Selberg integrals & symmetric functions

Outline

- History of the Selberg integral
- Interlude on symmetric functions
- Some recent integrals of Selberg-type involving symmetric functions
- Generalisations of these (w/ Ole Warnaar & Eric Rains)

1941

Selberg uses a variant of his integral in a paper.

Hilfsatz I ergibt nun auf $g(z)\prod_{i=1}^{p} z(z-1)\dots(z-l_i^{(0)}+1)$ angewandt, dass dieser Ausdruck, und mithin auch $g(z)$ selbst, ein Polynom ist.

§ 2.

Hilfsatz III

Es sei m und ν positiv und ganz, ausserdem sei $m \geqq 2(p-1)\nu$, dann gilt

$$\frac{\nu!(2\nu)!\dots(p\nu)!(m-(p-1)\nu)!(m-p\nu)!\dots(m-2(p-1)\nu)!}{p!\,\nu!^p \prod_{i=1}^{p}(z-(i-1)\nu)(z-(i-1)\nu-1)\dots(z-m+(i-1)\nu)} = \sum_{0\leq a_i\leq m} c_{a_1\dots a_p}\prod_{i=1}^{p}\frac{a_i!}{z\dots(z-a_i)}$$

wo die $c_{a_1 a_2\dots a_p}$ sämmtlich ganze Zahlen sind.

Beweis: Es gilt[1]) wenn $\Delta\left(\frac{1}{y}\right)$ die Diskriminante $\prod_{i<j}\left(\frac{1}{y_i}-\frac{1}{y_j}\right)$ der p Grössen $\frac{1}{y_1}, \frac{1}{y_2}, \dots, \frac{1}{y_p}$, bedeutet, für $R(z) > m \geqq 2(p-1)\nu$,

$$\frac{\nu!(2\nu)!\dots(p\nu)!(m-(p-1)\nu)!(m-p\nu)!\dots(m-2(p-1)\nu)!}{p!\,\nu!^p\prod_{i=1}^{p}(z-(i-1)\nu)(z-(i-1)\nu-1)\dots(z-m+(i-1)\nu)} = \tag{11}$$

$$=\int_0^\infty\dots\int_0^\infty\frac{(y_1 y_2\dots y_p)^m\,\Delta^{2\nu}\left(\frac{1}{y}\right)}{((1+y_1)(1+y_2)\dots(1+y_p))^{z+1}}\,dy_1\,dy_2\dots dy_p,$$

entwickeln wir hier $\Delta^{2\nu}\left(\frac{1}{y}\right)$ nach Potenzen der Grössen y_i, erhalten wir

$$(y_1 y_2\dots y_p)^m\,\Delta^{2\nu}\left(\frac{1}{y}\right) = \sum_{0\leq a_i\leq m} c_{a_1 a_2\dots a_p} y_1^{a_1} y_2^{a_2}\dots y_p^{a_p},$$

wo die $c_{a_1 a_2\dots a_p}$ ganze Zahlen sind. Wird dies in (11) eingetragen, erhalten wir wegen

$$\int_0^\infty \frac{y_i^{a_i}}{(1+y_i)^{z+1}}\,dy_i = \frac{a_i!}{z(z-1)\dots(z-a_i)},$$

den Hilfsatz.

[1]) Leider habe ich die Formel (11) nirgends in der Litteratur finden können, ein Beweis hier zu bringen scheint aber nicht angebracht, da die Arbeit sonst zu sehr anschwellen würde; sollte sich aber herausstellen, dass die Formel neu wäre, beabsichtige ich später ein Beweis zu veröffentlichen.

1944

"Remarks on a multiple integral"

14.

