Universal character factorisations

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Outline

1. An old theorem of D. E. Littlewood

2. Littlewood's theorem in different guises

3. Recent generalisations by Ayyer and Kumari for other group characters

4. Extra stuff

A theorem of Littlewood

VII. *If the numbers of the sequence*

$$\lambda_1+rm-1, \quad \lambda_2+rm-2, \quad \lambda_3+rm-3, \quad \ldots, \quad \lambda_{rm}$$

congruent respectively to 0, 1, 2,..., $r-1$ (mod r) *are not all equal,* $\{\lambda\} = 0$. *If they are equal and those congruent to* q (mod r) *are*

$$r[\mu_{q1}+m-1]+q, \quad r[\mu_{q2}+m-2]+q, \quad \ldots, \quad r\mu_{qm}+q,$$

then

$$\{\lambda\} = \theta\{\mu_{01}, \mu_{02}, \ldots, \mu_{0m}\}'\{\mu_{11}, \ldots, \mu_{1m}\}' \ldots \{\mu_{r-1,1}, \ldots, \mu_{r-1,m}\}',$$

where $\{\lambda\}$ *denotes an S-function of* $f(x^r)$ *and* $\{\mu\}'$ *an S-function of* $f(x)$.

From *The Theory of Group Characters and Matrix Representations of Groups* (1940) page 133.

Dictionary:

$\lambda \longrightarrow$ partition with at most *rm* nonzero entries

$\theta \longrightarrow$ sign of a particular permutation

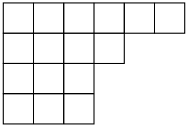
$\{\lambda\} \longrightarrow$ Littlewood's notation for the Schur function s_λ

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence

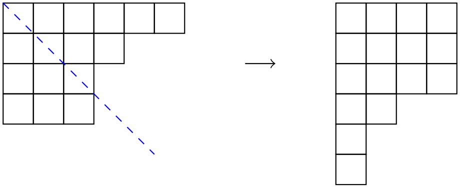
$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \cdots$$

such that only fintely $\lambda_i \neq 0$. The number of nonzero λ_i, writen $\ell(\lambda)$, is called the length and the sum is $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$.

For example $\lambda = (6, 4, 3, 3)$ has $\ell(\lambda) = 4$ and $|\lambda| = 16$. Its Young diagram is given by



The conjugate partition is obtained by reflecting the Young diagram in the "main diagonal"



so that $(4, 4, 4, 2, 1, 1)$ is the conjugate of $(6, 4, 3, 3)$.

Littlewood's $\{\lambda\}$ are Schur functions, usually first defined by the bialternant formula

$$s_\lambda(x_1,\ldots,x_n) = \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}.$$

Note the denominator is the Vandermonde determinant

$$\det_{1\leqslant i,j\leqslant n}(x_i^{n-j}) = \prod_{1\leqslant i<j\leqslant n}(x_i - x_j),$$

so the ratio is a polynomial, and moreover is homogeneous of degree $|\lambda|$ and symmetric in the x_i.

For example

$$s_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3.$$

For $\lambda = (r)$ and $\lambda = (1^r)$, a single row or column of length r, the Schur functions reduce to the complete homogeneous and elementary symmetric functions respectively:

$$s_{(r)}(X) =: h_r(X) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r} x_{i_1} \cdots x_{i_r},$$

$$s_{(1^r)}(X) =: e_r(X) = \sum_{1 \leqslant i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

Here we let $X = (x_1, x_2, \dots)$ be an arbitrary *countable* set of variables.

The Schur functions can also be defined in this generality, for example by the Jacobi–Trudi and Nägelsbach–Kostka identities

$$s_\lambda = \det_{1 \leqslant i,j \leqslant n} (h_{\lambda_i - i + j}) = \det_{1 \leqslant i,j \leqslant m} (e_{\lambda'_i - i + j}),$$

where $\ell(\lambda) \leqslant n$ and $\ell(\lambda') \leqslant m$.

Fact: Any symmetric function can be written as a polynomial in either the h_r or the e_r. The ring generated by them is the ring of symmetric functions denoted Λ.

Let $t \geqslant 2$ be an integer, ζ a primitive t-th root of unity, and $X = (x_1, \dots, x_n)$. Then Littlewood's result may be phrased as follows.

