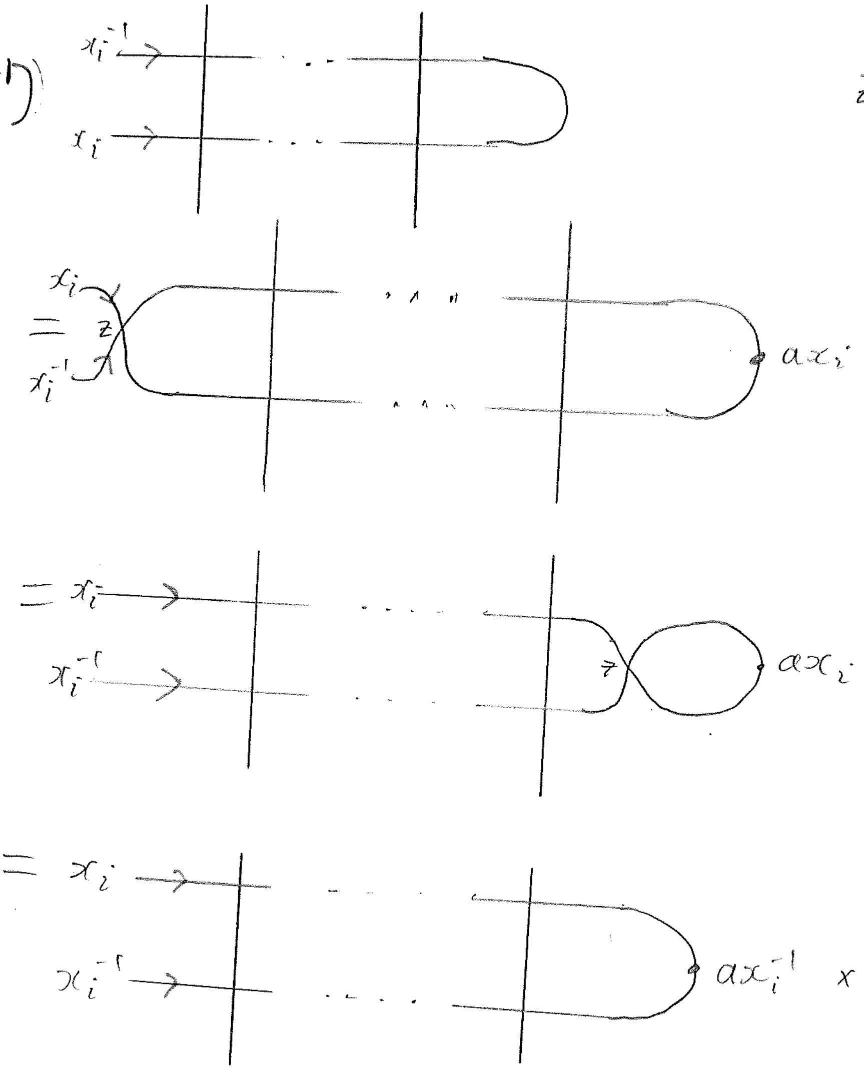
Using the reflection equation (Lemma 2)



Removing the crossings (after shifting back?) shows the desired symmetry.

To show the symmetry in $x_i \mapsto x_i^{-1}$ we note

$\sigma(a z^{-1})$



$z = a^{-1} x_i^2$

$\times\ \sigma(a^2 x_i^2)$

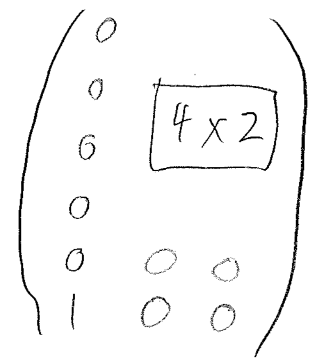
□

Lemma 5 If $x_1 = a y_1$ $(y_1 = a x_1)$, then

$$\frac{Z_U(n; \vec{x}_n, \vec{y}_n)}{Z_U(n-1; \vec{x}_{n-1}, \vec{y}_{n-1})} = \sigma(a^2)\sigma(a^2 x_1^2)\sigma(b y_1^{-1}) \prod_{n \geq i \geq 2} \sigma(a x_i^{-1} y_1)\sigma(x_1^{-1} y_i)\sigma(a x_i y_1)\sigma(a x_i^{-1} y_i^{-1}).$$

Proof If $x_1 = a y_1$, then our bottom-left entry is forced to be a 1, which in turn forces the bottom two rows and the first column to all be zero.

e.g.



