The Selberg integral and Macdonald polynomials

Seamus Albion
The University of Queensland

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This talk is about...

1. Generalisations of the Selberg integral, itself a k-dimensional generalisation of Euler's beta integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

2. Summation formulae for Macdonald polynomials, symmetric functions which are q, t-deformations of the Schur functions.

3. How to get from one to the other!

Selberg's integral and $\mathfrak{sl}_2$

Writing $\boldsymbol{t} := (t_1, \dots, t_k)$, Selberg's formula states

$$\begin{aligned} S_k(\alpha, \beta; \gamma) &:= \int_{[0,1]^k} |\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\, \mathrm{d}\boldsymbol{t} \\ &= \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(k+i-2)\gamma)\Gamma(1+\gamma)} \end{aligned}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and

$$\mathrm{Re}(\gamma) > -\min\{1/k, \mathrm{Re}(\alpha)/(k-1), \mathrm{Re}(\beta)/(k-1)\}.$$

Here $\Delta(\boldsymbol{t})$ denotes the (type A) Vandermonde product

$$\Delta(\boldsymbol{t}) = \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j).$$

This leads to a natural and fruitful association with A_{k-1} (Macdonald conjectures).

The Selberg integral has played a role in analytic number theory, random matrix theory, and conformal field theory.

First we will describe a connection with representation theory, associating $S_k(\alpha, \beta; \gamma)$ to $\mathfrak{sl}_2$ (A_1).

Let V_λ, V_μ be two irreducible $\mathfrak{sl}_2$-modules with highest weights λ, μ and $\Omega \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ the Casimir element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h.$$

Then we may state the Knizhnik–Zamolodchikov (KZ) equations as

$$\frac{\partial u}{\partial z} = \gamma \frac{\Omega}{z - w} u,$$
$$\frac{\partial u}{\partial w} = \gamma \frac{\Omega}{w - z} u,$$

where $\gamma \in \mathbb{C}$ and $u(z, w)$ is a function of the form

$$u : \mathbb{C}^2 \longrightarrow V_\lambda \otimes V_\mu.$$

If $u(z,w)$ takes values in the space of singular vectors of weight $\lambda+\mu-2k$ then Schechtman and Varchenko (1991) showed that

$$u(z,w) = \sum_{r=0}^{k} u_r(z,w)(f^i v_\lambda \otimes f^{r-i} v_\mu)$$

where the coordinate functions $u_r(z,w)$ are given by

$$u_r(z,w) := (z-w)^{\lambda\mu\gamma} \int_C A_r(z,w;\boldsymbol{t}) \Delta^{2\gamma}(\boldsymbol{t}) \prod_{i=1}^{k} (t_i - z)^{-\lambda\gamma} (t_i - w)^{-\mu\gamma} \, \mathrm{d}\boldsymbol{t},$$

and $A_r(z,w;\boldsymbol{t})$ is some explicitly known rational function. The domain of integration is the simplex

$$C = \{\boldsymbol{t} \in \mathbb{R}^k : z \leqslant t_k \leqslant \cdots \leqslant t_1 \leqslant w\}.$$

In general it is not known how to compute the above integral. However, when $r = 0$ the evaluation follows from the Selberg integral

$$u_0(z, w) = \frac{(-1)^a (z-w)^b}{k!} S_k(1-\lambda\gamma, -\mu\gamma; \gamma),$$

for some constants a, b involving k, λ, μ, γ. For $z = 0, w = 1$ this is simply the Selberg integral.

The previous derivation of solutions $u(z, w)$ to the KZ equations may be generalised to an arbitrary simple Lie algebra $\mathfrak{g}$ of rank n. Letting $\alpha_1, \ldots, \alpha_n$ denote the simple roots then the coordinate functions for $z = 0, w = 1$ now involve the scaled master function

$$\Phi(\boldsymbol{t}) = \prod_{i=1}^{k} t_i^{(\lambda, \alpha_{t_i})} (1-t_i)^{(\mu, \alpha_{t_i})} \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{(\alpha_{t_i}, \alpha_{t_j})}$$

where $k := k_1 + \cdots + k_n$ and for each $1 \leqslant r \leqslant n$ we define $\alpha_{t_i} := \alpha_r$ if

$$k_1 + \cdots + k_{r-1} < i \leqslant k_1 + \cdots + k_r.$$

For $\mathfrak{g} = \mathfrak{sl}_2$ the scaled master function is the integrand of the Selberg integral. This motivated Mukhin and Varchenko (2000) to make the following remarkable conjecture.

