From symmetric functions to hypergeometric integrals

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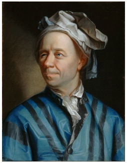
What this talk is about



Euler

Hypergeometric integrals

Selberg

&

Jack

Symmetric functions

Macdonald

Hypergeometry

We call a series $\sum_k c_k$ hypergeometric if the ratio of consecutive terms is a rational function of the index. Examples include your favourite Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1},$$

which have termwise ratios

$$k+1, \qquad -\frac{(2k+1)x^2}{2k+3}.$$

Of course the series may be finite

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \qquad \frac{c_{k+1}}{c_k} = \frac{(n-k)x}{k+1}.$$

The most important classical hypergeometric series is Gauss' hypergeometric function

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, $$

where $(a)_k$ is the Pochhammer symbol

$$ (a)_k = (a)(a+1)\cdots(a+k-1). $$

This has an equivalent expression as an integral due to Euler

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,\mathrm{d}t, $$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. This involves the gamma function

$$ \Gamma(x) = \int_0^{\infty} t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t, \quad \mathrm{Re}(x) > 0. $$

These are both examples of hypergeometric integrals! But they're not the one we're looking for.

Sending $x \to 1^-$ Gauss was able to sum his series representation, giving

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

where $\mathrm{Re}(c-a-b) > 0$.

Equating Gauss' and Euler's expressions and doing some rearranging tells us

$$ \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}. $$

This is the beta integral, due to Euler,

$$ \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, $$

with $\alpha = b$ and $\beta = c - a - b$.

The three levels

Elliptic hypergeometric series — (p, q)

c_{k+1}/c_k is an elliptic function of k

$\downarrow$ $p \to 0$

Basic hypergeometric series — (q)

c_{k+1}/c_k is a rational function of q^k

$\downarrow$ $q \to 1$

Hypergeometric series

c_{k+1}/c_k is a rational function of k

Level two: basic hypergeometric series

Let $p(n)$ denote the number of partitions of a nonnegative integer n, i.e., the number of ways of writing n as a sequence of nonnegative integers $\lambda_1, \lambda_2, \ldots$ such that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

and

$$|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots = n.$$

The study of q-series was initiated by Euler who gave his famous generating function for $p(n)$

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{i \geq 0} \frac{1}{1-q^i}.$$

We may write this succinctly in q-series notation. For $n \in \mathbb{N} \cup \{\infty\}$ define the q-Pochhammer symbol (or q-shifted factorial) by

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Hence

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

The study of q-series was invigorated by Heine who defined a q-analogue of Gauss' ${}_2F_1$

$$
{}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; x, q\right] := \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{x^k}{(q;q)_k}.
$$

He proved a transformation formula for this series that is a q-analogue of Euler's integral formula for the ${}_2F_1$.

This motivated Thomae (and later Jackson) to define the q-integral

$$
\int_0^1 f(t)\, \mathrm{d}_q t = (1-q) \sum_{k=0}^{\infty} f(q^k) q^k,
$$

where $f(t)$ is any function for which the right-hand side exists, as well as the q-gamma function

$$
\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}.
$$

Our goal is to obtain a q-analogue of the beta integral. This is in fact hidden behind the q-binomial thoerem

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; x, q\right] = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}. $$

How? Set $a = q^\beta$ and $z = q^\alpha$ then rearrange using elementary q-series identities to obtain

$$ (1-q)\sum_{k=0}^{\infty} q^{k(\alpha-1)}(q^{k+1};q)_{\beta-1} = (1-q)\frac{(q^{\alpha+\beta};q)_\infty (q;q)_\infty}{(q^\alpha;q)_\infty (q^\beta;q)_\infty (q;q)_\infty^2}. $$

The left-hand side is a q-integral with $f(t) = t^{\alpha-1}(qt;q)_{\beta-1}$ and the right-hand side is a ratio of q-gamma functions. Hence we may write the identity as

$$ \int_0^1 t^{\alpha-1}(qt;q)_{\beta-1}\, \mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}. $$

The Selberg integral

In 1941/1944 Atle Selberg proved a remarkable multidimensional generalisation of Euler's beta integral.

