PROOF OF SOME LITTLEWOOD IDENTITIES CONJECTURED BY LEE, RAINS AND WARNAAR

SEAMUS P. ALBION

Abstract. We prove a novel pair of Littlewood identities for Schur functions, recently conjecture by Lee, Rains and Warnaar in the Macdonald case, in which the sum is over partitions with empty 2-core. As a byproduct we obtain a new Littlewood identity in the spirit of Littlewood's original formulae.

Keywords: empty 2-core, Koornwinder polynomials, Littlewood identities, Schur functions

1. Introduction

The classical Littlewood identities are the following three summation formulae for Schur functions

$$\sum_{\lambda} s_\lambda(x) = \prod_{i\geqslant 1} \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_ix_j} \tag{1.1a}$$

$$\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x) = \prod_{i\geqslant 1} \frac{1}{1-x_i^2} \prod_{i<j} \frac{1}{1-x_ix_j} \tag{1.1b}$$

$$\sum_{\substack{\lambda \\ \lambda' \text{ even}}} s_\lambda(x) = \prod_{i<j} \frac{1}{1-x_ix_j}. \tag{1.1c}$$

Each sum is over all partitions λ satisfying the given condition where λ even (resp. λ' even) means λ has even rows (resp. even columns). These were first written down together by Littlewood [15, p. 238], however (1.1a) was already known to Schur [26]. The identities (1.1) have since afforded many far-reaching generalisations and have found applications in areas such as combinatorics, representation theory and elliptic hypergeometric series. In particular there are many generalisations of (1.1) at the Schur level [3, 7, 10, 11, 12, 20, 21, 27]. Also see [24] for comprehensive references to the literature.

The purpose of this note is to prove the Schur function case of a pair of Littlewood identities for Macdonald polynomials recently conjectured by Lee, Rains and Warnaar [14, Conjecture 9.5]. To state these we need some notation. Denote the multiset of hook lengths of a partition λ by $\mathcal{H}_\lambda$. We refine this by writing $\mathcal{H}_\lambda^{\mathrm{e/o}}$ for the submultiset of even/odd hook lengths. The standard infinite q-shifted factorial is given by $(a;q)_\infty := \prod_{i\geqslant 0}(1-aq^i)$

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and we define a statistic

$$b(\lambda) := \sum_{(i,j)\in\lambda} (-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda_i - i),$$

in terms of the Young diagram of λ; see Subsection 2.1 below.

Theorem 1.1. *There holds*

$$\sum_{\substack{\lambda\\ 2\text{-core}(\lambda)=0}} q^{b(\lambda)} \frac{\prod_{h\in\mathcal{H}^{\mathrm{o}}_\lambda}(1-q^h)}{\prod_{h\in\mathcal{H}^{\mathrm{e}}_\lambda}(1-q^h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j}, \tag{1.2}$$

and

$$\sum_{\substack{\lambda\\ 2\text{-core}(\lambda)=0}} q^{b(\lambda')} \frac{\prod_{h\in\mathcal{H}^{\mathrm{o}}_\lambda}(1-q^h)}{\prod_{h\in\mathcal{H}^{\mathrm{e}}_\lambda}(1-q^h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{(q^2x_i^2;q^2)_\infty}{(qx_i^2;q^2)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j}. \tag{1.3}$$

The condition 2-core$(\lambda) = 0$ generalises both the even row/column conditions of (1.1b) and (1.1c). Indeed, by Lemma 2.2 we have that $b(\lambda) = 0$ if and only if λ is even. Thus in the $q \to 0$ limit (1.2) and (1.3) collapse to (1.1b) and (1.1c) respectively. In this sense these identities are in the spirit of Kawanaka's identity [12, Theorem 1.1]

$$\sum_\lambda \prod_{h\in\mathcal{H}_\lambda}\left(\frac{1+q^h}{1-q^h}\right)s_\lambda(x) = \prod_{i\geqslant 1}\frac{(-qx_i;q)_\infty}{(x_i;q)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j},$$

since this reduces to (1.1a) when $q = 0$. Unlike Kawanaka's identity one can make sense of the $q \to 1$ limit of (1.2) and (1.3). In either case we obtain the following Littlewood identity.

Corollary 1.2. *We have*

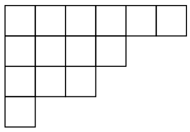
$$\sum_{\substack{\lambda\\ 2\text{-core}(\lambda)=0}} \frac{\prod_{h\in\mathcal{H}^{\mathrm{o}}_\lambda}(h)}{\prod_{h\in\mathcal{H}^{\mathrm{e}}_\lambda}(h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{1}{(1-x_i^2)^{1/2}}\prod_{i<j}\frac{1}{1-x_ix_j}.$$

The outline of the paper is as follows. In the next section we give preliminaries regarding partitions, Schur functions and Koornwinder polynomials. In Section 3 we prove a pair of vanishing integrals for Schur functions again conjectured by Lee, Rains and Warnaar in the Macdonald case [14, Conjecture 9.2]. Then, in Section 4, we follow the techniques of [24] to prove the bounded analogues of Theorem 1.1 conjectured in [14, Conjecture 9.4]. The theorem then follows by taking an appropriate limit. We conclude with a derivation of Corollary 1.2.