Conjecture (Mukhin & Varchenko)

If the space of singular vectors of weight $\lambda + \mu - \sum_{i=1}^{n} k_i \alpha_i$ is one-dimensional then

$$\int_C |\Phi(\boldsymbol{t})|^\gamma = \text{product of gamma functions.}$$

Neither the domain of integration C nor the form the product of gamma functions should take is specified by the conjecture.

The first nontrivial case of the conjecture was resolved by Tarasov and Varchenko (2003) who dealt with $\mathfrak{g} = \mathfrak{sl}_3$. A uniform approach for $\mathfrak{sl}_n$ was developed by Warnaar (2009) using Macdonald polynomial theory.

Symmetric functions

Let Λ denote the ring of symmetric functions over $\mathbb{Q}$ in countably many indeterminates $X = (x_1, x_2, \ldots)$. An algebraic basis for Λ is given by the power sums

$$p_r(X) = x_1^r + x_2^r + x_3^r + \cdots, \quad r \in \mathbb{Z}_{>0},$$

and $p_0(X) := 1$. This is extended to partitions by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}.$$

One defines a scalar product (the Hall scalar product) on Λ by demanding that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda.$$

This is equivalent to the Cauchy identity

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(X) p_\lambda(Y) = \prod_{x \in X} \prod_{y \in Y} \frac{1}{1 - xy}.$$

Another important basis for Λ are the Schur functions, most simply defined in the case of n indeterminates as a ratio of determinants

$$s_\lambda(x_1,\ldots,x_n)=\frac{\det(x_j^{\lambda_i+n-i})_{1\leqslant i,j\leqslant n}}{\det(x_j^{n-i})_{1\leqslant i,j\leqslant n}},$$

and their definition may be extended to the case of countably many indeterminates.

Like the power sums they are orthogonal (indeed orthonormal) under the Hall scalar product

$$\langle s_\lambda, s_\mu\rangle=\delta_{\lambda\mu},$$

and so also satisfy a Cauchy identity

$$\sum_\lambda s_\lambda(X)s_\lambda(Y)=\prod_{x\in X}\prod_{y\in Y}\frac{1}{1-xy}.$$

Macdonald polynomials

In the late 1980s Macdonald introduced a new basis for Λ over the field $\mathbb{Q}(q, t)$ which is unfortunately difficult to define. Instead we will discuss some fundamental properties of this basis.

Define the q, t-Hall scalar product on $\Lambda_{\mathbb{Q}(q,t)}$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i \geqslant 1} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$

The Macdonald polynomials $P_\lambda(X; q, t)$ are orthogonal under this scalar product

$$\langle P_\lambda, P_\mu \rangle_{q,t} = \delta_{\lambda\mu} \frac{c'_\lambda(q, t)}{c_\lambda(q, t)},$$

where $c_\lambda(q, t)$ and $c'_\lambda(q, t)$ are generalised hook polynomials.
If we set $q = t$ then we recover the Schur functions

$$s_\lambda(X) = P_\lambda(X; q, q).$$

For $n \in \mathbb{N} \cup \{\infty\}$ define the q-shifted factorial by

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

As with the power sums and Schur functions, the orthogonality of the Macdonald polynomials implies they also satisfy a Cauchy identity

$$\sum_{\lambda} \frac{c'_\lambda(q,t)}{c_\lambda(q,t)} P_\lambda(X) P_\lambda(Y) = \prod_{x \in X} \prod_{y \in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty}.$$

Upon suitable specialisation, the above identity simplifies dramatically to the well-known q-binomial theorem

$$\sum_{r=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

Taking appropriate definitions of the q-integral and q-gamma function, this identity is a q-analogue of the beta integral.

