CONTINUITY OF THE SOLUTION MAP FOR HYPERBOLIC POLYNOMIALS

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Abstract. Hyperbolic polynomials are monic real-rooted polynomials. By Bronshtein's theorem, the increasingly ordered roots of a hyperbolic polynomial of degree d with $C^{d-1,1}$ coefficients are locally Lipschitz and this solution map "coefficients-to-roots" is bounded. We prove continuity of the solution map from hyperbolic polynomials of degree d with C^d coefficients to their increasingly ordered roots with respect to the C^d structure on the source space and the Sobolev $W^{1,q}$ structure, for all $1 \le q < \infty$, on the target space. Continuity fails for $q = \infty$. As a consequence, we obtain continuity of the local surface area of the roots.

Contents

1. Introduction

Determining the optimal regularity of the roots of polynomials whose coefficients depend smoothly on parameters is a much studied problem with a long history. It has important applications in various fields such as partial differential equations and perturbation theory.

In this paper, we focus on the class of monic *hyperbolic polynomials* for which the regularity problem has a special flavor; the general case is treated in [11]. A monic real polynomial of degree d is called hyperbolic if all its d roots (counted with multiplicities) are real. By Bronshtein's theorem [3], any continuous system of the roots of a $C^{d-1,1}$ family of hyperbolic polynomials of degree d is actually locally

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Lipschitz continuous (i.e. $C^{0,1}$); in general, this is optimal. There are uniform bounds for the Lipschitz constants of the roots in terms of the $C^{d-1,1}$ norms of the coefficients (see Theorem 4.1).

Thus we have a bounded *solution map* that takes hyperbolic polynomials of degree d with $C^{d-1,1}$ coefficients to $C^{0,1}$ systems of their roots. This will be made precise below.

The purpose of this paper is to investigate the continuity of the solution map.

1.1. **Hyperbolic polynomials and the solution map.** A monic polynomial of degree d,

$$P_a(Z) = Z^d + \sum_{j=1}^d a_j Z^{d-j} \in \mathbb{R}[Z],$$

is called *hyperbolic* if all its d roots are real. In the following, we will identify the polynomial P_a with its coefficient vector $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Then the set of all hyperbolic polynomials of degree d is the image of the map $\sigma = (\sigma_1, \ldots, \sigma_d) : \mathbb{R}^d \to \mathbb{R}^d$, where

$$\sigma_j(x_1, \ldots, x_d) = (-1)^j \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}$$

is the j-th elementary symmetric function (up to sign). It is a closed semialgebraic subset of $\mathbb{R}^d$ which we equip with the trace topology. We denote this space by $\operatorname{Hyp}(d)$ and call it the *space of hyperbolic polynomials of degree* d.

For $a \in \operatorname{Hyp}(d)$, let $\lambda_1^\uparrow(a) \le \cdots \le \lambda_d^\uparrow(a)$ denote the increasingly ordered roots of P_a. Then

$$\lambda^\uparrow = (\lambda_1^\uparrow, \ldots, \lambda_d^\uparrow) : \operatorname{Hyp}(d) \to \mathbb{R}^d$$

is a continuous map, see [1, Lemma 4.1].

Let $U \subseteq \mathbb{R}^m$ be open. Let $C^{d-1,1}(U, \operatorname{Hyp}(d))$ denote the set of $C^{d-1,1}$ maps $a : U \to \mathbb{R}^d$ such that $a(U) \subseteq \operatorname{Hyp}(d)$. Thus $a \in C^{d-1,1}(U, \operatorname{Hyp}(d))$ amounts to a hyperbolic polynomial P_a of degree d whose coefficients are $C^{d-1,1}$ functions defined on U. We equip $C^{d-1,1}(U, \operatorname{Hyp}(d))$ with the trace topology of the natural Fréchet topology on $C^{d-1,1}(U, \mathbb{R}^d)$. Note that $C^{d-1,1}(U, \operatorname{Hyp}(d))$ is a closed nonlinear subset of $C^{d-1,1}(U, \mathbb{R}^d)$. Then Bronshtein's theorem (see also Theorem 4.1) implies that the *solution map*

$$(\lambda^\uparrow)_* : C^{d-1,1}(U, \operatorname{Hyp}(d)) \to C^{0,1}(U, \mathbb{R}^d), \quad a \mapsto \lambda^\uparrow \circ a, \tag{1.1}$$

is well-defined and bounded.