2. Partitions and (BC_n)-symmetric functions

2.1. **Partitions.** A partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a weakly decreasing sequence of nonnegative integers such that finitely many λ_i are nonzero. The sum of the entries is denoted $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ and if $|\lambda| = n$ we say λ is a partition of n, written $\lambda \vdash n$. Nonzero entries are called parts, and the number of parts is called the length, denoted $l(\lambda)$. We denote by $\mathscr{P}$ the set of all partitions and by $\mathscr{P}_n$ the set of all partitions with length at most n. In particular $\mathscr{P}_0 = \{0\}$ where 0 denotes the unique partition of zero. If

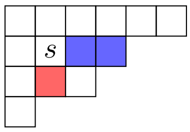
$\lambda \in \mathscr{P}_n$ we write $\lambda + \delta$ for the partition $(\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n)$. The number $m_i(\lambda)$ of occurrences of an integer i as a part of λ is called the multiplicity. Sometimes we express a partition in terms of its multiplicities as $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} 3^{m_3(\lambda)} \dots)$. We write $\mu \subset \lambda$ if the partition μ is contained in λ, i.e. if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. If $\lambda \subseteq (m^n)$ for some nonnegative integers m, n, then we write $(m^n) - \lambda$ for the complement of λ inside (m^n), that is, $(m^n) - \lambda := (m - \lambda_n, m - \lambda_{n-1}, \dots, m - \lambda_1)$. A partition is identified with its Young diagram, which is the left-justified array of squares with λ_i squares in row i with i increasing downward. For example



is the Young diagram of $(6, 4, 3, 1)$. The conjugate of a partition, written λ', is obtained by reflecting the Young diagram in the main diagonal, so that $(6, 4, 3, 1)' = (4, 3, 3, 2, 1, 1)$. The arm and leg lengths of a square $s = (i, j) \in \lambda$ are given by

$$a(s) := \lambda_i - j \quad \text{and} \quad l(s) := \lambda'_j - i,$$

which is the number of boxes strictly to the right or below s respectively. The hook length is the sum of these including s itself, so that $h(s) := a(s) + l(s) + 1$. Using the same example as above,



where we have marked the square $s = (2, 2)$ with $a(s) = 1$, $l(s) = 2$ and $h(s) = 4$. As in the introduction we denote the multiset of hook lengths of λ by $\mathcal{H}_\lambda$. This is further refined as $\mathcal{H}^{\mathrm{e}}_\lambda$ and $\mathcal{H}^{\mathrm{o}}_\lambda$, the collection of hook lengths which are even or odd respectively. In terms of these we define the hook polynomials

$$H_\lambda(q) := \prod_{h \in \mathcal{H}_\lambda} (1 - q^h)$$

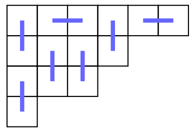
$$H^{\mathrm{e/o}}_\lambda(q) := \prod_{h \in \mathcal{H}^{\mathrm{e/o}}_\lambda} (1 - q^h),$$

which are invariant under conjugation of λ. For $z \in \mathbb{C}$ we also need the content polynomials

$$C_\lambda(z; q) := \prod_{(i,j) \in \lambda} (1 - z q^{j-i})$$

$$C^{\mathrm{e/o}}_\lambda(z; q) := \prod_{\substack{(i,j) \in \lambda \\ i + j \text{ even/odd}}} (1 - z q^{j-i}).$$

In this paper we will frequently encounter partitions with empty 2-core, written 2-core$(\lambda) = 0$. One definition of such partitions is that their diagrams may be tiled by dominoes. Our running example of $(6, 4, 3, 1)$ has empty 2-core since it admits the following tiling



We will now give some equivalent formulations of partitions with empty 2-core, which all easily follow by induction on $|\lambda|$.

Lemma 2.1. *For $\lambda \in \mathscr{P}_{2n}$ the following are equivalent:*

(1) $2\text{-core}(\lambda) = 0$.
(2) $|\mathcal{H}^{\mathrm{o}}_\lambda| = |\mathcal{H}^{\mathrm{e}}_\lambda| = n$.
(3) *For any integer m such that $2m \geqslant n$, the sequence*

$$\lambda_1 + 2m - 1, \lambda_2 + 2m - 2, \dots, \lambda_{2m-1} + 1, \lambda_{2m}$$

contains n even and n odd integers.

The equivalence of (1) and (3) is the content of [14, Lemma 2.1]. Claim (2) also follows from Littlewood's decomposition of a partition into r-cores and r-quotients in the case of empty 2-core [16].

2.2. **Auxiliary results.** Here we prove some properties of the statistic $b(\lambda)$. Firstly, as we have already used in the introduction, we have the following characterisation of the vanishing of $b(\lambda)$.

Lemma 2.2. *Let $2\text{-core}(\lambda) = 0$. Then $b(\lambda) \geqslant 0$ with $b(\lambda) = 0$ if and only if λ is even.*

Proof. If λ is even then $b(\lambda) = 0$ since the number of even/odd hook lengths in each row is equal. Assume that λ is not even. Then the number of odd λ_i is even. Let $\lambda_{i_1}, \lambda_{i_2}$ be the first two odd rows of λ. Since $2\text{-core}(\lambda) = 0$ these must be separated by an even number of even rows (possibly zero). Ignoring the remaining rows, the contribution to $b(\lambda)$ from the rows up to λ_{i_2} is thus

$$\lambda_{i_1} - \lambda_{i_2} + i_2 - i_1 + 2 \sum_{j=i_1+1}^{i_2-1} (-1)^{i_1+j-1} (\lambda_j - j).$$

Since the numbers $\lambda_i - i$ are strictly decreasing this sum must be nonnegative. In fact it is positive since $\lambda_{i_1} - \lambda_{i_2} + i_2 - i_1$ is positive. The next nonzero contribution to $b(\lambda)$ will come from the next pair of odd rows. Thus repeating the above shows us that if λ is not even then $b(\lambda) > 0$. $\square$

Note that $b((2,1,1,1)) = 0$, so that $b(\lambda)$ may vanish for partitions with nonempty 2-core.