Bemerkninger om et multipelt integral

Norsk matematisk tidsskrift B. 26 (1944), 71 – 78

1. Den naturligste generalisasjon av det første Eulerske integral[1])

(1) $\int_0^1 u^{x-1}(1-u)^{y-1}\,du = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ (gyldig for $\Re x > 0$, $\Re y > 0$)[2])

til $p > 1$ dimensjoner, er utvilsomt gitt ved Dirichlets formel[3])

(2) $$\underset{\substack{u_i>0 \\ u_1+\dots+u_p<1}}{\int\dots\int} u_1^{x_1-1}u_2^{x_2-1}\dots u_p^{x_p-1}(1-u_1-\dots-u_p)^{x_{p+1}-1}du_1\dots du_p =$$

$$= \frac{\Gamma(x_1)\,\Gamma(x_2)\dots\Gamma(x_{p+1})}{\Gamma(x_1+x_2+\dots+x_{p+1})},$$

som gjelder for $\Re x_i > 0$, $i = 1, 2, \dots, p+1$.

I dette arbeid skal vi betrakte et annet integral som også kan anses som en generalisasjon av Eulers, for $p > 2$ inneholder det riktignok ikke så mange vilkårlige parametere som Dirichlets integral. Vi skal vise at *når p er et helt positivt tall og*

$$\Re x > 0,\ \Re y > 0,\ \Re z > -\min\left\{\frac{1}{p}, \frac{\Re x}{p-1}, \frac{\Re y}{p-1}\right\},$$

har vi

(3) $$\int_0^1\dots\int_0^1 (u_1u_2\dots u_p)^{x-1}\{(1-u_1)(1-u_2)\dots$$

$$(1-u_p)\}^{y-1}\,|\Delta(u)|^{2z}\,du_1\dots du_p =$$

$$= \prod_{\nu=1}^{p} \frac{\Gamma(1+\nu z)\,\Gamma(x+(\nu-1)z)\,\Gamma(y+(\nu-1)z)}{\Gamma(1+z)\,\Gamma(x+y+(p+\nu-2)z)}.$$

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[1]) Se f. eks. *C. Størmer*, Forelesninger over Gammafunksjonen, § 22 s. 51.

[2]) $\Re w$ betegner her og i det følgende realdelen av w.

[3]) *C. Størmer*, Forelesninger over Gammafunksjonen, § 26 s. 56.

From 1944 – 1979 the Selberg paper received one citation. The story of it's rediscovery is somewhat remarkable.

In 1963, Dyson & Mehta were led to conjecture the following (Random matrix $\gamma = 1/2, 1, 2$)

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\sum_{i=1}^{n} t_i^2/2\right) \prod_{1 \le i < j \le n} |t_i - t_j|^{2\gamma} dt$$

$$= \prod_{i=1}^{n} \frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}. \qquad (M)$$

Mehta included this conjecture in his 1967 book on random matrices and, in 1974, submitted it as an open problem to SIAM Review. Due to this popularising the above is referred to as Mehta's integral.

A proof was uncovered in 1979 when Bombieri, who was at IAS with both Dyson & Selberg, asked both about an integral similar to the Selberg integral. Making the connection, he was able to resolve the conjecture.

Since 1979 the Selberg integral has found continual applications in

- Random matrix theory

- Constant terms (Macdonald conjectures)
- Number theory (Riemann zeroes)
- Special functions (Orthogonal polynomials)
- Combinatorics (Stanley, Kim & Stanton)
- K-Z equations & Conformal field theory

2. Symmetric functions

Note the integrand of (5) is a symmetric function:

$$\prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma}.$$

We will be interested in multiplying the integrand by a symmetric function.

One of the many proofs of (5) is due to Aomoto, who proved that

$$\int_{[0,1]^n} |\Delta(t)|^{2\gamma} \prod_{i=1}^{r} t_i \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1}\, dt_1 \cdots dt_n$$

$$= S_n(\alpha,\beta;\gamma) \prod_{i=1}^{r} \frac{\alpha+(n-i)\gamma}{\alpha+\beta+(2n-i-1)\gamma}.$$