Let $\boldsymbol{t} := (t_1, \ldots, t_n)$. Then Selberg's formula states

$$\begin{aligned} S_n(\alpha, \beta; \gamma) &:= \int_{[0,1]^n} |\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \, \mathrm{d}\boldsymbol{t} \\ &= \prod_{i=1}^{n} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2n-i-1)\gamma)\Gamma(1+\gamma)} \end{aligned}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and

$$\mathrm{Re}(\gamma) > -\min\{1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1)\}.$$

Here $\Delta(\boldsymbol{t})$ denotes the (type A) Vandermonde product

$$\Delta(\boldsymbol{t}) = \prod_{1 \le i < j \le n} (t_i - t_j).$$

Since Selberg's original paper (which went almost unknown until the late 1970s), the integral has played important roles in analytic number theory, random matrix theory, conformal field theory, combinatorics... It is often referred to as one of the most important hypergeometric integrals.

But can we prove it hypergeometrically?

Answer: Yes! Using a different type of basic hypergeometric series based on symmetric functions.

Symmetric functions

Let $X = (x_1, \ldots, x_n)$. For this whirlwind tour we work over the $\mathbb{Q}[X]$. The ring of symmetric functions is the subring $\Lambda_n \subseteq \mathbb{Q}[X]$ of all elements invariant under the action of the symmetric group, i.e., for any $w \in \mathfrak{S}_n$,

$$f(x_1, \ldots, x_n) = f(x_{w(1)}, \ldots, x_{w(n)}).$$

An example of a family of symmetric functions are the power sums

$$p_r(X) = x_1^r + \cdots + x_n^r, \quad r \in \mathbb{Z}_{>0},$$

and $p_0(X) := 1$. This extended to partitions by

$$p_\lambda(X) = p_{\lambda_1}(X) p_{\lambda_2}(X) \cdots p_{\lambda_n}(X),$$

where we assume λ has at most n nonzero parts. The power sums form a basis for Λ_n over $\mathbb{Q}$ (but not over $\mathbb{Z}$).

The most important (linear) basis for Λ_n is given by the Schur functions, most simply defined as a ratio of determinants

$$s_\lambda(X) = \frac{\det(x_j^{\lambda_i+n-i})_{1\le i,j\le n}}{\det(x_j^{n-i})_{1\le i,j\le n}}.$$

For example

$$\begin{aligned}
s_{(1)}(x_1,x_2,x_3) &= x_1+x_2+x_3,\\
s_{(1,1)}(x_1,x_2,x_3) &= x_1x_2+x_1x_3+x_2x_3,\\
s_{(2)}(x_1,x_2,x_3) &= x_1x_2+x_1x_3+x_2x_3+x_1^2+x_2^2+x_3^2,\\
s_{(2,1)}(x_1,x_2,x_3) &= x_1x_2^2+x_1x_3^2+x_1^2x_2+x_1^2x_3+x_2^2x_3+x_2x_3^2+2x_1x_2x_3.
\end{aligned}$$

In the late 1980s, Ian Macdonald introduced a new class of symmetric functions with two parameters q, t, denoted $P_\lambda(X;q,t)$, that generalise the Schur functions. Unfortunately they are notoriously difficult to define. Note that when $q = t$ they reduce to the Schur functions

$$P_\lambda(X;q,q) = s_\lambda(X).$$

Kaneko–Macdonald basic hypergeometric series

In the 20th century Milne and others had been studying hypergeometric series with symmetric function argument. More explicitly

$$\sum_{k\geq 0} f_k(\boldsymbol{a}, \boldsymbol{b}; q) x^k \quad \xrightarrow{\text{replace with}} \quad \sum_{\lambda} g_\lambda(\boldsymbol{a}, \boldsymbol{b}; q, t) P_\lambda(X; q, t)$$

Note that the new series are multivariate hypergeometric series

In particular, Kaneko and Macdonald both considered hypergeometric series generalising the q-binomial theorem and discovered that

$${}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; X, q, t\right] := \sum_{\lambda} \frac{t^{n(\lambda)}(a; q, t)_\lambda}{c'_\lambda(q, t)} P_\lambda(X; q, t) = \prod_{x\in X} \frac{(ax; q)_\infty}{(x; q)_\infty},$$

where X is now an arbitrary alphabet.