1.2. **The main results.** We will see in Example 1.5 that the solution map (1.1) is *not* continuous.

However, the solution map $(\lambda^\uparrow)_*$ becomes continuous if we restrict it to $C^d(U, \operatorname{Hyp}(d))$, carrying the trace topology of the natural Fréchet topology on $C^d(U, \mathbb{R}^d)$, and relax the topology on the target space: for $1 \le q < \infty$ let $C^{0,1}_q(U, \mathbb{R}^d)$ denote the set $C^{0,1}(U, \mathbb{R}^d)$ equipped with the trace topology of the inclusion $C^{0,1}(U, \mathbb{R}^d) \to W^{1,q}_{\text{loc}}(U, \mathbb{R}^d)$. See Section 2 for precise definitions of the function spaces.

The following theorem, which is our main result, solves Open Problem 3.8 in [12].

Theorem 1.1. *Let $U \subseteq \mathbb{R}^m$ be open. The solution map*
$$(\lambda^\uparrow)_* : C^d(U, \mathrm{Hyp}(d)) \to C^{0,1}_q(U, \mathbb{R}^d), \quad a \mapsto \lambda^\uparrow \circ a,$$
is continuous, for all $1 \le q < \infty$.

As a corollary, we find that the solution map on $C^d(U, \mathrm{Hyp}(d))$ is continuous into the Hölder space $C^{0,\alpha}(U, \mathbb{R}^d)$ with its natural topology, for all $0 < \alpha < 1$.

Corollary 1.2. *Let $U \subseteq \mathbb{R}^m$ be open. The solution map*
$$(\lambda^\uparrow)_* : C^d(U, \mathrm{Hyp}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d), \quad a \mapsto \lambda^\uparrow \circ a,$$
is continuous, for all $0 < \alpha < 1$.

The essential work for the proof of Theorem 1.1 happens in dimension $m = 1$ of the parameter space. The passage from one to several parameters is rather easy. The following is the main technical result of the paper.

Theorem 1.3. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $a_n \to a$ in $C^d(I, \mathrm{Hyp}(d))$, i.e., for each relatively compact open interval $I_1 \Subset I$,*
$$\|a - a_n\|_{C^d(\overline{I}_1, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{1.2}$$
Then $\{\lambda^\uparrow \circ a_n : n \ge 1\}$ is a bounded set in $C^{0,1}(I, \mathbb{R}^d)$ and, for each relatively compact open interval $I_0 \Subset I$ and each $1 \le q < \infty$,
$$\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{W^{1,q}(I_0, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{1.3}$$

The proof of Theorem 1.3 is based on the dominated convergence theorem. The domination follows from Bronshtein's theorem which we recall in Theorem 4.1. We will show in Theorem 5.1 that, for almost every $x \in I$,
$$(\lambda^\uparrow \circ a_n)'(x) \to (\lambda^\uparrow \circ a)'(x) \quad \text{as } n \to \infty.$$
To this end, we will develop a version of Bronshtein's theorem at a single point, see Theorem 4.5.

Note that by Egorov's theorem we may conclude that $(\lambda^\uparrow \circ a_n)' \to (\lambda^\uparrow \circ a)'$ almost uniformly on I as $n \to \infty$, i.e., for each $\epsilon > 0$ there exists a measurable subset $E \subseteq I$ with $|E| < \epsilon$ such that $(\lambda^\uparrow \circ a_n)' \to (\lambda^\uparrow \circ a)'$ uniformly on $I \setminus E$. In general, the convergence is not uniform on the whole interval I; see Example 1.5.

For later reference, we state a simple consequence of Theorem 1.3.

Corollary 1.4. *Let $I \subseteq \mathbb{R}$ be an open interval and $I_0 \Subset I$ a relatively compact open subinterval. If $a_n \to a$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$, then*
$$\big\| \|(\lambda^\uparrow \circ a)'\|_2 - \|(\lambda^\uparrow \circ a_n)'\|_2 \big\|_{L^q(I_0)} \to 0 \quad \text{as } n \to \infty,$$
and
$$\|(\lambda^\uparrow \circ a_n)'\|_{L^q(I_0, \mathbb{R}^d)} \to \|(\lambda^\uparrow \circ a)'\|_{L^q(I_0, \mathbb{R}^d)} \quad \text{as } n \to \infty,$$
for all $1 \le q < \infty$.