Theorem (Littlewood (1940), version II)

Let λ be a partition of length at most nt. If the set

$$\{\lambda_1 + nt - 1, \lambda_2 + nt - 2, \dots, \lambda_{nt}\}$$

contains n integers of residue r modulo t for each $0 \leqslant r \leqslant t-1$ then

$$s_\lambda(X, \zeta X, \dots, \zeta^{t-1} X) = \pm \prod_{r=0}^{t-1} s_{\lambda^{(r)}}(X^t),$$

where the $\lambda^{(r)}$ are some partitions defined in terms of λ and t. Else,

$$s_\lambda(X, \zeta X, \dots, \zeta^{t-1} X) = 0.$$

The t-core and t-quotient of λ are hiding behind this statement.

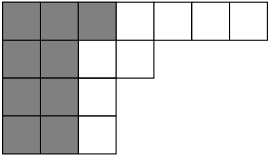
Cores and quotients

The *t*-core of a partition was defined by Nakayama in 1940 in terms of the removal of ribbons (or border strips, skew hooks) from its Young diagram.

The *t*-quotient was defined by Littlewood in 1951. However his student Farahat showed that his construction is equivalent to the star diagrams of Nakayama, Osima, Robinson and Staal.

We say μ is contained in λ if its Young diagram fits inside the Young diagram of λ, written $\mu \subseteq \lambda$.

So $(3,2,2,2) \subseteq (7,4,3,3)$, and we form the skew shape by removing μ's Young diagram from λ's.

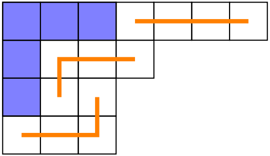


What remains here is a skew shape with no 2×2 square. Such a shape is called a ribbon. Since it has 8 cells, it is an 8-ribbon. The height of this ribbon is 3, with definition

$$\mathrm{ht}(\lambda) = \#\mathrm{rows} - 1.$$

For a fixed t, Nakayama considered successively removing t-ribbons from a Young diagram, so that at each step a valid Young diagram remains.

Let's try with $\lambda = (7, 4, 3, 3)$ and $t = 4$:

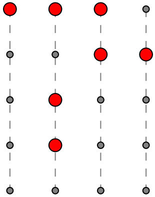


Theorem (Nakayama): Applying this procedure produces a unique partition with no hook of length t. This partition is called the t-core of λ.

For our example 4-core$((7, 4, 3, 3)) = (3, 1, 1)$.

A different description of this procedue—which also naturally explains the t-quotinet—is due to James (but appears in the book of James and Kerber).

Lay out the integers $\{0, 1, 2, 3, \ldots\}$ in infinite columns according to their residue modulo t.



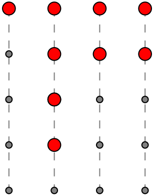
Given a partition λ and an integer $n \geqslant \ell(\lambda)$, place a bead at position $\lambda_i + n - i$ for each $1 \leqslant i \leqslant n$. This is called a bead configuration.

In the above, for $\lambda = (7, 4, 3, 3)$ and $n = 7$, place beads at

$$\{13, 9, 7, 6, 2, 1, 0\}.$$

The key observation:

Moving a bead up one space $\quad\longleftrightarrow\quad$ Removing a t-ribbon from λ

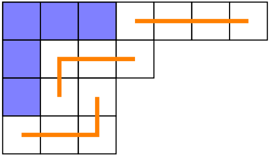


$$\{13, 9, 7, 6, 2, 1, 0\} \longrightarrow (7, 4, 3, 3)$$

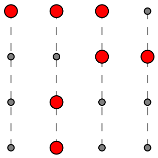
$$\{13, 9, 6, 3, 2, 1, 0\} \longrightarrow (7, 4, 2)$$

$$\{13, 6, 5, 3, 2, 1, 0\} \longrightarrow (7, 1, 1)$$

$$\{9, 6, 5, 3, 2, 1, 0\} \longrightarrow (3, 1, 1)$$

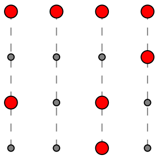


The t-quotient is obtained by reading each column as a bead configuration in the bead configuration for λ:



$$\big(\{0\},\{3,2,0\},\{1,0\},\{1\}\big) \longrightarrow \big(\varnothing,(1,1),\varnothing,(1)\big) =: \big(\lambda^{(0)},\lambda^{(1)},\lambda^{(2)},\lambda^{(3)}\big)$$

Changing n, as long as $n \geqslant \ell(\lambda)$, cyclically permutes the elements of the quotient. In what follows we fix n as a multiple of t.



Note in this case t-core(λ) is empty if and only if each column contains the same number of beads.