Using a similar (but more complicated) process as before, one may use the Kaneko–Macdonald q-binomial theorem to prove a q-analogue of the Selberg integral, known as the Askey–Habseiger–Kadell integral

$$\int_{[0,1]^n} \Delta(\boldsymbol{t};q,2\gamma) \prod_{i=1}^{n} t_i^{\alpha-1}(qt_i;q)_{\beta-1} \;\mathrm{d}_q\boldsymbol{t}$$
$$= q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} \prod_{i=1}^{n} \frac{\Gamma_q(\alpha+(i-1)\gamma)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma_q(\alpha+\beta+(n-i-2)\gamma)\Gamma_q(1+\gamma)},$$

where

$$\Delta(\boldsymbol{t};q,2\gamma) := \prod_{1\le i<j\le n} t_i^{2\gamma}(t_j q^{1-\gamma}/t_i;q)_{2\gamma},$$

and γ is a positive integer.

Taking the limit as $q \to 1^-$ we obtain the Selberg integral. The case of general γ may be obtain either through analytic continuation or a different limiting procedure.

Generalised Selberg integrals

A one-parameter deformation of the Schur functions are given by the Jack symmetric functions, which can be defined as a limit of the Macdonald polynomials

$$P_\lambda^{(1/\gamma)}(X) = \lim_{q\to 1} P_\lambda(X; q, q^\gamma).$$

Macdonald conjectured and Kadell proved the following generalised Selberg integral with a Jack polynomial in the integrand.

$$\int_{[0,1]^n} P_\lambda^{(1/\gamma)}(\boldsymbol{t}) |\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1}\, \mathrm{d}\boldsymbol{t}$$
$$= P_\lambda^{(1/\gamma)}(1,1,\dots,1) \prod_{i=1}^{n} \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2n-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)}$$

In 2010, Alday, Gaiotto, and Tachikawa (AGT) conjectured a relationship between conformal blocks in Liouville field theory and $\mathcal{N} = 2$ supersymmetric gauge theory. In their following proof of the AGT relation for SU(n) by Alba, Fateev, Litvinov, and Tarnopolskiy required an integral over a pair of Jack polynomials

$$\int_{[0,1]^n} P_\lambda^{(1/\gamma)}(\boldsymbol{t}) P_\mu^{(1/\gamma)}[\boldsymbol{t} + \beta/\gamma - 1] |\Delta(\boldsymbol{t})|^{2\gamma} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \, \mathrm{d}\boldsymbol{t}$$

$$= P_\lambda^{(1/\gamma)}[n] P_\mu^{(1/\gamma)}[n + \beta/\gamma - 1]$$

$$\times \prod_{i=1}^{n} \frac{\Gamma(\alpha + (k-i)\gamma + \lambda_i)\Gamma(\beta + (i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha + \beta + (2n - \ell - i - 1)\gamma + \lambda_i)\Gamma(1+\gamma)}$$

$$\times \prod_{i=1}^{n} \prod_{j=1}^{\ell} \frac{\Gamma(\alpha + \beta + (2n - i - j - 1)\gamma + \lambda_i + \mu_j)}{\Gamma(\alpha + \beta + (2n - i - j)\gamma + \lambda_i + \mu_j)},$$

which they could prove only in some special cases.

Using the full Cauchy identity for Macdonald polynomials (a generalisation of the Kaneko–Macdonald q-binomial theorem), we were able to prove the AFLT integral in generality.

Beyond

Through a representation-theoretic interpretation of the Selberg integral, one may think of $S_n(\alpha, \beta; \gamma)$ as associated to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In 2009, Warnaar extended the Selberg integral to $\mathfrak{sl}_n(\mathbb{C})$.

Using extensions of the previous symmetric functions approach one may extend the AFLT integral to $\mathfrak{sl}_n(\mathbb{C})$.

Unfortunately this is not enough for the SU(n) AGT conjecture, which requires an $\mathrm{sl}_n(\mathbb{C})$ Selberg integral over n Jack polynomials

$$\int_C P^{(1/\gamma)}_{\lambda^{(1)}} \cdots P^{(1/\gamma)}_{\lambda^{(n)}} \times |\Phi(\boldsymbol{t}^{(1)}), \dots, \boldsymbol{t}^{(n)})|^{\gamma} \, \mathrm{d}\boldsymbol{t}$$

$$= \text{“Product of gamma functions”}.$$

The End