Proof. Let us set $\lambda := \lambda^\uparrow \circ a$ and $\lambda_n := \lambda^\uparrow \circ a_n$. Then
$$\begin{aligned} \big| \|\lambda'\|_{L^q(I_0,\mathbb{R}^d)} - \|\lambda_n'\|_{L^q(I_0,\mathbb{R}^d)} \big| &= \big| \big\| \|\lambda'\|_2 \big\|_{L^q(I_0)} - \big\| \|\lambda_n'\|_2 \big\|_{L^q(I_0)} \big| \\ &\le \big\| \|\lambda'\|_2 - \|\lambda_n'\|_2 \big\|_{L^q(I_0)} \le \big\| \|\lambda' - \lambda_n'\|_2 \big\|_{L^q(I_0)} = \|\lambda' - \lambda_n'\|_{L^q(I_0,\mathbb{R}^d)} \end{aligned}$$
so that the assertions follow from (1.3). □

We will interpret the results for hyperbolic polynomials as special versions of the general theorems of [11] in Section 7.1. In the general case, it is natural to consider the *unordered* d-tuple of roots because there is no canonical choice of a parameterization of the roots by continuous functions. If the parameter space has dimension ≥ 2, then continuous selections of the roots might not even exist.

As further applications, we deduce local convergence of the surface area of the graphs of the single roots $\lambda_j^\uparrow \circ a_n$, for $1 \le j \le d$, as $n \to \infty$ (see Corollary 7.6) and an approximation result by hyperbolic polynomials with all roots distinct (see Corollary 7.8).

1.3. **On the optimality of the results.** The following example shows that the solution map $(\lambda^\uparrow)_*$ is not continuous with respect to the $C^{0,1}$ topology on the target space.

Example 1.5. Let $g(x) := x^2$ and $g_n(x) := x^2 + 1/n^2$, $n \ge 1$. Then, for all $k \in \mathbb{N}$ and each bounded open interval $I \subseteq \mathbb{R}$, $\|g - g_n\|_{C^k(\overline{I})} = 1/n^2 \to 0$ as $n \to \infty$. Let f and f_n be the positive square roots of g and g_n, respectively: $f(x) := |x|$ and $f_n(x) := \sqrt{x^2 + 1/n^2}$. Then, for each bounded open interval $I \subseteq \mathbb{R}$ containing 0,

$$|f - f_n|_{C^{0,1}(\overline{I})} \ge \sup_{0<x\in I} \Big| \frac{(f(x) - f_n(x)) - (f(0) - f_n(0))}{x} \Big|$$
$$= \sup_{0<x\in I} \Big| \frac{x - \sqrt{x^2 + \frac{1}{n^2}} + \frac{1}{n}}{x} \Big| \ge \Big| \frac{\frac{1}{n} - \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} + \frac{1}{n}}{\frac{1}{n}} \Big| = 2 - \sqrt{2},$$

for large enough n. Observe that

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}$$

tends pointwise to $f'(x) = \operatorname{sgn}(x)$ for all $x \ne 0$ but not uniformly on any neighborhood of 0:

$$f_n'(\pm\tfrac{1}{n}) = \pm\frac{1}{\sqrt{2}}.$$

This also violates the first conclusion of Corollary 1.4 for $q = \infty$.

Notice that the example also shows that not every continuous (thus $C^{0,1}$) system of the roots of g is the limit of a continuous system of the roots of g_n: each continuous system of the roots of g_n tends to $\pm|x|$, none to $\pm x$.

Remark 1.6. We do not know if the continuity results in Theorem 1.1, Corollary 1.2, Theorem 1.3, and Corollary 1.4 still hold for the solution map $(\lambda^\uparrow)_*$ on $C^{d-1,1}(U, \operatorname{Hyp}(d))$ (instead of $C^d(U, \operatorname{Hyp}(d))$).