Theorem (Littlewood, 1951): For each $t \geqslant 2$ the map

$$\begin{aligned} \phi_t : \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \ldots, \lambda^{(t-1)})\big) \end{aligned}$$

is a bijection such that

$$|\lambda| = |t\text{-core}(\lambda)| + t\big(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|\big).$$

Our previous example gives

$$\phi_4\big((7,4,3,3)\big) = \big((3,1,1), ((1), \varnothing, (1,1), \varnothing)\big)$$

and

$$17 = 5 + 3 \times 4.$$

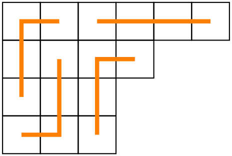
If t-core(λ) is *empty*, then the Young diagram has a decomposition

$$\varnothing = \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(k-1)} \subseteq \nu^{(k)} = \lambda,$$

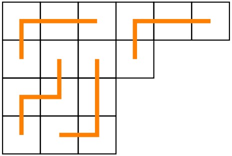
where $k = |\lambda|/t$ and $\nu^{(r)}/\nu^{(r-1)}$ is a t-ribbon. It is a non-obvious fact that for any such ribbon decomposition the sign

$$\mathrm{sgn}_t(\lambda) := (-1)^{\sum_{r=1}^{k} \mathrm{ht}(\nu^{(r)}/\nu^{(r-1)})}$$

is the same. This has been shown both by van Leeuwen and Pak.



$$\mathrm{sgn}_t\big((6,4,3,3)\big) = (-1)^6 = 1$$



$$\mathrm{sgn}_t\big((6,4,3,3)\big) = (-1)^6 = 1$$

Putting this all together...

Theorem (Littlewood (1940), version III)

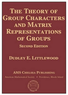
Let λ be a partition. If t-core(λ) is empty, then

$$s_\lambda(X, \zeta X, \ldots, \zeta^{t-1}X) = \mathrm{sgn}_t(\lambda) \prod_{r=0}^{t-1} s_{\lambda^{(r)}}(X^t).$$

Else, if t-core(λ) is non-empty, then

$$s_\lambda(X, \zeta X, \ldots, \zeta^{t-1}X) = 0.$$

Littlewood's theorem has been rediscovered many times, both in this form and as a statement for the characters of the symmetric group.

In 

Littlewood defines what he calls *"S-functions of series"*. The h_r may be defined by the generating function

$$\prod_{i=1}^{n} \frac{1}{1 - zx_i} = \sum_{r \geqslant 0} h_r(x_1, \ldots, x_n) z^n.$$

The "S-functions" (Schur functions) of this series are the ordinary Schur functions $s_\lambda(x_1, \ldots, x_n)$, defined by the Jacobi–Trudi determinant

$$s_\lambda(x_1, \ldots, x_n) = \det_{1 \leqslant i,j \leqslant n}(h_{\lambda_i - i + j})$$

By considering other series $f(z)$, one essentially chooses different values for the h_r, resulting in different Schur functions.

For example if we choose

$$f(z) = \prod_{i=1}^{n} \frac{1 - zy_i}{1 - zx_i}$$

then the associated $h_r^{f(z)}$ are

$$\sum_{k=0}^{r} (-1)^{r-k} h_k(x_1, \ldots, x_n) e_{r-k}(y_1, \ldots, y_n).$$

From now on we will work with the series

$$f(z) := \prod_{i \geqslant 1} \frac{1}{1 - x_i z} = \sum_{r \geqslant 0} h_r(X) z^r,$$

where $X = (x_1, x_2, \dots)$ and as before

$$h_r(X) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r} x_{i_1} \cdots x_{i_r}.$$

Littlewood's theorem is really about comparing the Schur functions obtained from the series

$$f(z^t) = \sum_{r \geqslant 0} h_r(X) z^{tr}$$

with those obtained from $f(z)$. We can encode this by an operator

$$\begin{aligned} \varphi_t : \Lambda &\longrightarrow \Lambda \\ h_r &\longmapsto \begin{cases} h_{r/t} & \text{if } t \text{ divides } r \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Of course by equating coefficients in

$$\sum_{r\geqslant 0} h_r(X)z^{tr} = \prod_{i\geqslant 1} \frac{1}{1-x_i z^t}$$
$$= \prod_{i\geqslant 1}\prod_{r=0}^{t-1} \frac{1}{1-\zeta^r x_i^{1/t} z}$$
$$= \sum_{r\geqslant 0} h_r\big(X^{1/t}, \zeta X^{1/t}, \ldots, \zeta^{t-1}X^{1/t}\big) z^r,$$

we see that

$$\varphi_t h_r(X) = h_r\big(X^{1/t}, \zeta X^{1/t}, \ldots, \zeta^{t-1}X^{1/t}\big).$$