Lemma 2.3. *For $\lambda \in \mathscr{P}_n$ there holds*

$$b(\lambda) = \sum_{(i,j) \in \lambda + \delta} (-1)^{\lambda_i - i - j + 1} (\lambda_i - i) - \sum_{1 \leqslant i < j \leqslant 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_i - i).$$

Moreover, if $2\text{-core}(\lambda) = 0$, then

$$b(\lambda') = \frac{|\lambda|}{2} - n^2 - n + \sum_{1 \leqslant i < j \leqslant 2n} (-1)^{\lambda_i - \lambda_j + j - i} (\lambda_j - j).$$

Proof. We interpret the definition of $b(\lambda)$ as a sum over the diagram of λ where each square has weight $(-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda_i-i)$. In the diagram of $\lambda+\delta$ place the integer $(-1)^{\lambda_i-i-j+1}(\lambda_i-i)$ in box (i,j). Summing over i,j gives the first sum on the right. To identify the second sum, we remove the columns with index $\lambda_j+2n-j+1$ for $2\leqslant j\leqslant 2n$ whose entries are $(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i)$. The remaining diagram is that of λ with entries $(-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda_i-i)$, which shows the first identity.

The proof of the second identity is similar. Note that $b(\lambda')$ may be written as

$$b(\lambda')=\sum_{(i,j)\in\lambda}(-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda'_j-j).$$

We thus fill the diagram of $\lambda+\delta$ with integers $(-1)^{\lambda_i-i-j+1}(2n-j)$, so that removing the same columns as before now gives

$$b(\lambda')=\sum_{(i,j)\in\lambda+\delta}(-1)^{\lambda_i-i-j+1}(2n-j)-\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}(j-\lambda_j-1).$$

A simple calculation shows that for 2-core$(\lambda)=0$,

$$\sum_{(i,j)\in\lambda+\delta}(-1)^{\lambda_i-i-j+1}(2n-j)+\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}=\frac{|\lambda|}{2}-n^2-n,$$

completing the proof. □

2.3. **Schur functions.**

For completeness we give a definition of the Schur functions in terms of the classical ratio of alternants. Let $\lambda\in\mathscr{P}_n$, then the Schur function is defined as

$$s_\lambda(x):=\frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{\lambda_j+n-j}\big)}{\det_{1\leqslant i,j\leqslant n}\big(x_i^{n-j}\big)},$$

and $s_\lambda(x):=0$ for $l(\lambda)>n$. The set of the $s_\lambda(x)$ indexed over $\mathscr{P}_n$ forms a $\mathbb{Z}$-basis for Λ_n, the ring of symmetric functions in n variables. We also use the Schur functions in countably many variables $x_1,x_2,x_3,\dots$, such as in Theorem 1.1. The set of such $s_\lambda(x)$ when indexed over all partitions λ form a $\mathbb{Z}$-basis for the ring Λ of symmetric functions in infinitely many variables [18].

Several of the results we need below are best stated in terms of Macdonald polynomials, which are a q,t-analogue of the Schur functions [18]. We simply note that the Macdonald polynomials $P_\lambda(x;q,t)$ are a basis for Λ_n with coefficients in $\mathbb{Q}(q,t)$ and reduce to the Schur functions when $q=t$, i.e., $P_\lambda(x;q,q)=s_\lambda(x)$.

2.4. **Koornwinder polynomials and integrals.**

The Koornwinder polynomials are a family of BC_n-symmetric functions depending on six parameters first introduced by Koornwinder [13] as a multivariate analogue of the Askey–Wilson polynomials [1]. In what follows $x=(x_1,\dots,x_n)$, $x^{\pm}=(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$ and for a single-variable function $g(x_i)$ we set

$$\begin{aligned}g\big(x_i^{\pm}\big)&:=g(x_i)g\big(x_i^{-1}\big)\\ g\big(x_i^{\pm}x_j^{\pm}\big)&:=g(x_ix_j)g\big(x_i^{-1}x_j\big)g\big(x_ix_j^{-1}\big)g\big(x_i^{-1}x_j^{-1}\big).\end{aligned}$$

Below the function will be one of $g(x_i) = (x_i; q)_\infty$ or $g(x_i) = (1 - x_i)$. Also for the infinite q-shifted factorial we adopt the usual multiplicative notation

$$(a_1, \dots, a_n; q)_\infty := (a_1; q)_\infty \cdots (a_n; q)_\infty.$$

Let $W := \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the group of signed permutations on n letters. A Laurent polynomial $f(x) \in \mathbb{C}[x^{\pm}]$ is called BC_n-symmetric if it is invariant under the natural action of W on the n variables where the reflections act by $x_i \mapsto 1/x_i$. For $\lambda \in \mathscr{P}_n$ define the orbit-sum indexed by λ as

$$m_\lambda^{\mathrm{BC}}(x) := \sum_\alpha x^\alpha,$$

where the sum is over all elements of the W-orbit of λ, the reflections act on sequences by $\alpha_i \mapsto -\alpha_i$, and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The orbit-sums form a basis for the ring Λ_n^{BC} of BC_n-symmetric functions. For $q, t, t_0, t_1, t_2, t_3 \in \mathbb{C}$ with $|q|, |t|, |t_0|, |t_1|, |t_2|, |t_3| < 1$, define the Koornwinder density by

$$\Delta(x; q, t; t_0, t_1, t_2, t_3) := \prod_{i=1}^n \frac{(x_i^{\pm 2}; q)_\infty}{\prod_{r=0}^3 (t_r x_i^{\pm}; q)_\infty} \prod_{1 \leqslant i < j \leqslant n} \frac{(x_i^{\pm} x_j^{\pm}; q)_\infty}{(t x_i^{\pm} x_j^{\pm}; q)_\infty}.$$