By symmetry, this evaluation is the same when

$$\prod_{i=1}^{r} t_i \longmapsto \prod_{i\in I} t_i \qquad I \subseteq \{1,\dots,n\}.$$

$$|I| = r$$

Using this fact we can symmetrize the integrand to obtain

$$\int_{[0,1]^n} e_r(t_1,\dots,t_n)\,|\Delta(t)|^{2\gamma} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1}\,dt_1\cdots dt_n$$

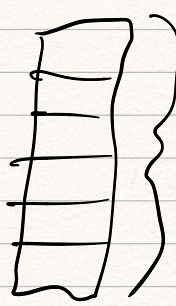
$$= \binom{n}{r} S_n(\alpha,\beta;\gamma) \prod_{i=1}^{r} \frac{\alpha+(n-i)\gamma}{\alpha+\beta+(2n-i-1)\gamma}. \quad (A)$$

where

$$e_r(t_1,\dots,t_n) := \sum_{1\le i_1<\cdots<i_r\le n} t_{i_1}\cdots t_{i_r},$$

the r-th elementary symmetric function. Alternatively,

$$e_r(X_n) = s_{(1^r)}(X_n) \leftarrow \text{Schur function.}$$

where $(1^r) =$  r boxes. $\qquad X_n := (x_1,\dots,x_n)$

Recall that

$$s_\lambda(X_n) := \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j}\right)}{\Delta(X_n)}.$$

Other important classes are

$$h_r(X_n) := \sum_{1\le i_1\le\cdots\le i_r\le n} x_{i_1}\cdots x_{i_r} \quad \text{(complete homogeneous)}$$

and

$$p_r(X_n) := x_1^r + \cdots + x_n^r. \qquad \text{(power sums)}$$

Let Λ_n denote the algebra of symmetric functions in n variables. Then $\dim \Lambda_n$ is the number of partitions w/ at most n parts. If we extend $p_r(X) := x_1^r + x_2^r + \cdots$ then the algebra we get is

$$\Lambda := \mathbb{Q}[p_1, p_2, \ldots],$$

of symmetric f^{ns} in countably many variables.
Plethystic notation makes use of this fact to define operations on alphabets.

Let $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$. We can define the sum of two alphabets by demanding that

$$p_r[X+Y] := p_r[X] + p_r[Y].$$

Of course $p_r[x] = x^r$, so that this forces

$$f[X] = f[x_1 + x_2 + x_3 + \cdots] = f(x_1, x_2, \ldots)$$

for any $f \in \Lambda$.
We can also subtract alphabets by defining

$$p_r[X-Y] := p_r[X] - p_r[Y].$$

So

$$p_r[X+Y-Y] = p_r[X] + p_r[Y] - p_r[Y]$$

$$= pr[X].$$

Note that

$$pr[nX] = pr[\underbrace{X + \cdots + X}_{n \text{ times}}] = n\, pr[X].$$

In particular

$$pr[\underbrace{1 + \cdots + 1}_{n \text{ times}}] = pr[n].$$

So in general $f(\underbrace{1, \ldots, 1}_{n \text{ times}}) = f[n]$. This is further extended to allow for $f[z]$ to be computed for any $z \in \mathbb{C}$.

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3. Back to Selberg

In 1987 Macdonald conjectured an extension of Aomoto's integral.

Let

$$\langle f \rangle^n_{\alpha,\beta;\gamma} := \frac{\int_{[0,1]^n} f(t) |\Delta(t)|^{2\gamma} \prod_{i=1}^{n} t_i^{\alpha-1} (1-t_i)^{\beta-1} \, dt}{S_n(\alpha, \beta; \gamma)}$$

Then:

$$\langle P_\lambda^{(1/\gamma)} \rangle^n_{\alpha,\beta;\gamma} = P_\lambda^{(1/\gamma)}[n] \prod_{i \geq 1} \frac{(\alpha + (k-i)\gamma)_{\lambda_i}}{(\alpha + \beta + (2k-i-1))_{\lambda_i}}$$

where $(a)_n := a(a+1) \cdots (a+n-1)$.

This was proved soon after by Kadell.