What is left forms a $(2n-2) \times (n-1)$ UASM. The factor comes from the weight of the fixed vertices. $\square$

Lemma 6 Both sides of (*) are Laurent polynomials in y_1, which agree at the $2n$ points $y_1 = a x_i^{\pm 1}$ for $1 \leq i \leq n$. Hence they are equal.

Proof Omitted. $\square$

To conclude the proof of the theorem we combine all of the above Lemmas. This gives an inductive proof that both sides are equal.

4 Proof of the enumeration of UASMs

Let $\omega_3 = e^{\frac{i\pi}{3}}$. As the matrix $M_U(n;1,1)$ is singular we are forced to take a limit. Let

$$\vec{q}(k) = \left(q^{\frac{k+1}{2}}, q^{\frac{k+2}{2}}, \ldots, q^{\frac{k+n}{2}}\right)$$

and

$$\vec{q}(0) := \vec{q}.$$

Then $\lim_{q \to 1} \vec{q}(k) = \vec{1}$.

Then using Kuperberg's <u>Theorem 16</u> we have

$$M_U(n;\vec{q},\vec{q}(n))_{i,j} = \frac{q^{\frac{n+j+i}{2}} - q^{\frac{-n-j-i}{2}}}{q^{\frac{3n+3j+3i}{2}} - q^{\frac{3n-3j-3i}{2}}} - \frac{q^{\frac{n+j-i}{2}} - q^{\frac{-n-j+i}{2}}}{q^{\frac{3n+3j-3i}{2}} - q^{\frac{-3n-3j+3i}{2}}}$$

$$= \frac{-q^{i+j+n}(1-q^{2i})(1-q^{2j+2n})}{(q^{2i}+q^{2j+2n}+q^{i+j+n})(1+q^{i+j+n}+q^{i+j+n})}.$$

Consequently

$$\det M_U(n;\vec{q},\vec{q}(n))$$

$$= \frac{\prod_{1\le i<j\le 2n}\left(q^{\frac{3j-3i}{2}} - q^{\frac{3i-3j}{2}}\right)\prod_{i=1}^{2n+1}\prod_{j=1}^{n}\left(q^{\frac{6j-3i+1}{2}} - q^{\frac{3i-6j+1}{2}}\right)}{\prod_{i,j=1}^{n}\left(q^{\frac{3n+3j-3i}{2}} - q^{\frac{-3n-3j+3i}{2}}\right)\left(q^{\frac{3n+3j+3i}{2}} - q^{\frac{-3n-3j-3i}{2}}\right)}$$

Now using

$$A_U(2n;x,y) = \frac{Z_U(n;\vec{x},\vec{y})^{\vec{1}\ \vec{1}}}{\sigma(a)^{4\binom{n}{2}}\sigma(a^2)^n\sigma(ba^{-1})^n}$$

with $x=y=1$, $a=\omega_3$, $b=\omega_4$ and sending $q\to 1$ we obtain the desired result.

5 Quantum symmetric pairs

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involution on the Dynkin diagram of $\mathfrak{g}$. Then the Lie subalgebra of "fixed points" is given by

$$\mathfrak{k} = \{ x \in \mathfrak{g} \mid \theta(x) = x \}$$

and the pair $(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair.
In the case there is a "q-analogue" of $U(\mathfrak{k})$, denoted B, then we call

$$(U_q(\mathfrak{g}), B)$$

a quantum symmetric pair.

B is called a right coideal subalgebra if

$$\Delta(B) \subseteq B \otimes U_q(\mathfrak{g}).$$

Now in the quantum affine case $B \subseteq U_q(\widehat{\mathfrak{gl}}_n)$, while an R-matrix acts on V_z as per Paige's talk,

$$K(z) : V_z \longrightarrow V_{z^{-1}}.$$