This further allows to define an inner product on Λ_n^{BC} by

$$\langle f, g \rangle_{q,t;t_0,t_1,t_2,t_3}^{(n)} := \int_{\mathbb{T}^n} f(x) g(x^{-1}) \Delta(x; q, t; t_0, t_1, t_2, t_3) \, \mathrm{d}T(x),$$

where $\mathbb{T}^n$ is the standard n-torus and the measure $T(x)$ is given by

$$\mathrm{d}T(x) := \frac{1}{2^n n! (2\pi \mathrm{i})^n} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

The Koornwinder polynomials are defined to be the unique BC_n-symmetric functions satisfying

$$K_\lambda = m_\lambda^{\mathrm{BC}} + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu^{\mathrm{BC}},$$

where $c_{\lambda\mu} \in \mathbb{C}(q, t, t_0, t_1, t_2, t_3)$, and for which

$$\langle K_\lambda, K_\mu \rangle_{q,t;t_0,t_1,t_2,t_3}^{(n)} = 0 \qquad \text{if } \lambda \neq \mu.$$

Note that $\mu \leqslant \lambda$ denotes the extension of the usual dominance order to all partitions $\lambda, \mu \in \mathscr{P}$: $\mu \leqslant \lambda$ if and only if $\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i$ for all $i \geqslant 1$. The Koornwinder polynomials satisfy many nice properties such as the quadratic norm evaluation and evaluation symmetry [4, 25]. The key identity we need is [24, Equation (2.6.9)] (see also [22, Corollary 7.2.1])

$$\begin{aligned} \lim_{m \to \infty} & (x_1 \dots x_n)^m K_{(m^n) - \lambda}(x; q, t; t_0, t_1, t_2, t_3) \\ &= P_\lambda(x; q, t) \prod_{i=1}^n \frac{(t_0 x_i, t_1 x_i, t_2 x_i, t_3 x_i; q)_\infty}{(x_i^2; q)_\infty} \prod_{1 \leqslant i < j \leqslant n} \frac{(t x_i x_j; q)_\infty}{(x_i x_j; q)_\infty}. \end{aligned} \tag{2.1}$$

We will only use this for $\lambda = 0$, in which case $P_0(x; q, t) = 1$.

For a basis $\{f_\lambda\}$ of Λ_n^{BC} we write $[f_\lambda]g$ for the coefficient of f_λ in the expansion $g = \sum_\lambda c_\lambda f_\lambda$ where the c_λ lie in some coefficient ring. The virtual Koornwinder integral of a BC_n-symmetric function f is defined as

$$I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) := [K_0(x; q, t; t_0, t_1, t_2, t_3)] f.$$

This is extended to allow for symmetric function arguments via the homomorphism $\Lambda_{2n} \longrightarrow \Lambda_n^{\mathrm{BC}}$ for which $f(x_1,\dots,x_n) \mapsto f(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$. Of course since $K_0 = 1$ the orthogonality of the Koornwinder polynomials allows us to express this as

$$I_K^{(n)}(f;q,t;t_0,t_1,t_2,t_3) = \frac{\langle f,1\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3}}{\langle 1,1\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3}}.$$

Note that the denominator has the explicit evaluation

$$\langle 1,1\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3} = \prod_{i=1}^{n} \frac{(t,t_0t_1t_2t_3t^{n+i-2};q)_\infty}{(q,t^i;q)_\infty \prod_{0\leqslant r<s\leqslant 3}(t_rt_st^{i-1};q)_\infty},$$

which is Gustafson's generalised Askey–Wilson integral [9]. The virtual Koornwinder integral can be evaluated for many choices of the argument f, see [14, 22, 23, 24]. In particular, the vanishing integrals of the next section may be expressed in terms of virtual Koornwinder integrals. We need one final identity involving virtual Koornwinder integrals [24, Proposition 4.2]. To state this conveniently, let

$$f_\lambda^{(m)}(q,t;t_0,t_1,t_2,t_3) := [P_\lambda(x;q,t)](x_1\cdots x_n)^m K_{(m^n)}(x;q,t;t_0,t_1,t_2,t_3).$$

Proposition 2.4. *For nonnegative integers n,m and $\lambda \subseteq (2m)^n$,*

$$f_\lambda^{(m)}(q,t;t_0,t_1,t_2,t_3) = (-1)^{|\lambda|} I_K^{(m)}\big(P_{\lambda'}(t,q);t,q;t_0,t_1,t_2,t_3\big).$$

3. Vanishing integrals

In this section we evaluate a pair of vanishing integrals for Schur functions conjectured by Lee, Rains and Warnaar in the Macdonald case [14, Conjecture 9.2].