In the proof of Theorem 1.3, we need the convergence of the coefficient vectors in C^d only on the accumulation points of the preimage under a of the discriminant locus. If this preimage is the union of an open set and a set of measure zero, then for (1.3) it is enough that a_n tends to a in $C^{d-1,1}$. Thus, for a potential counterexample a has to meet the discriminant locus in a Cantor-like set with positive measure.

1.4. **Structure of the paper.** We fix notation and recall facts on function spaces in Section 2 and provide the necessary background on hyperbolic polynomials in Section 3. In Section 4, we prove a version of Bronshtein's theorem at a single point (Theorem 4.5). It provides bounds for the derivative of the roots that are crucial for the proof of Theorem 1.3 which is carried out in Section 5. Then Theorem 1.1 and Corollary 1.2 are deduced in Section 6. The last Section 7 is dedicated to applications.

Notation. The m-dimensional Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$. If not stated otherwise, 'measurable' means 'Lebesgue measurable' and 'almost everywhere' means 'almost everywhere with respect to Lebesgue measure'. For measurable $E \subseteq \mathbb{R}^m$, we usually write $|E| = \mathcal{L}^m(E)$. We shall also use the k-dimensional Hausdorff measure $\mathcal{H}^k$.

For $1 \le p \le \infty$, $\|x\|_p$ denotes the p-norm of $x \in \mathbb{R}^d$. If $f : E \to \mathbb{R}^d$, for measurable $E \subseteq \mathbb{R}^m$, is a measurable map, then we set

$$\|f\|_{L^p(E,\mathbb{R}^d)} := \big\| \|f\|_2 \big\|_{L^p(E)}.$$

For us a set is countable if it is either finite or has the cardinality of $\mathbb{N}$.

2. Function spaces

Let us fix notation and recall background on the function spaces used in this paper.

2.1. Hölder–Lipschitz spaces.

Let $U \subseteq \mathbb{R}^m$ be open and $k \in \mathbb{N}$. Then $C^k(U)$ is the space of k-times continuously differentiable real valued functions with its natural Fréchet topology. If U is bounded, then $C^k(\overline{U})$ denotes the space of all $f \in C^k(U)$ such that each $\partial^\alpha f$, $0 \le |\alpha| \le k$, has a continuous extension to the closure $\overline{U}$. Endowed with the norm

$$\|f\|_{C^k(\overline{U})} := \max_{|\alpha| \le k} \sup_{x \in U} |\partial^\alpha f(x)|$$

it is a Banach space. For $0 < \gamma \le 1$, we consider the Hölder–Lipschitz seminorm

$$|f|_{C^{0,\gamma}(\overline{U})} := \sup_{x,y \in U,\, x \ne y} \frac{|f(x) - f(y)|}{\|x - y\|_2^\gamma}.$$

For $k \in \mathbb{N}$ and $0 < \gamma \le 1$, we have the Banach space

$$C^{k,\gamma}(\overline{U}) := \{f \in C^k(\overline{U}) : \|f\|_{C^{k,\gamma}(\overline{U})} < \infty\},$$

where

$$\|f\|_{C^{k,\gamma}(\overline{U})} := \|f\|_{C^k(\overline{U})} + \max_{|\alpha| = k} |\partial^\alpha f|_{C^{0,\gamma}(\overline{U})}.$$

We write $C^{k,\gamma}(U)$ for the space of C^k functions on U that belong to $C^{k,\gamma}(\overline{V})$ for each relatively compact open $V \Subset U$, with its natural Fréchet topology.

2.2. Lebesgue spaces.

Let $U \subseteq \mathbb{R}^m$ be open and $1 \le p \le \infty$. We denote by $L^p(U)$ the Lebesgue space with respect to the m-dimensional Lebesgue measure $\mathcal{L}^m$, and $\| \cdot \|_{L^p(U)}$ is the corresponding L^p-norm. For Lebesgue measurable sets $E \subseteq \mathbb{R}^n$ we also write $|E| = \mathcal{L}^m(E)$. We remark that for continuous functions $f : U \to \mathbb{R}$ we have (and use interchangeably) $\|f\|_{L^\infty(U)} = \|f\|_{C^0(\overline{U})}$.

