Elliptic Selberg integrals

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1. The Selberg integral

The Selberg integral needs almost no introduction. It was published by Selberg in 1944, and has since played an important role in many different areas of mathematics. For $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$ and $\mathrm{Re}(\gamma)$ sufficiently large, Selberg's integral is

$$S_k(\alpha,\beta;\gamma) := \int_{[0,1]^k} \prod_{i=1}^k x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |x_i-x_j|^{2\gamma}\,\mathrm{d}\mathbf{x}$$
$$= \prod_{i=1}^k \frac{\Gamma(\beta+(i-1)\gamma)\Gamma(\alpha+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(k+i-2)\gamma)\Gamma(1+\gamma)},$$

where $\mathbf{x}:=(x_1,\dots,x_k)$.

3. Selberg integrals and the AGT conjecture

In their verification of the AGT conjecture for SU(2), Alba, Fateev, Litvinov and Tarnopolsky (AFLT) needed to evaluate the Selberg integral with a pair of Jack polynomials inserted into the integrand. These are symmetric functions which are a limiting case of the Macdonald polynomials.

Let $\langle f[\mathbf{x}]\rangle^k_{\alpha,\beta;\gamma}$ be the result of integrating the symmetric function f against the Selberg density and normalising by $S_k(\alpha,\beta;\gamma)$ (the Selberg average). Then the AFLT integral is

$$\Big\langle \tilde{P}^{(1/\gamma)}_\lambda[\mathbf{x}]\tilde{P}^{(1/\gamma)}_\mu[\mathbf{x}+\beta/\gamma-1]\Big\rangle^k_{\alpha,\beta;\gamma}$$
$$=\prod_{i=1}^k \frac{(\alpha+(k-i)\gamma)_{\lambda_i}}{(\alpha+\beta+(k-i-1)\gamma)_{\lambda_i}} \prod_{i,j=1}^k \frac{(\alpha+\beta+(2k-i-j-1)\gamma)_{\lambda_i+\mu_j}}{(\alpha+\beta+(2k-i-j)\gamma)_{\lambda_i+\mu_j}},$$

where $\tilde{P}^{(1/\gamma)}_\lambda$ is a normalised Jack polynomial (the normalisation is the polynomial evaluated at $x_i=1$ for all $1\leqslant i\leqslant k$) and $(a)_n:=\Gamma(a+n)/\Gamma(a)$ is the ordinary Pochhammer symbol.

Note: The second Jack polynomial is modified by a certain plethystic substitution.

5. An elliptic AFLT integral

In recent joint work with Rains and Warnaar we gave an elliptic analogue of the above AFLT integral in which the role of the Jack polynomials is played by the elliptic interpolation functions defined to the right.

The integral is most easily stated in terms of the elliptic symbol

$$\Delta^0_\lambda(a|b_1,\dots,b_\ell;t;p,q)$$
$$:=\prod_{r=1}^k\prod_{s=1}^\ell \frac{\Gamma_{p,q}(t^{1-r}p^{\lambda^{(1)}_r}b_s)\Gamma_{p,q}(t^{1-r}q^{\lambda^{(2)}_r}b_s)\Gamma^2_{p,q}(pqt^{1-r}a/b_s)}{\Gamma_{p,q}(pqt^{1-r}p^{\lambda^{(1)}_r}a/b_s)\Gamma_{p,q}(pqt^{1-r}q^{\lambda^{(2)}_r}a/b_s)\Gamma^2_{p,q}(t^{1-r}b_s)}.$$

Also, in analogy with the above, let $\langle\cdot\rangle^k_{t_1,\dots,t_6;t;p,q}$ be the elliptic Selberg average. Then again with the balancing condition $t^{2k-2}t_1\cdots t_6=pq$ we have that