Kadell was able to go one step further, and also evaluated

$$\langle P_\lambda^{(1/\gamma)} P_\mu^{(1/\gamma)} \rangle^n_{\alpha,\gamma;\gamma}$$

The case

$$\langle s_\lambda s_\mu \rangle^n_{\alpha,1;1}$$

was already known to Hua (1963 at least).

In 2011, while verifying the AGT conjecture for rank 1, Alba, Fateev, Litvinov and Tarnopolsky computed the integral

$$\langle P_\lambda^{(1/\gamma)}[t]\, P_\mu^{(1/\gamma)}[t+\beta/\gamma-1] \rangle^n_{\alpha,\beta;\gamma}$$

$$= P_\lambda^{(1/\gamma)}[n]\, P_\mu^{(1/\gamma)}[n+\beta/\gamma-1]$$

$$\times \prod_{i=1}^{n} \frac{(\alpha+(n-i)\gamma)_{\lambda_i}}{(\alpha+\beta+(n-i-1)\gamma)_{\lambda_i}}$$

$$\times \prod_{i,j=1}^{n} \frac{(\alpha+\beta+(2n-i-j-1)\gamma)_{\lambda_i+\mu_j}}{(\alpha+\beta+(2n-i-j)\gamma)_{\lambda_i+\mu_j}}.$$

AFLT give a (partial) proof based on the Okounkov–Olshanski formula.

We show that the above is a limiting case of the Cauchy identity for Macdonald polynomials.

Structural generalisations

Selberg integral

$\swarrow$ $\searrow$

q-Selberg integral
(Askey, Habsieger, Kadell)

$\downarrow$

elliptic Selberg integral
(van Diejen, Spiridonov, Rains)

A_n Selberg integral
(Tarasov, Varchenko, Warnaar)

$\downarrow$

elliptic A_n Selberg int.

All of these admit AFLT-type analogues!

The proofs of these analogues require new indentities for Macdonald polynomials.

$$P_\lambda(X;q,t) \xrightarrow[\substack{(q,q^\gamma)\\ \lim_{q\to 1}}]{} P_\lambda^{(1/\gamma)}(X).$$

Why? The beta integral has a q-analogue,

$$\sum_{i=1}^{\infty} \frac{(a;q)_i}{(q;q)_i} z^i = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (q\text{-Bin})$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$$

$n \in \mathbb{N} \cup \{\infty\}$. This is the q-binomial theorem.

Setting $(a, z) \mapsto (q^\beta, q^\alpha)$ this may be rewritten as

$$\int_0^1 t^{\alpha-1} (qt;q)_{\beta-1} \, d_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

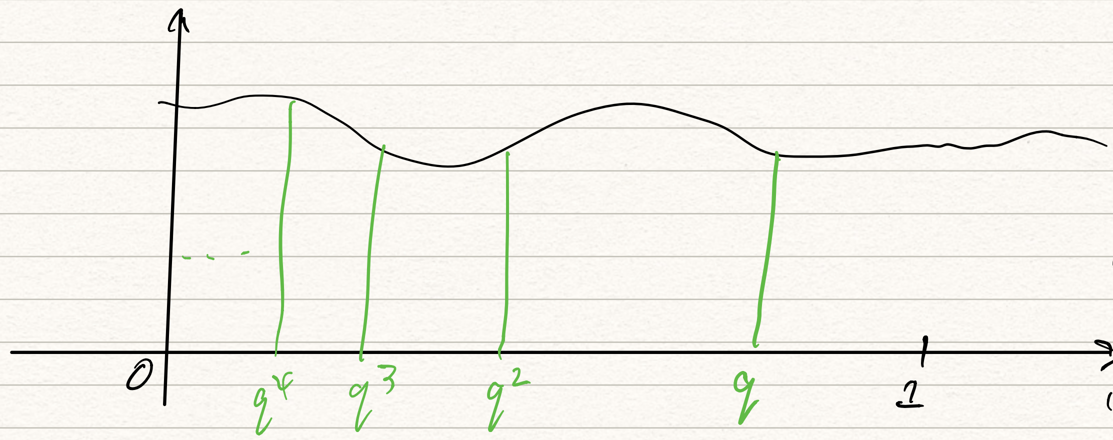
with

$$\Gamma_q(\alpha) := (1-q)^{1-\alpha} \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}$$

and

$$\int_0^1 f(t)\, d_q t := (1-q) \sum_{k=0}^{\infty} f(q^k) q^k$$

the q-integral of $f(t)$ $\quad (0<q<1)$



$q \to 1$ recovers the Riemann integral on $[0,1]$.