2.3. **Sobolev spaces.** For $k \in \mathbb{N}$ and $1 \le q \le \infty$, we consider the Sobolev space

$$W^{k,q}(U) := \{f \in L^q(U) : \partial^\alpha f \in L^q(U) \text{ for } |\alpha| \le k\},$$

where $\partial^\alpha f$ are distributional derivatives. Endowed with the norm

$$\|f\|_{W^{k,q}(U)} := \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^q(U)}$$

it is a Banach space. We will also use

$$W^{k,q}_{\text{loc}}(U) := \{f \in L^q_{\text{loc}}(U) : \partial^\alpha f \in L^q_{\text{loc}}(U) \text{ for } |\alpha| \le k\}$$

with its natural topology.

2.4. **A result on composition.** In the following proposition we use the norm

$$\|f\|_{C^k(\overline{U},\mathbb{R}^\ell)} := \max_{0 \le j \le k} \sup_{x \in U} \|d^j f(x)\|_{L_j(\mathbb{R}^m,\mathbb{R}^\ell)}$$

on the space $C^k(\overline{U},\mathbb{R}^\ell) := (C^k(\overline{U},\mathbb{R}))^\ell$, where $U \subseteq \mathbb{R}^m$ and $L_j(\mathbb{R}^m,\mathbb{R}^\ell)$ is the space of j-linear maps with j arguments in $\mathbb{R}^m$ and values in $\mathbb{R}^\ell$.

Proposition 2.1. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^\ell$ be open, bounded, and convex. Let $\psi \in C^{k+1}(\overline{V},\mathbb{R}^p)$. Then $\psi_* : C^k(\overline{U},V) \to C^k(\overline{U},\mathbb{R}^p)$, $\psi_*(\varphi) := \psi \circ \varphi$, is well-defined and continuous. More precisely, for φ_1, φ_2 in a bounded subset B of $C^k(\overline{U},V)$,*

$$\|\psi_*(\varphi_1) - \psi_*(\varphi_2)\|_{C^k(\overline{U},\mathbb{R}^p)} \le C\, \|\psi\|_{C^{k+1}(\overline{V},\mathbb{R}^p)} \|\varphi_1 - \varphi_2\|_{C^k(\overline{U},\mathbb{R}^\ell)},$$

where $C = C(k,B)$.

This result must be well-known; a short proof can be found in [11, Appendix A.2].

3. Hyperbolic polynomials

In this section, we recall basic facts on hyperbolic polynomials that will be used below. Proofs can be found in [12] or in [7].

3.1. **Tschirnhausen form.** We say that a monic polynomial

$$P_a(Z) = Z^d + \sum_{j=1}^d a_j Z^{d-j}$$

is in *Tschirnhausen form* if $a_1 = 0$. Every polynomial P_a can be put in Tschirnhausen form by the substitution (called *Tschirnhausen transformation*)

$$P_{\tilde a}(Z) = P_a(Z - \tfrac{a_1}{d}) = Z^d + \sum_{j=2}^d \tilde a_j Z^{d-j}.$$

For clarity, we consistently equip the coefficients of polynomials in Tschirnhausen form with a 'tilde'. Note that

$$\tilde a_j = \sum_{i=0}^j C_i a_i a_1^{j-i}, \quad 2 \le j \le d, \tag{3.1}$$

where the C_i are universal constants and $a_0 = 1$. For a polynomial $P_{\tilde a}$ in Tschirnhausen form we have

$$-2\tilde a_2 = \lambda^\uparrow_1(\tilde a)^2 + \cdots + \lambda^\uparrow_d(\tilde a)^2.$$

Consequently, for a hyperbolic polynomial $P_{\tilde{a}}$ in Tschirnhausen form,

$$\tilde{a}_2 \le 0.$$

Lemma 3.1 ([12, Lemma 2.4]). *The coefficients of a hyperbolic polynomial $P_{\tilde{a}}$ in Tschirnhausen form satisfy*

$$|\tilde{a}_j|^{1/j} \le \sqrt{2}\,|\tilde{a}_2|^{1/2}, \quad j = 1, \ldots, d.$$

As a consequence, $\tilde{a} = (0, \tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_d) = 0$ if and only if $\tilde{a}_2 = 0$.