$$\Big\langle R^*_\lambda(\mathbf{z};t_1,t_2;t;p,q)R^*_\mu(\mathbf{z};t_4/t,t_5/t;t_3t_4t_5/t,t_6;t;p,q)\Big\rangle^k_{t_1,\dots,t_6;t;p,q}$$
$$=\prod_{r=3}^6\Delta^0_\lambda(t^{k-1}t_1/t_2|t^{k-1}t_1t_r)\prod_{r=4}^5\Delta^0_\mu(t^{k-2}t_3t_4t_5/t_6|t^{k-1}t_3t_r)$$
$$\times\frac{\Delta^0_\mu(t^{k-2}t_3t_4t_5/t_6|t^{k-2}t_1t_3t_4t_5\langle\boldsymbol{\lambda}\rangle_{k;t;p,q}/t_6)}{\Delta^0_\mu(t^{k-2}t_3t_4t_5/t_6|t^{k-1}t_1t_3t_4t_5\langle\boldsymbol{\lambda}\rangle_{k;t;p,q}/t_6)}$$

2. The elliptic Selberg integral

Both the integrand and the evaluation of the elliptic Selberg integral involve the elliptic gamma function, which for $|p|,|q|<1$ is defined as

$$\Gamma_{p,q}(z):=\prod_{i,j=0}^\infty \frac{1-p^{i+1}q^{j+1}/z}{1-zp^iq^j}.$$

In addition with $|t|,|t_1|,\dots,|t_6|<1$ such that the balancing condition $t^{2k-2}t_1\cdots t_6=pq$ holds the elliptic Selberg integral is

$$S_k(t_1,\dots,t_6;t;p,q):=\varkappa_k\int_{\mathbb{T}^k}\prod_{i=1}^k\frac{\Gamma_{p,q}(t)\prod_{r=1}^6\Gamma_{p,q}(t_rz_i^{\pm})}{\Gamma_{p,q}(z_i^{\pm2})}\prod_{1\leqslant i<j\leqslant k}\frac{\Gamma_{p,q}(tz_i^{\pm}z_j^{\pm})}{\Gamma_{p,q}(z_i^{\pm}z_j^{\pm})}\frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}$$
$$=\prod_{i=1}^k\Big(\Gamma_{p,q}(t^i)\prod_{1\leqslant r<s\leqslant 6}\Gamma_{p,q}(t^{i-1}t_rt_s)\Big),$$

where $\varkappa_k:=(p;p)^k_\infty(q;q)^k_\infty/2^kk!(2\pi\mathrm{i})^k$. For $k=1$ this reduces to Spiridonov's elliptic beta integral

$$\varkappa_1\int_{\mathbb{T}}\frac{\prod_{r=1}^6\Gamma_{p,q}(t_rz^{\pm})}{\Gamma_{p,q}(z^{\pm2})}\frac{\mathrm{d}z}{z}=\prod_{1\leqslant r<s\leqslant 6}\Gamma_{p,q}(t_rt_s).$$

Also, taking a careful limit of $S_k(t_1,\dots,t_6;t;p,q)$ produces the non-elliptic Selberg integral. Note that the integral is symmetric in p and q.

4. Elliptic interpolation functions

It is natural to ask for a lift of the AFLT integral to the elliptic level. In this setting the Jack polynomials are replaced by a pair of elliptic interpolation functions.

Let $\boldsymbol{\mu}$ be a bipartition, that is, a pair of partitions $(\mu^{(1)},\mu^{(2)})$ both with at most k parts. Define the associated spectral vector by

$$\langle\boldsymbol{\mu}\rangle_{k;t;p,q}:=\big(p^{\mu^{(1)}_1}q^{\mu^{(2)}_1}t^{k-1},\dots,p^{\mu^{(1)}_1}q^{\mu^{(2)}_1}t^0\big).$$

The elliptic interpolation function

$$R^*_{\boldsymbol{\mu}}(z_1,\dots,z_k;a,b;t;p,q)$$

indexed by $\boldsymbol{\mu}$ is a BC_k-symmetric function with the important property that for any $\boldsymbol{\lambda}$ with $\boldsymbol{\mu}\not\subseteq\boldsymbol{\lambda}$ (read: $\mu^{(1)}\not\subseteq\lambda^{(1)}$ and $\mu^{(2)}\not\subseteq\lambda^{(2)}$)

$$R^*_{\boldsymbol{\mu}}(\langle\boldsymbol{\lambda}\rangle_{k;t;p,q};a,b;t;p,q)=0.$$

These functions are an elliptic lift of Okounkov's BC_k interpolation Macdonald polynomials.