Getting back to Selberg

In order to obtain the Selberg integral a similar (but much more complicated) procedure may be carried out

$$\begin{tikzcd}
\text{Cauchy identity for Macdonald polynomials} \arrow[d, "\text{Specialisation \& } t = q^{\gamma}"] \\
\text{“Multidimensional } q\text{-Selberg integral”} \arrow[d, "q \to 1"] \\
\text{The Selberg integral } S_k(\alpha, \beta; \gamma)
\end{tikzcd}$$

Kadell's integral

Define the Jack polynomials by

$$P_\lambda^{(1/\gamma)}(X) := \lim_{q\to 1} P_\lambda(X;q,q^\gamma).$$

Then Macdonald conjectured and Kadell proved

$$\int_{[0,1]^k} P_\lambda^{(1/\gamma)}(\boldsymbol{t})|\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^k t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}\boldsymbol{t}$$
$$= P_\lambda^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}}) \prod_{i=1}^k \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)}.$$

If $\lambda = (1^r)$ then the integral is a symmetrised version of Aomoto's integral.

The Hua–Kadell integral

In 1993, for $\beta = \gamma$ Kadell (and for much earlier for $\gamma = 1$, Hua) extended this by adding a second Jack polynomial

$$\begin{aligned}
&\int_{[0,1]^n} P_\lambda^{(1/\gamma)}(\boldsymbol{t}) P_\mu^{(1/\gamma)}(\boldsymbol{t}) |\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\gamma-1}\,\mathrm{d}\boldsymbol{t} \\
&\quad = P_\lambda^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}}) P_\mu^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}}) \\
&\qquad \times \prod_{i=1}^{k} \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\gamma+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\gamma+(2k-\ell-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)} \\
&\qquad \times \prod_{i,j=1}^{k} \frac{\Gamma(\alpha+\gamma+(2k-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma(\alpha+\gamma+(2k-i-j)\gamma+\lambda_i+\mu_j)}.
\end{aligned}$$

The AGT Conjecture

In 2010, Alday, Gaiotto, and Tachikawa conjectured a relationship between conformal blocks in Liouville field theory and $\mathcal{N}=2$ supersymmetric gauge theory.

For SU(2) this was verified by Alba, Fateev, Litvinov, and Tarnopolskiy (2010), who required a Selberg integral over a pair of Jack polynomials which removes the restriction $\beta=\gamma$ coming from the Hua–Kadell integral.

Further, Matsuo and Zhang (2011) showed that a verification of the SU(n) AGT conjecture would require an $\mathfrak{sl}_n$ Selberg integral generalising Warnaar's $\mathfrak{sl}_n$ Kadell integral.

The AFLT integral

In order to state the AFLT integral we require a little plethystic notation. Let $X = (x_1, x_2, x_3, \ldots)$ and $Y = (y_1, y_2, y_3, \ldots)$ be countable alphabets. Then we may define symmetric functions on the sum/difference of alphabets using the power sums

$$p_r[X+Y] := p_r[X] + p_r[Y], \quad p_r[X-Y] := p_r[X] - p_r[Y],$$

and extending to an arbitrary $f \in \Lambda$ using the fact that the p_r generate Λ as a $\mathbb{Q}(q,t)$-algebra.

In particular for k a positive integer

$$p_r[kX] = p_r[\underbrace{X + \cdots + X}_{k \text{ times}}] = k\, p_r[X].$$

This may be extended to any $z \in \mathbb{C}$ by

$$p_r[zX] = z\, p_r[X].$$

Note that we write

$$p_r[X+z] = p_r[X] + z, \quad f[\underbrace{1 + 1 + \cdots + 1}_{k times}] = f[k].$$

The AFLT integral may therefore be stated as

$$\begin{aligned}
&\int_{[0,1]^k} P_\lambda^{(1/\gamma)}(\boldsymbol{t}) P_\mu^{(1/\gamma)}[\boldsymbol{t}+\beta/\gamma-1] |\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^k t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}\boldsymbol{t} \\
&\quad = P_\lambda^{(1/\gamma)}[k] P_\mu^{(1/\gamma)}[k+\beta/\gamma-1] \\
&\qquad \times \prod_{i=1}^k \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-\ell-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)} \\
&\qquad \times \prod_{i=1}^k \prod_{j=1}^\ell \frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\lambda_i+\mu_j)},
\end{aligned}$$

where ℓ is an arbitrary integer such that $\ell(\mu) \leqslant \ell$.