Definition 3.2. Let $\operatorname{Hyp}_T(d)$ denote the space of monic hyperbolic polynomials of degree d in Tschirnhausen form and $\operatorname{Hyp}_T^0(d)$ the compact subspace of polynomials $P_{\tilde{a}}$ with $\tilde{a}_2 = -1$, i.e.,

$$\operatorname{Hyp}_T(d) = \{\tilde{a} \in \operatorname{Hyp}(d) : \tilde{a}_1 = 0\},$$
$$\operatorname{Hyp}_T^0(d) = \{\tilde{a} \in \operatorname{Hyp}_T(d) : \tilde{a}_2 = -1\}.$$

3.2. **Splitting.** Let us recall a simple consequence of the inverse function theorem.

Lemma 3.3 (E.g. [12, Lemma 2.5]). *Let $P_a = P_b P_c$, where P_b and P_c are monic real polynomials without common (complex) root. Then we have $P = P_{b(P)} P_{c(P)}$ for analytic mappings $P \mapsto b(P) \in \mathbb{R}^{\deg P_b}$ and $P \mapsto c(P) \in \mathbb{R}^{\deg P_c}$, defined for P near P_a in $\mathbb{R}^{\deg P_a}$, with the given initial values.*

Let $P_{\tilde{a}} \in \operatorname{Hyp}_T(d)$ be such that $\tilde{a} \neq 0$. Then the polynomial

$$Q_{\underline{a}}(Z) := |\tilde{a}_2|^{-d/2} P_{\tilde{a}}(|\tilde{a}_2|^{1/2} Z) = Z^d - Z^{d-2} + \sum_{j=3}^{d} |\tilde{a}_2|^{-j/2} \tilde{a}_j Z^{d-j}$$

belongs to $\operatorname{Hyp}_T^0(d)$. By Lemma 3.3, we have

$$Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}},$$

on some open ball $B(\tilde{a}, r) \subseteq \mathbb{R}^d$ such that $d_b := \deg Q_{\underline{b}} < d$, $d_c := \deg Q_{\underline{c}} < d$, and

$$\underline{b}_i = \psi_i(|\tilde{a}_2|^{-3/2}\tilde{a}_3, \ldots, |\tilde{a}_2|^{-d/2}\tilde{a}_d), \quad i = 1, \ldots, \deg Q_{\underline{b}},$$

where ψ_i are real analytic functions; likewise for $\underline{c}_i$. If $Q_{\underline{a}}$ is hyperbolic, then also $Q_{\underline{b}}$ and $Q_{\underline{c}}$ are hyperbolic; we restrict our attention to the set $B(\tilde{a}, r) \cap \operatorname{Hyp}_T(d)$. If $\underline{\lambda}_1 \le \cdots \le \underline{\lambda}_d$ are the roots of $Q_{\underline{a}}$, then we assume that, on $B(\tilde{a}, r) \cap \operatorname{Hyp}_T(d)$, $\underline{\lambda}_1 \le \cdots \le \underline{\lambda}_{d_b}$ are the roots of $Q_{\underline{b}}$ and $\underline{\lambda}_{d_b+1} \le \cdots \le \underline{\lambda}_d$ are the roots of $Q_{\underline{c}}$; this follows from continuity of the map $\lambda^{\uparrow}$ and the simple topology of $\operatorname{Hyp}_T(d)$, cf. [12, Theorem 8.1].

The splitting $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ induces a splitting

$$P_{\tilde{a}} = P_b P_c, \quad \text{on } B(\tilde{a}, r),$$

where

$$b_i = |\tilde{a}_2|^{i/2} \psi_i(|\tilde{a}_2|^{-3/2}\tilde{a}_3, \ldots, |\tilde{a}_2|^{-d/2}\tilde{a}_d), \quad i = 1, \ldots, \deg P_b. \tag{3.2}$$

The coefficients $\tilde{b}_i$ of $P_{\tilde{b}}$, resulting from P_b by the Tschirnhausen transformation, have an analogous representation.

$$\tilde{b}_i = |\tilde{a}_2|^{i/2} \tilde{\psi}_i(|\tilde{a}_2|^{-3/2}\tilde{a}_3, \ldots, |\tilde{a}_2|^{-d/2}\tilde{a}_d), \quad i = 1, \ldots, \deg P_b. \tag{3.3}$$

Shrinking $r > 0$ slightly, we may assume that all partial derivatives of ψ_i and $\tilde{\psi}_i$ of all orders are bounded on $B(\tilde{a}, r)$.