For $a,b,q \in \mathbb{C}$ with $|a|,|b|,|q|<1$ we define

$$I_\lambda^{(n)}(a,b;q) := \frac{1}{Z_n(a,b;q)} \int_{\mathbb{T}^n} s_\lambda\big(x_1^\pm,\dots,x_n^\pm\big) \prod_{i=1}^{n} \frac{(x_i^{\pm 2};q)_\infty}{(ax_i^{\pm 2},bx_i^{\pm 2};q^2)_\infty} \times \prod_{1\leqslant i<j\leqslant n} \big(1-x_i^\pm x_j^\pm\big)\,\mathrm{d}T(x),$$

where λ is a partition with length at most $2n$ and the normalising factor is given by

$$\begin{aligned} Z_n(a,b;q) &:= \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{(x_i^{\pm 2};q)_\infty}{(ax_i^{\pm 2},bx_i^{\pm 2};q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \big(1-x_i^\pm x_j^\pm\big)\,\mathrm{d}T(x) \\ &= \prod_{i=1}^{n} \frac{(abq^{n+i-2};q)_\infty}{(q^i,-aq^{i-1},-bq^{i-1};q)_\infty (abq^{2i-2};q^2)_\infty^2}. \end{aligned}$$

Note that in terms of virtual Koornwinder integrals this is

$$I_\lambda^{(n)}(a,b;q) = I_K^{(n)}(s_\lambda;q,q,a^{1/2},-a^{1/2},b^{1/2},-b^{1/2}).$$

Lee, Rains and Warnaar prove the following properties of the above integral [14, Proposition 9.3].

Proposition 3.1. *For $a, b, q \in \mathbb{C}$ with $|a|, |b|, |q| < 1$ and λ a partition of length at most $2n$ the integral $I_\lambda^{(n)}(a, b; q)$ vanishes unless* 2-core$(\lambda) = 0$. *Furthermore*

$$\text{(3.1a)}\quad I_\lambda^{(n)}(q, q; q) = \prod_{i=1}^{n} \frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}} \times \operatorname*{Pf}_{1\leqslant i,j\leqslant 2n}\left(\frac{q^{(\lambda_i-\lambda_j+j-i-1)/2}}{1-q^{\lambda_i-\lambda_j+j-i}}\chi(\lambda_i-\lambda_j+j-i \text{ odd})\right),$$

and

$$\text{(3.1b)}\quad I_\lambda^{(n)}(1, q^2; q) = \frac{1}{2^{n-1}(1+q^n)} \prod_{i=1}^{n} \frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}} \times \operatorname*{Pf}_{1\leqslant i,j\leqslant 2n}\left(\frac{1+q^{\lambda_i-\lambda_j+j-i}}{1-q^{\lambda_i-\lambda_j+j-i}}\chi(\lambda_i-\lambda_j+j-i \text{ odd})\right).$$

Lee, Rains and Warnaar also give a conjectural Macdonald polynomial analogue of this proposition [14, Conjecture 9.2]. There the generalisations of (3.1) are explicit products. Our next proposition gives the evaluation of the Pfaffians in the previous proposition, verifying the conjecture of Lee, Rains and Warnaar for $q = t$.

Proposition 3.2. *For λ with length at most $2n$ and 2-core$(\lambda) = 0$,*

$$\text{(3.2)}\qquad I_\lambda^{(n)}(q, q; q) = q^{b(\lambda')} \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}$$

and

$$\text{(3.3)}\qquad I_\lambda^{(n)}(1, q^2; q) = q^{b(\lambda)} \frac{1+q^{n+2b(\lambda')-2b(\lambda)}}{1+q^n} \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}.$$

Proof. Since the structure of the Pfaffians is similar, we focus on the second identity, and evaluate (3.1b).

Fix a partition $\lambda \in \mathscr{P}_{2n}$ with empty 2-core. Define the set $J \subseteq \{1, \dots, 2n\}$ as the collection of integers j for which column j has a nonzero entry in the first row, and set $I := \{1, \dots, 2n\} \setminus J$. Since 2-core$(\lambda) = 0$ it follows that $|I| = |J| = n$. The elements of I and J are labeled by i_k and j_k respectively, where $1 \leqslant k \leqslant n$ and ordered naturally. With this established we define the $n \times n$ matrix

$$M_{k,\ell} := \frac{1+q^{\lambda_{i_k}-\lambda_{j_\ell}+j_\ell-i_k}}{1-q^{\lambda_{i_k}-\lambda_{j_\ell}+j_\ell-i_k}}.$$

The Pfaffian in (3.1b) may be expressed in terms of the determinant of M. Indeed, by rearranging rows and columns we see that

$$\begin{aligned}\operatorname*{Pf}_{1\leqslant i,j\leqslant 2n}&\left(\frac{1+q^{\lambda_i-\lambda_j+j-i}}{1-q^{\lambda_i-\lambda_j+j-i}}\chi(\lambda_i-\lambda_j+j-i \text{ odd})\right)\\ &= (-1)^{\binom{n+1}{2}+\sum_{j\in J} j} \operatorname{Pf}\begin{pmatrix} 0 & -M^t \\ M & 0 \end{pmatrix}\\ &= (-1)^{n+\sum_{j\in J} j} \det M.\end{aligned}$$

The determinant may be simply evaluated by applying a generalisation of Cauchy's double alternant which may be found in [5, Example 3.1; $a=0$]:

$$\det_{1\leqslant i,j\leqslant n}\left(\frac{bx_i+cy_j}{x_i+y_j}\right)=(b-c)^{n-1}\left(b\prod_{i=1}^n x_i+(-1)^{n-1}c\prod_{i=1}^n y_i\right)\\ \times\frac{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)}{\prod_{i,j=1}^n(x_i+y_j)}.$$