A Macdonald polynomial verision of (q-Bin) (and its rank n extension) are the key parts of the proofs.

An integrals

Another appearance of the Selberg integral is as solutions to the Knizhnik–Zamolodchikov equations. This generalises the appearance of the beta integral as a special case of a solution to

$$x(1-x)\frac{d^2F}{dx^2} + (c-(a+b+1)x)\frac{dF}{dx} - abF = 0.$$

Indeed, Gauss's solution is

$${}_2F_1\left[\begin{matrix} a \quad b \\ c \end{matrix}; x\right] = \sum_{n\geq 0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} x^n$$

Euler's solution

$${}_2F_1\left[\begin{matrix} a \quad b \\ c \end{matrix}; x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times (1-xt)^{-a} \, dt$$

At $x=1$ this reduces to

$${}_2F_1\left[\begin{matrix} a \quad b \\ c \end{matrix}; 1\right] = \frac{\cancel{\Gamma(c)}\,\Gamma(c-a-b)}{\Gamma(c-a)\,\cancel{\Gamma(c-b)}}$$

$$= \frac{\cancel{\Gamma(c)}}{\Gamma(b)\,\cancel{\Gamma(c-b)}} \int_0^1 t^{b-1}(1-t)^{c-a-b-1} \, dt$$

Beta integral with $(\alpha, \beta) \mapsto (b, c-a-b)$.

Now let $\mathfrak{g}$ be a simple Lie algebra and V_λ, V_μ highest weight reps. of $\mathfrak{g}$. The KZ equations are

$$\kappa \frac{\partial u}{\partial z} = \frac{\Omega}{z-w} u, \quad \kappa \frac{\partial u}{\partial w} = \frac{\Omega}{w-z} u$$

where $u(z,w) : V_\lambda \otimes V_\mu \to \mathbb{C}$ and Ω is the Casimir element.

In 1991 Schechtman & Varchenko solved these equations (for specific $u(z,w)$) in terms of generalised Selberg integrals.

This inspired the following remarkable conjecture.

Conjecture (Mukhin & Varchenko 2000)

For any simple Lie algebra, if a certain space is one-dim., then

$$\int_{?} |\Phi(t)|^{\gamma} = \prod \text{Gamma functions.}$$

For A_1, the scaled master function is

$$\Phi(t) = \prod_{i=1}^{n} t_i^{\frac{\alpha-1}{\gamma}} (1-t_i)^{\frac{\beta-1}{\gamma}} \prod_{1 \le i < j \le n} (t_i - t_j)^2$$

The conjecture has only been resolved in type A by Tarasov & Varchenko (2003) and Warnaar (2009).

An : Rank n

$$k_1 \le k_2 \le k_3 \le \cdots \cdots \le k_{n-2} \le k_{n-1} \le k_n$$

$P_{\lambda}^{(1/\gamma)}[t^{(1)}]$ $\qquad\qquad\qquad\qquad$ $P_{\mu}^{(1/\gamma)}[t^{(n)} + \beta/\gamma - 1]$

$1 \le r \le n-1$ $\qquad \prod_{i=1}^{k_r} (t_i^{(r)})^{\alpha_r - 1} \cdot |\Delta(t^{(r)})|^{2\gamma}$