Furthermore, since the roots of $P_{\tilde a}$ are given by $\lambda_j := |\tilde a_2|^{1/2} \cdot \underline{\lambda}_j$, for $1 \le j \le d$, we have that, on $B(\tilde a, r) \cap \mathrm{Hyp}_T(d)$, $\lambda_1 \le \cdots \le \lambda_{d_b}$ are the roots of P_b and $\lambda_{d_b+1} \le \cdots \le \lambda_d$ are the roots of P_c.

Lemma 3.4 ([12, Lemma 3.13]). *In this situation, we have* $|\tilde b_2| \le 4\,|\tilde a_2|$.

Definition 3.5. For each $d \ge 2$ fix the following data. Choose a finite cover of $\mathrm{Hyp}^0_T(d)$ by balls $B_1, \ldots, B_s$ such that on each of these balls we have a splitting $P_{\tilde a} = P_b P_c$ as above together with analytic functions ψ_i and $\tilde\psi_i$, and we fix this splitting. There exists $r \in (0,1)$ such that for each $\underline{p} \in \mathrm{Hyp}^0_T(d)$ there is $1 \le i \le s$ with $B(\underline{p}, r) \subseteq B_i$. We refer to this data as a *universal splitting of hyperbolic polynomials of degree d in Tschirnhausen form* and to r as the *radius of the splitting*.

4. Bronshtein's theorem and a variant at a single point

We recall Bronshtein's theorem in Theorem 4.1. We shall need a version at a single point with a suitable bound for the derivative of the roots. This version is given in Theorem 4.5.

4.1. **Bronshtein's theorem.**

The following result is a version of Bronshtein's theorem [3] with uniform bounds due to [7], see also [12, Theorem 3.2].

Theorem 4.1. *Let $I \subseteq \mathbb{R}$ be an open interval and $a \in C^{d-1,1}(I, \mathrm{Hyp}(d))$. Then any continuous root $\lambda \in C^0(I)$ of P_a is locally Lipschitz and, for any pair of relatively compact open intervals $I_0 \Subset I_1 \Subset I$,*

$$|\lambda|_{C^{0,1}(\bar I_0)} \le C \max_{1 \le j \le d} \|a_j\|^{1/j}_{C^{d-1,1}(\bar I_1)}, \tag{4.1}$$

with $C = C(d)\,\max\{\delta^{-1}, 1\}$, where $\delta := \mathrm{dist}(I_0, \mathbb{R} \setminus I_1)$.

A multiparameter version follows easily; see [7] and [12, Theorem 3.4].

4.2. **Reclusive points.**

For the formulation of Theorem 4.5 we introduce the notion of reclusive points.

Definition 4.2. Let $I \subseteq \mathbb{R}$ be an open interval and $\tilde a \in C^{d-1,1}(I, \mathrm{Hyp}_T(d))$; recall that this means $\tilde a \in C^{d-1,1}(I, \mathbb{R}^d)$ and $\tilde a(I) \subseteq \mathrm{Hyp}_T(d)$. Let $x_0 \in I$ be such that $\tilde a_2(x_0) \ne 0$. Then $P_{\tilde a}$ splits in a neighborhood of x_0. We may assume that it is a *full splitting*: if $\{\lambda_1, \ldots, \lambda_k\}$ are the distinct roots of $P_{\tilde a(x_0)}$ with multiplicities $\{m_1, \ldots, m_k\}$ then

$$P_{\tilde a} = P_{b_1} P_{b_2} \cdots P_{b_k} \quad \text{on a neighborhood of } x_0, \tag{4.2}$$

where $\deg P_{b_j} = m_j$ and $P_{b_j(x_0)}(Z) = (Z - \lambda_j)^{m_j}$, for all $1 \le j \le k$. So, after Tschirnhausen transformation $b_j \rightsquigarrow \tilde b_j$, $\tilde b_{j,2}(x_0) = 0$ for all $1 \le j \le k$.