We wish apply this with $(b,c,x_i,y_i)\mapsto(1,-1,q^{\lambda_i-i},-q^{\lambda_i-i})$ for $1\leqslant i\leqslant n$. After doing so and cancelling minus signs, the evaluation may be expressed as

$$I_\lambda^{(n)}(1,q^2;q)\\ =\frac{\prod_{i\in I}q^{\lambda_i-i}+\prod_{j\in J}q^{\lambda_j-j}}{1+q^n}\prod_{i=1}^n\frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}}\\ \times\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \lambda_i-\lambda_j+j-i\text{ even}}}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{q^{\lambda_j-j}}\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \lambda_i-\lambda_j+j-i\text{ odd}}}\frac{q^{\lambda_j-j}}{1-q^{\lambda_i-\lambda_j+j-i}}.$$

The terms of the form $1-q^x$ can be simplified thanks to the identity [18, p. 10–11]

$$\frac{C_\lambda(q^{2n};q)}{H_\lambda(q)}=\prod_{s\in\lambda}\frac{1-q^{n+c(s)}}{1-q^{h(s)}}=\frac{\prod_{1\leqslant i<j\leqslant n}1-q^{\lambda_i-\lambda_j+j-i}}{\prod_{i=1}^n(q;q)_i},$$

where $l(\lambda)\leqslant n$. Restricting all products to even/odd exponents implies that

$$\frac{C_\lambda^{\mathrm{e}}(q^{2n};q)H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n};q)H_\lambda^{\mathrm{e}}(q)}\\ =\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \lambda_i-\lambda_j+j-i\text{ even}}}(1-q^{\lambda_i-\lambda_j+j-i})\prod_{\substack{1\leqslant i<j\leqslant 2n\\ \lambda_i-\lambda_j+j-i\text{ odd}}}\frac{1}{1-q^{\lambda_i-\lambda_j+j-i}}\\ \times\prod_{i=1}^n\frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}}.$$

It remains to show that the powers of q agree in the prefactor. Since

$$\prod_{i\in I}q^{\lambda_i-i}+\prod_{j\in J}q^{\lambda_j-j}=\prod_{\substack{i=1\\ \lambda_i-i\text{ even}}}^{2n}q^{\lambda_i-i}+\prod_{\substack{i=1\\ \lambda_i-i\text{ odd}}}^{2n}q^{\lambda_i-i},$$

this may be reduced to the pair of identities

$$b(\lambda)=\sum_{\substack{i=1\\ \lambda_i-i\text{ even}}}^{2n}(\lambda_i-i)+\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_j-j),$$

$$n+2b(\lambda')-2b(\lambda)=\sum_{\substack{i=1\\ \lambda_i-i\text{ odd}}}^{2n}(\lambda_i-i)-\sum_{\substack{i=1\\ \lambda_i-i\text{ even}}}^{2n}(\lambda_i-i).$$

In the first of these write

$$\sum_{\substack{i=1\\ \lambda_i-i\text{ even}}}^{2n}(\lambda_i-i)=\sum_{(i,j)\in\lambda+\delta}(-1)^{\lambda_i-i-j+1}(\lambda_i-i)+\sum_{i=1}^{2n}(\lambda_i-i)$$
$$=b(\lambda)+\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i)+\sum_{i=1}^{2n}(\lambda_i-i),$$

where in the second equality we have applied Lemma 2.3. Since

$$\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i)+\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i)+\sum_{i=1}^{2n}(\lambda_i-i)$$
$$=\sum_{i,j=1}^{2n}(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i)$$
$$=0,$$

the first identity follows. For the second identity, a similar rewriting using Lemma 2.3 shows us that

$$\sum_{\substack{i=1\\ \lambda_i-i\text{ odd}}}^{2n}(\lambda_i-i)-\sum_{\substack{i=1\\ \lambda_i-i\text{ even}}}^{2n}(\lambda_i-i)$$
$$=-2\sum_{(i,j)\in\lambda+\delta}(-1)^{\lambda_i-i-j+1}(\lambda_i-i)-\sum_{i=1}^{2n}(\lambda_i-i)$$
$$=-2b(\lambda)-|\lambda|+2n^2+n-2\sum_{1\leqslant i<j\leqslant 2n}(-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i)$$
$$=n+2b(\lambda')-2b(\lambda).$$

This finishes the evaluation of (3.1b). The evaluation of (3.1a) is almost identical except one directly applies the second identity of Lemma 2.3 to compute the exponent of q in the prefactor. □

4. Bounded Littlewood identities

Here we use the integral evaluations of the previous section to prove a bounded analogue of Theorem 1.1. This is followed by proofs of the theorem and Corollary 1.2.

4.1. **A bounded analogue of Theorem 1.1.** Bounded Littlewood identities are generalisations of ordinary Littlewood identities in which the largest part of the indexing partition has an upper bound, say m, such that sending m to infinity recovers an ordinary (unbounded) Littlewood identity. The first example of such an identity was discovered by Macdonald [17] where he used a bounded analogue of (1.1a) to prove the MacMahon and Bender–Knuth conjectures on plane partitions [2, 19]. Bounded analogues of the remaining two classical identities (1.1b) and (1.1c) were obtained by Désarménien, Proctor and Stembridge [7, 21, 27] and Okada [20] respectively. A host of other bounded identities for Hall–Littlewood and Macdonald polynomials may be found in [24] and references therein. We now state the bounded analogue of Theorem 1.1.