Theorem (Littlewood (1940), version IV)

We have that $\varphi_t s_\lambda = 0$ unless t-core(λ) is empty, in which case

$$\varphi_t s_\lambda = \mathrm{sgn}_t(\lambda) \prod_{r=0}^{t-1} s_{\lambda^{(r)}}.$$

The proof of this version is quite simple. First observe that in

$$\varphi_t s_\lambda = \det_{1\leqslant i,j\leqslant nt}(\varphi_t h_{\lambda_i-i+j})$$

the only matrix entries which survive are those for which $\lambda_i - i + j$ is a multiple of t.

Rearrange the rows so that the remainders modulo t of $\lambda_i - i$ are increasing, and the indices are themselves decreasing withing each class. Then do the same for the columns with $-j$.

What remains is a block-diagonal matrix where each block as dimension $m_r \times n$ where m_r is the number of beads in the r-th column of the bead configuration. An entry in block r is

$$h_{(\lambda_i-i+j)/t} = h_{\lambda^{(r)}-i'+j'}$$

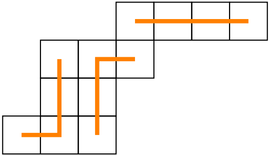
where $1 \leqslant i' \leqslant m_r$ and $1 \leqslant j' \leqslant n$.
For an example $\varphi_3 s_{(6,5,2,2,2,1)} = -s^2_{(2,1)}$.

Before moving on, let us remark that there is a version for the skew Schur functions

$$s_{\lambda/\mu} := \det_{1 \leqslant i,j \leqslant n} (h_{\lambda_i - \mu_j - i + j}).$$

Theorem Farahat (1954) & Macdonald (1995): For any skew shape λ/μ, $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is tilable by t-ribbons, in which case

$$\varphi_t s_{\lambda/\mu} = \mathrm{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$



The proof is (almost) identical to the straight shape case!

Inspired by a rediscovery of Littlewood's theorem by Prasad, Ayyer and Kumari proved similar factorisation theorems for the characters of the groups $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{O}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$.

These characters, which we denote by sp_λ, o_λ and so_λ respectively, are Laurent polynomials in $x_1,\dots,x_n$ which are symmetric under permutation and inversion of variables.

Rather than working with the characters directly, in this context it is much better to work with the universal characters as defined in part by Weyl, and in part by Koike and Terada. These are lifts of the symmetric Laurent polynomials to symmetric functions:

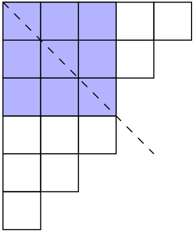
$$\mathrm{sp}_\lambda := \frac{1}{2}\det_{1\leqslant i,j\leqslant n}\left(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\right)$$

$$\mathrm{o}_\lambda := \det_{1\leqslant i,j\leqslant n}\left(h_{\lambda_i-i+j}-h_{\lambda_i-i-j}\right)$$

$$\mathrm{so}^{\pm}_\lambda := \det_{1\leqslant i,j\leqslant n}\left(h_{\lambda_i-i+j}\pm h_{\lambda_i-i-j+1}\right)$$

Specialising to $x_1^{\pm},\dots,x_n^{\pm}$ give *actual* characters of the labelled groups.

The Durfee square of a partition is the largest square which fits inside the Young diagram



Call the side lengths $d(\lambda)$. The Frobenius notation for a partition records how many cells are below/to the right of each cell on the main diagonal. For example

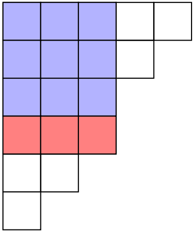
$$(5, 4, 3, 3, 2, 1) = (4, 2, 0 \mid 5, 3, 1).$$

So any pair of strictly decreasing nonnegative integer sequences $a_1, \ldots, a_d$, $b_1, \ldots, b_d$ determine a partition with $d \times d$ Durfee square.

Ayyer and Kumari were led to what they call z-asymmetric partitions, which are those of the form

$$(a_1, \ldots, a_d \mid a_1 + z, \ldots, a_d + z)$$

for some integers d and z.



is 1-asymmetric. A 0-asymmetric partition is called self-conjugate.

These partitions naturally arise in the proof of the factorisations for the universal characters, as we saw for the Schur functions.