To make sense of the plethystic substitution in the AFLT integral we actually need the "hybrid" functions

$$R^*_{\boldsymbol{\mu}}(z_1,\dots,z_k;v_1,\dots,v_{2\ell};a,b;t;p,q),$$

which are additionally $\mathfrak{S}_{2\ell}$-symmetric in the v_i. These are a scaled instance of certain skew interpolation functions.

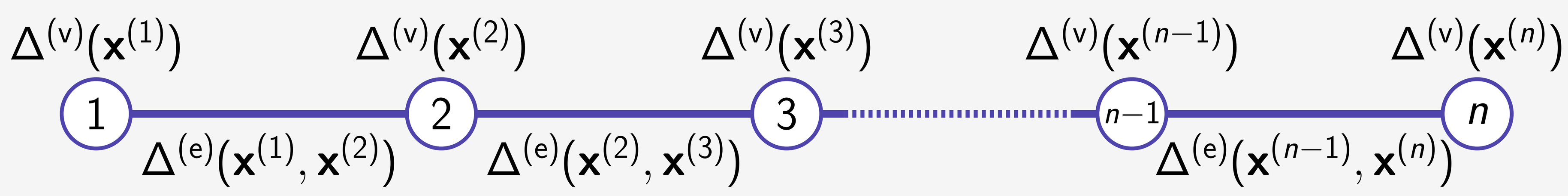
6. Vandermonde products

Below we will use the "vertex" and "edge" Vandermonde products

$$\Delta^{(v)}(x_1,\dots,x_k):=\prod_{1\leqslant i<j\leqslant k}(x_i-x_j),$$
$$\Delta^{(e)}(x_1,\dots,x_k;y_1,\dots,y_\ell):=\prod_{i=1}^k\prod_{j=1}^\ell(x_i-y_j).$$

7. From A_1 to A_n

Together with Rains and Warnaar we have shown that all of the integrals above admit generalisations in the form of A_n Selberg integrals. For the ordinary Selberg integral this was already done in 2009 by Warnaar and earlier for A_2 by Tarasov and Varchenko.



The recipe for the integrand of the A_n Selberg integral:

- pick a sequence of integers $0\leqslant k_1\leqslant\cdots\leqslant k_n$,
- each vertex r gets a set of variables $\mathbf{x}^{(r)}$ with $|\mathbf{x}^{(r)}|=k_r$, a complex parameter α_r and a "vertex" Vandermonde product (exponent 2γ).
- each edge gets an "edge" Vandermonde pairing $\mathbf{x}^{(s)}$ and $\mathbf{x}^{(s+1)}$ (exponent $-\gamma$),
- vertex n gets β_n (on the right set $\beta_r:=1$ if $1\leqslant r<n$).

See Ole's talk for the elliptic version!

$$\int_{C^{k_1,\dots,k_n}_\gamma[0,1]}\prod_{r=1}^n\prod_{i=1}^{k_r}\big(x^{(r)}_i\big)^{\alpha_r-1}\big(1-x^{(r)}_i\big)^{\beta_r-1}\prod_{r=1}^n\big|\Delta^{(v)}(\mathbf{x}^{(r)})\big|^{2\gamma}\prod_{r=1}^{n-1}\big|\Delta^{(e)}(\mathbf{x}^{(r)},\mathbf{x}^{(r+1)})\big|^{-\gamma}\,\mathrm{d}\mathbf{x}^{(1)}\cdots\mathrm{d}\mathbf{x}^{(n)}$$
$$=\prod_{r=1}^n\prod_{i=1}^{k_r}\frac{\Gamma(\beta_r+(i-k_{r+1}-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}$$
$$\times\prod_{1\leqslant r\leqslant s\leqslant n}\prod_{i=1}^{k_r-k_{r-1}}\frac{\Gamma(\alpha_r+\cdots+\alpha_s+(r-s+i-1)\gamma)}{\Gamma(\alpha_r+\cdots+\alpha_s+\beta_s+(k_s-k_{s+1}+r-s+i-2)\gamma)}$$