Theorem 4.1. *For m, n nonnegative integers,*

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda')} \frac{C^{\mathrm{e}}_{\lambda}(q^{-2m};q) H^{\mathrm{o}}_{\lambda}(q)}{C^{\mathrm{o}}_{\lambda}(q^{-2m};q) H^{\mathrm{e}}_{\lambda}(q)} s_\lambda(x) = (x_1 \cdots x_n)^m K_{(m^n)}\big(x;q,q;q^{1/2},-q^{1/2},q^{1/2},-q^{1/2}\big) \tag{4.1}$$

and

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} \frac{q^{2b(\lambda')-b(\lambda)}+q^{m+b(\lambda)}}{1+q^m} \frac{C^{\mathrm{e}}_{\lambda}(q^{-2m};q) H^{\mathrm{o}}_{\lambda}(q)}{C^{\mathrm{o}}_{\lambda}(q^{-2m};q) H^{\mathrm{e}}_{\lambda}(q)} s_\lambda(x) = (x_1 \cdots x_n)^m K_{(m^n)}(x;q,q;1,-1,q,-q). \tag{4.2}$$

These identities are indeed bounded since $C^{\mathrm{e}}_{\lambda}(q^{-2m};q)$ vanishes if $\lambda_1 > 2m$. Since, by [14, Lemma 4.1], the Koornwinder polynomials on the right reduce to classical group characters for $q=0$, one recovers the previously mentioned Désarménien–Proctor–Stembridge and Okada identities respectively in this case. The Koornwinder polynomials for $q=t$ on the right-hand side may alternatively be expressed as a ratio of determinants of Askey–Wilson polynomials [1]; see e.g. [6, Definition 4.1]. This, however, does not seem to shed light on a more explicit expression for the evaluation of these sums. In particular, the specialisations of $K_{(m^n)}$ above are not contained in [14, Lemma 4.1].

The following argument is sketched in [14, §9], but we give the details in the Schur case. Assuming the Macdonald polynomial version of the vanishing integrals [14, Conjecture 9.2], the same argument gives the conjectural Littlewood identities.

Proof of Theorem 4.1. The goal is to find an expression for the coefficient of $s_\lambda(x)$ in the Schur expansion of the right-hand side. By Proposition 2.4 this coefficient is

$$f^{(m)}_{\lambda}(x;q,q,t_0,t_1,t_2,t_3) = I^{(m)}_{K}(s_{\lambda'}(x);q,q;t_0,t_1,t_2,t_3).$$

If we specialise $(t_0,t_1,t_2,t_3)=(q^{1/2},-q^{1/2},q^{1/2},-q^{1/2})$ then this reduces to

$$f^{(m)}_{\lambda}\big(x;q,q;q^{1/2},-q^{1/2},q^{1/2},-q^{1/2}\big) = (-1)^{|\lambda|} I^{(m)}_{\lambda'}(q,q;q).$$

The integral on the right is (3.2), as desired, and vanishes unless 2-core$(\lambda)=0$. In this case the sign cancels since $|\lambda|$ is even and we obtain

$$(-1)^{|\lambda|} I^{(m)}_{\lambda'}(q,q;q) = q^{b(\lambda)} \frac{C^{\mathrm{e}}_{\lambda'}(q^{2m};q) H^{\mathrm{o}}_{\lambda'}(q)}{C^{\mathrm{o}}_{\lambda'}(q^{2m};q) H^{\mathrm{e}}_{\lambda'}(q)}.$$

By [14, Lemma 2.3] we may alternatively express this as

$$q^{b(\lambda)} \frac{C^{\mathrm{e}}_{\lambda'}(q^{2m};q) H^{\mathrm{o}}_{\lambda'}(q)}{C^{\mathrm{o}}_{\lambda'}(q^{2m};q) H^{\mathrm{e}}_{\lambda'}(q)} = q^{b(\lambda')} \frac{C^{\mathrm{e}}_{\lambda}(q^{-2m};q) H^{\mathrm{o}}_{\lambda}(q)}{C^{\mathrm{o}}_{\lambda}(q^{-2m};q) H^{\mathrm{e}}_{\lambda}(q)} \tag{4.3}$$

This establishes (4.1). For (4.2) the same procedure applies with the substitution $(t_0,t_1,t_2,t_3)=(1,-1,q,-q)$ and by using the integral (3.3). □

4.2. **Proof of Theorem 1.1.** With the bounded identities established we may take the $m \to \infty$ limit of both identities to obtain their unbounded counterparts. For the Koornwinder side we use (2.1) with $(\lambda, q, t) = (0, q, q)$ and $(t_0, t_1, t_2, t_3) = (q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2})$ or $(t_0, t_1, t_2, t_3) = (1, -1, q, -q)$. In the case of (4.1) this yields

$$\begin{aligned}\lim_{m\to\infty} &(x_1 \dots x_n)^m K_{(m^n)}\big(x; q, q; q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2}\big) \\ &= \prod_{i=1}^{n} \frac{(q^{1/2}x_i, -q^{1/2}x_i, q^{1/2}x_i, -q^{1/2}x_i; q)_\infty}{(x_i^2; q)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1 - x_i x_j} \\ &= \prod_{i=1}^{n} \frac{(qx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1 - x_i x_j},\end{aligned}$$

where we have used

$$(a, -a; q)_\infty = (a^2; q^2)_\infty.$$

For the limit of the summand we use it in conjugate form (4.3) so that

$$\lim_{m\to\infty} q^{b(\lambda)} \frac{C^{\mathrm{e}}_{\lambda'}(q^{2m}; q) H^{\mathrm{o}}_\lambda(q)}{C^{\mathrm{o}}_{\lambda'}(q^{2m}; q) H^{\mathrm{e}}_\lambda(q)} = q^{b(\lambda)} \frac{H^{\mathrm{o}}_\lambda(q)}{H^{\mathrm{e}}_\lambda(q)}.$$

Thus we have proved (1.2). As before the same procedure yields (1.3).