Theorem: We have that $\varphi_t \mathrm{sp}_\lambda = 0$ unless t-core(λ) is 1-asymmetric, in which case

$$\varphi_t \mathrm{sp}_\lambda = \mathrm{sgn}_t^{\mathrm{sp}}(\lambda; nt)\mathrm{sp}_{\lambda^{(t-1)}} \prod_{r=0}^{\lfloor (t-3)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-2)}} \times \begin{cases} \mathrm{so}_{\lambda^{((t-2)/2)}} & \text{if } t \text{ is even,} \\ 1 & \text{if } t \text{ is odd.} \end{cases}$$

Theorem: We have that $\varphi_t \mathrm{o}_\lambda = 0$ unless t-core(λ) is (-1)-asymmetric, in which case

$$\varphi_t \mathrm{o}_\lambda = \mathrm{sgn}_t^{\mathrm{o}}(\lambda; nt)\mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-1)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & \text{if } t \text{ is even,} \\ 1 & \text{if } t \text{ is odd.} \end{cases}$$

Theorem: We have that $\varphi_t \mathrm{so}_\lambda = 0$ unless t-core(λ) is self-conjugate, in which case

$$\varphi_t \mathrm{so}_\lambda = \mathrm{sgn}_t^{\mathrm{so}}(\lambda; nt) \prod_{r=0}^{\lfloor (t-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-1)}} \times \begin{cases} 1 & \text{if } t \text{ is even,} \\ \mathrm{so}_{\lambda^{((t-1)/2)}} & \text{if } t \text{ is odd.} \end{cases}$$

These factorisations involve a fourth universal character

$$\mathrm{rs}_{\lambda,\mu}(X;Y) := \det_{1\leqslant i,j\leqslant n+m} \begin{pmatrix} \big(h_{\lambda_i-i+j}(X)\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}(X)\big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} \\ \big(h_{\mu_i-i-j+1}(Y)\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} & \big(h_{\mu_i-i+j}(Y)\big)_{1\leqslant i,j\leqslant m} \end{pmatrix}$$

For $X = Y$ we write this simply as $\mathrm{rs}_{\lambda,\mu}$. This was first defined by Balantekin and Bars (also Cummins and King), and again by Koike. It appears frequently in the physics literature under the name "composite Schur function".

This arises from the rational representations of $\mathrm{GL}(n,\mathbb{C})$. It is, in a sense, the "correct" universal character analogue of the Schur function in $2n$ variables

$$s_\nu\big(x_1^{\pm},\dots,x_n^{\pm}\big).$$

The key identity is

$$s_{(\lambda_1,\dots,\lambda_n)}(x_1,\dots,x_n) = (x_1\cdots x_n)s_{(\lambda_1-1,\dots,\lambda_n-1)}(x_1,\dots,x_n),$$

which allows for Schur functions to be extended to weakly decreasing sequences of integers with length exactly n.

Given a pair of partitions λ, μ such that $\ell(\lambda) + \ell(\mu) \leqslant n$, define the staircase

$$[\lambda, \mu] = (\underbrace{\lambda_1, \lambda_2, \dots, -\mu_2, -\mu_1}_{\text{length } n}).$$

Littlewood showed that

$$\mathrm{rs}_{\lambda,\mu}(x_1, \dots, x_n; 1/x_1, \dots, 1/x_n) = s_{[\lambda,\mu]}(x_1, \dots, x_n),$$

i.e., in this case $\mathrm{rs}_{\lambda,\mu}$ is, up to a power of $x_1 \cdots x_n$, a Schur function. The right-hand side is the character of a rational representation of $\mathrm{GL}(n, \mathbb{C})$ indexed by $[\lambda, \mu]$.

The proofs of the factorisation $\varphi_t \mathrm{sp}_\lambda$ and its cousin proceed exactly as in the Schur case, the precise rearrangement of the determinants just requires a more careful analysis.