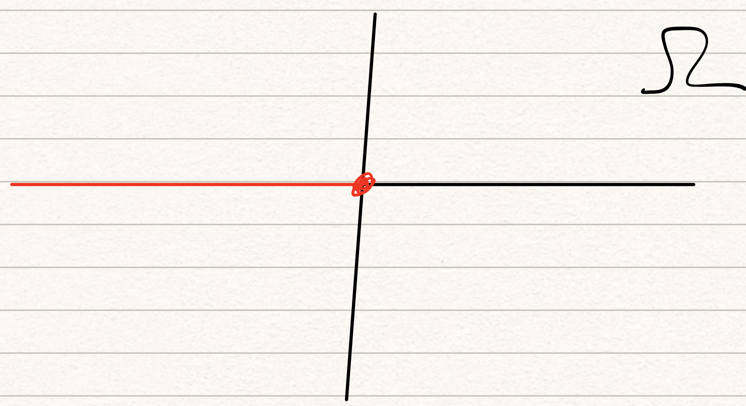
$r = n$ $\qquad \prod_{i=1}^{k_n} (t_i^{(n)})^{\alpha_r - 1} (1 - t_i^{(r)})^{\beta - 1} \cdot |\Delta(t^{(n)})|^{2\gamma}$

$$\prod_{i=1}$$

$1 \leq r \leq n-1$ $\quad \displaystyle\prod_{i=1}^{k_r} \prod_{j=1}^{k_{r+1}} \left| t_i^{(r)} - t_j^{(r+1)} \right|^{-\gamma}$ $\quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Again, the proof of the A_n Selberg integral and its extensions use Macdonald polynomial theory.

For $\gamma = 1$ the A_n integral diverges, but one can still make sense of the Schur case by passing to contour integration.



For $x \in \Omega^n$, $z \in \mathbb{C}^n$, define the Complex Schur f^n

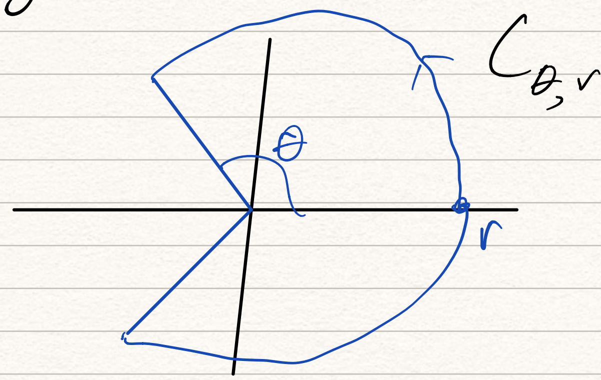
$$s^{(n)}(x; z) := \frac{\det_{1 \leq i,j \leq n} \left(x_i^{z_j} \right)}{\Delta(x)}$$

Symmetric in $x_1, \ldots, x_n$, and if $z_i = \lambda_i + n - i$,

$$s^{(n)}(x; z) \longrightarrow s_\lambda(x).$$

First defined by Kadell in 2000

The most important result for us is the reproducing integral.



$$\frac{(-1)^{\binom{k}{2}}}{k!\,(2\pi i)^k} \int_{C_{\theta,r}^k} S^{(k)}(x;z)\, S_\lambda[y-x]\, \Delta^2(x) \times \prod_{i=1}^{k} \prod_{j=1}^{l} (x_i - y_j)^{-1}\, dx$$

$$= \begin{cases} S^{(l)}(y;(z, \lambda_1 + l-k-1, \ldots, \lambda_{l-k}), & l(\lambda) \le l-k \\ 0 & \text{otherwise.} \end{cases}$$

This allows one to inductively prove cen A_n Selberg integral with $n+1$ Schur functions in the integrand. (originally conjectured by Zhang & Matsuo in the context of AGT.)

__Remarks__

This inductive argument is also how one proves

1. A_n Cauchy identities
2. Elliptic A_n Selberg integral

Complex Schur $\longmapsto$ "Elliptic interpolation kernel"

This similarly is a function depending on a pair of alphabets

$$K(x;y)$$

such that specialising one set gives the elliptic analogue of the Jack polynomial.