4.3. **Proof of Corollary 1.2.** In order to obtain Corollary 1.2 we take $q \to 1$ in either (1.2) or (1.3). Let $(a; q)_n = \prod_{k=0}^{n}(1 - aq^k)$. Then we may take the limit of the product-side of (1.2) by using

$$\begin{aligned}\lim_{q\to 1} \frac{(qx_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} &= \lim_{q\to 1} \sum_{n=0}^{\infty} \frac{(q; q^2)_n}{(q^2; q^2)_n} x_i^{2n} \\ &= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} x_i^{2n} \\ &= \frac{1}{(1 - x_i^2)^{1/2}},\end{aligned}$$

where in the first line we have applied the q-binomial theorem [8, Equation (1.3.2)]

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}.$$

The $q \to 1$ limit of the product-side of (1.3) gives the same result. The limit of either sum follows from the characterisation of partitions with empty 2-core in Lemma 2.1, namely that $|\mathcal{H}^{\mathrm{e}}_\lambda| = |\mathcal{H}^{\mathrm{o}}_\lambda|$.

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References

[1] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalise Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 219, iv+55.

[2] E. A. Bender and D. E. Knuth, *Enumeration of plane partitions*, J. Combin. Theory Ser. A **13** (1972), 40–54.

[3] D. M. Bressoud, *Elementary proofs of identities for Schur functions and plane partitions*, Ramanujan J. **4** (2000), no. 1, 6980.

[4] J. F. van Diejen, *Self-dual Koornwinder–Macdonald polynomials*, Invent. math. **126** (1996), 319–339.

[5] W. Chu, *Generalizations of the Cauchy determinant*, Publ. Math. Debrecen **58** (2001), no. 3, 353–365.

[6] S. Corteel, O. Mandelshtam, L. Williams, *Combinatorics of the two-species ASEP and Koornwinder moments*, Adv. Math. **321** (2017), 160–204.

[7] J. Désarménien, *La démonstration des identités de Gordon et MacMahon et de deux identités nouvelles*, Sém Lothar. Combin. **15** (1986), Art. B15a, 11pp.

[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia of Mathematics and its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.

[9] R. A. Gustafson, *A generalization of Selberg's beta integral*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 97–105.

[10] M. Ishikawa and M. Wakayama, *Applications of minor-summation formula II. Pfaffians and Schur polynomials*, J. Combin. Theory Ser. A **88** (1999), 136–157.

[11] F. Jouhet and J. Zeng, *Some new identities for Schur functions*, Adv. Appl. Math. **27** (2001), 493–509.

[12] N. Kawanaka, *A q-series identity involving Schur functions and related topics*, Osaka J. Math. **36** (1999), 157–176.

[13] T. H. Koornwinder, *Askey–Wilson polynomials for root systems of type BC*, in Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (Tampa, FL, 1991), Contemp. Math. **138**, Amer Math. Soc., Providence, RI, 1992, 189–204.

[14] C.-h. Lee, E. M. Rains and S. O. Warnaar, *An elliptic hypergeometric function approach to branching rules*, SIGMA **16** (2020), paper 142, 52 pp.

[15] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press, New York, 1940.

[16] D. E. Littlewood, *Modular representations of symmetric groups*, Proc. Roy. Soc. London. Ser. A **209** (1951), 333–353.

[17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, The Clarendon Press, Oxford University Press, New York, 1979.

[18] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., The Clarendon Press, Oxford University Press, New York, 1995.

[19] P. A. MacMahon, *Partitions of numbers whose graphs possess symmetry*, Trans. Cambridge Phil. Soc. **17** (1899), 149–170.

[20] S. Okada, *Applications of minor summation formulas to rectangular-shaped representations of classical groups*, J. Algebra **205** (1998), no. 2, 337–367,

[21] R. A. Proctor, *New symmetric plane partition identities from invariant theory work of De Concini and Procesi*, European J. Combin. **11** (1990), no. 3, 289–300.

[22] E. M. Rains, BC_n*-symmetric polynomials*, Transform. Groups **10** (2005), no. 1, 63–132.

[23] E. M. Rains and M. J. Vazirani, *Quadratic transformations of Macdonald and Koornwinder polynomials* Transform. Groups **12** (2007), no. 4, 725–759.

[24] E. M. Rains and S. O. Warnaar, *Bounded Littlewood identities*, Mem. Amer. Math. Soc. **270** (2021), no. 1317, vii+115 pp.

[25] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. Math. **150** (1999), 267–282.

[26] I. Schur, *Aufgabe 569*, Arch. Math. Phys. **27** (3) (1918), 163; Ges. Abhandlungen, Vol. 3, p. 456.

[27] J. R. Stembridge, *Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 469–498.

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090, Vienna, Austria

Email address: seamus.albion@univie.ac.at