Recall the involution $\omega : \Lambda \longrightarrow \Lambda$ on symmetric functions defined by $\omega h_r = e_r$. This satisfies

$$\begin{aligned} \omega s_{\lambda/\mu} &= s_{\lambda'/\mu'} \\ \omega \mathrm{sp}_\lambda &= \mathrm{o}_{\lambda'} \\ \omega \mathrm{so}_\lambda &= \mathrm{so}_{\lambda'}. \end{aligned}$$

Proposition: There holds

$$\omega\varphi_t s_{\lambda/\mu} = (-1)^{(t-1)(|\lambda^{(0)}/\mu^{(0)}|+\cdots+|\lambda^{(t-1)}/\mu^{(t-1)}|)}\varphi_t\omega s_{\lambda/\mu}$$

and

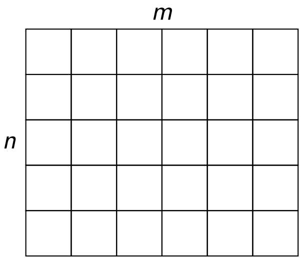
$$\omega\varphi_t \bullet_\lambda = (-1)^{(t-1)(|\lambda^{(0)}|+\cdots+|\lambda^{(t-1)}|)}\varphi_t\omega\bullet_\lambda$$

for $\bullet \in \{\mathrm{sp}, \mathrm{o}, \mathrm{so}\}$.

This implies that ω and φ_t commute for t odd.

Extras

Let (m^n) be a rectangular partition, e.g.



In 2009, Ciucu and Krattenthaler observed and proved that

$$s_{(2m)^n}(x_1^{\pm}, \ldots, x_n^{\pm}) = \mathrm{so}_{(m^n)}(x_1^{\pm}, \ldots, x_n^{\pm})\mathrm{so}^{-}_{(m^n)}(x_1^{\pm}, \ldots, x_n^{\pm}),$$

and

$$s_{(2m+1)^n}(x_1^{\pm}, \ldots, x_n^{\pm}) = \mathrm{sp}_{(m^n)}(x_1^{\pm}, \ldots, x_n^{\pm})\mathrm{o}_{((m+1)^n)}(x_1^{\pm}, \ldots, x_n^{\pm}).$$

These were generalised by Ayyer and Behrend to the identities

$$s_{(\lambda_1,\dots,\lambda_n,-\lambda_n,\dots,-\lambda_1)}(x_1^{\pm},\dots,x_n^{\pm}) \\ = \mathrm{so}_{(\lambda_1,\dots,\lambda_n)}(x_1^{\pm},\dots,x_n^{\pm})\mathrm{so}^{-}_{(\lambda_1,\dots,\lambda_n)}(x_1^{\pm},\dots,x_n^{\pm}),$$

and

$$s_{(\lambda_1+1,\dots,\lambda_n+1,-\lambda_n,\dots,-\lambda_1)}(x_1^{\pm},\dots,x_n^{\pm}) \\ = \mathrm{sp}_{(\lambda_1,\dots,\lambda_n)}(x_1^{\pm},\dots,x_n^{\pm})\mathrm{o}^{-}_{(\lambda_1+1,\dots,\lambda_n+1)}(x_1^{\pm},\dots,x_n^{\pm}).$$

These identities can be given universal character analogues of the form

$$\mathrm{rs}_{\lambda,\lambda} = \mathrm{so}_{\lambda}\mathrm{so}^{-}_{\lambda} \quad \text{and} \quad \mathrm{rs}_{\lambda+1^n,\lambda} = \mathrm{sp}_{\lambda}\mathrm{o}_{\lambda+1^n}.$$

Moreover, the proofs are short and sweet.

For example, in

$$\mathrm{rs}_{\lambda,\lambda} = \det_{1\leqslant i,j\leqslant 2n} \begin{pmatrix} \left(h_{\lambda_i-i+j}\right)_{1\leqslant i,j\leqslant n} & \left(h_{\lambda_i-i-j+1}\right)_{1\leqslant i,j\leqslant n} \\ \left(h_{\lambda_i-i-j+1}\right)_{1\leqslant i,j\leqslant n} & \left(h_{\lambda_i-i+j}\right)_{1\leqslant i,j\leqslant n} \end{pmatrix},$$

add the blocks on the right to the blocks on the left, and then subtract the blocks on the top from the blocks on the bottom, giving

$$\begin{aligned}\mathrm{rs}_{\lambda,\lambda} &= \det_{1\leqslant i,j\leqslant 2n} \begin{pmatrix} \left(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1}\right)_{1\leqslant i,j\leqslant n} & \left(h_{\lambda_i-i-j+1}\right)_{1\leqslant i,j\leqslant n} \\ 0 & \left(h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1}\right)_{1\leqslant i,j\leqslant n} \end{pmatrix} \\ &= \mathrm{so}_\lambda \mathrm{so}_\lambda^-.\end{aligned}$$

The other identities (including some not shown) have similarly short proofs.

S. P. Albion, *Universal characters twisted by roots of unity*, arXiv:2212.07343.