Using the full Cauchy identity for Macdonald polynomials it is possible to prove this formula in full generality. In doing so one also obtains a q-AFLT integral.

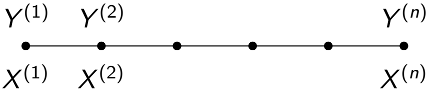
Higher rank Selberg integrals

We think of the Cauchy identity

$$\sum_{\lambda} \frac{c'_\lambda(q,t)}{c_\lambda(q,t)} P_\lambda(X;q,t) P_\lambda(Y;q,t) = \prod_{x\in X}\prod_{y\in Y} \frac{(txy;q)_\infty}{(xy;q)_\infty}$$

as associated to $\mathfrak{sl}_2$, where X and Y are attached to the single node of the Dynkin diagram:

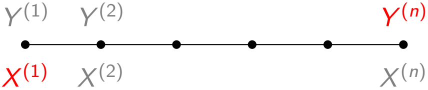
From this interpretation it is natural to consider an extension to $\mathfrak{sl}_{n+1}$ as



The corresponding sum is of the form

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{r=1}^{n} \frac{c'_{\lambda^{(r)}}(q,t)}{c_{\lambda^{(r)}}(q,t)} P_{\lambda^{(r)}}\big(X^{(r)}\big) P_{\lambda^{(r)}}\big(Y^{(r)}\big) \prod_{r=1}^{n-1} f^{(r)}_{\lambda^{(r)},\lambda^{(r+1)}}(q,t),$$

where $f^{(r)}$ is some function representing the edges of the Dynkin diagram. This evaluates in closed form in the case



where all but $X^{(1)}$ and $Y^{(n)}$ are specialised.

Such $\mathfrak{sl}_{n+1}$ Cauchy identities allows the extension of the AFLT integral to $\mathfrak{sl}_{n+1}$, which generalises previous results of Warnaar.

Unfortunately the previous Cauchy identity is not enough for the SU(n) AGT conjecture, which requires a Selberg integral of the form

$$\begin{aligned}&\mathcal{I}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma)\\&\qquad:=\int_C P^{(1/\gamma)}_{\lambda^{(1)}}[\boldsymbol{t}^{(1)}]\cdots P^{(1/\gamma)}_{\lambda^{(n+1)}}[\boldsymbol{t}^{(n)}+\beta/\gamma-1]|\Phi(\boldsymbol{t}^{(1)},\dots,\boldsymbol{t}^{(n)})|^{\gamma}\,\mathrm{d}\boldsymbol{t}.\end{aligned}$$

For $\gamma=1$, the Schur case, although the $\mathfrak{sl}_{n+1}$ Selberg integral diverges, it is possible to make sense of the Selberg average

$$\Big\langle s_{\lambda^{(1)}}[\boldsymbol{t}^{(1)}]\cdots s_{\lambda^{(n+1)}}[\boldsymbol{t}^{(n)}+\beta-1]\Big\rangle:=\lim_{\gamma\to 1}\frac{\mathcal{I}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma)}{\mathcal{I}_{0,\dots,0}(\alpha_1,\dots,\alpha_n,\beta;\gamma)},$$

where the denominator on the right is simply the $\mathfrak{sl}_{n+1}$ Selberg integral.

The Selberg average (conjecturally for $n>2$) evaluates as a product of gamma functions. For $n=1$ this is the AFLT integral. For $n=2$ the result follows from the inverse Pieri rule and some (very) complicated rational function identities.

The end