We say that $x_0 \in I$ is *reclusive for $\tilde a$* if

- x_0 is an isolated point of the zero set $Z_{\tilde a_2}$ of $\tilde a_2$
- or $x_0 \notin Z_{\tilde a_2}$ and x_0 is an isolated point of $Z_{\tilde b_{j,2}}$ for some $j \in \{1 \ldots, k\}$.

Note that x_0 is an isolated point of $Z_{\tilde b_{j,2}}$ if and only if x_0 is an isolated point of

$$E_{b_j} := \{x : \text{ all roots of } P_{b_j(x)} \text{ coincide}\}.$$

Lemma 4.3. *Let $I \subseteq \mathbb{R}$ be an open interval and $\tilde{a} \in C^{d-1,1}(I, \mathrm{Hyp}_T(d))$. Let $x_0 \in I$ be such that $\tilde{a}_2(x_0) \neq 0$ and assume that x_0 is not reclusive for $\tilde{a}$. If $P_{\tilde{a}} = P_b P_c$ is any splitting near x_0, then x_0 is not reclusive for $\tilde{b}$ and $\tilde{c}$ (which result from b and c by the Tschirnhausen transformation).*

Proof. After possibly reordering the factors in (4.2), we may assume that, on a neighborhood of x_0,

$$P_b = P_{b_1} \cdots P_{b_j} \quad \text{and} \quad P_c = P_{b_{j+1}} \cdots P_{b_k}.$$

The Tschirnhausen transformation $b \rightsquigarrow \tilde{b}$ effects a shift on all roots of P_b by $b_1/\deg P_b$ and retains the splitting

$$P_{\tilde{b}} = P_{\hat{b}_1} \cdots P_{\hat{b}_j}.$$

It follows that $E_{\hat{b}_i} = E_{b_i}$ for all $1 \le i \le j$. Suppose for contradiction that x_0 is reclusive for $\tilde{b}$. If x_0 is an isolated point of $Z_{\tilde{b}_2}$, then $j = 1$ and hence x_0 is reclusive for $\tilde{a}$. If $\tilde{b}_2(x_0) \neq 0$ and there is $i \in \{1, \ldots, j\}$ such that x_0 is an isolated point of $E_{\hat{b}_i} = E_{b_i}$, then again x_0 is reclusive for $\tilde{a}$. Since we assumed that x_0 is not reclusive for $\tilde{a}$, the assertion follows. □

Lemma 4.4. *Let $I \subseteq \mathbb{R}$ be an open interval and $\tilde{a} \in C^{d-1,1}(I, \mathrm{Hyp}_T(d))$. The set of all $x_0 \in I$ that are reclusive for $\tilde{a}$ is countable.*

Proof. Let $\lambda := \lambda^{\uparrow} \circ \tilde{a}$. Then λ is a curve in $\{x \in \mathbb{R}^d : x_1 \le x_2 \le \cdots \le x_d\}$. For $1 \le i < d$, let $\ell_i(x) := x_{i+1} - x_i$. If $x_0 \in I$ is reclusive for $\tilde{a}$, then there exist $1 \le i_1 < \cdots < i_k < d$ such that x_0 is an isolated point of $\{x \in I : \ell_{i_j}(\lambda(x)) = 0 \text{ for all } 1 \le j \le k\}$. The set of isolated points of the latter set is countable. The statement follows. □

4.3. **A version of Bronshtein's theorem at a single point.** For $x_0 \in \mathbb{R}$ and $r > 0$, let $I(x_0, r)$ denote the open interval centered at x_0 with radius r,

$$I(x_0, r) := \{x \in \mathbb{R} : |x - x_0| < r\}.$$

Its closure is denoted by $\overline{I}(x_0, r)$.

Theorem 4.5. *Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{a} \in C^{d-1,1}(\overline{I}(x_0, \delta), \mathrm{Hyp}_T(d))$. Assume that x_0 is not reclusive for $\tilde{a}$. Let $\lambda \in C^0(I(x_0, \delta))$ be a continuous root of $P_{\tilde{a}}$ and assume that $\lambda'(x_0)$ exists. Then*

$$|\lambda'(x_0)| \le C(d)\, A(\delta),$$

where

$$\begin{aligned} A(\delta) &:= 6 \max\{A_1(\delta), A_2(\delta)\}, \qquad (4.3)\\ A_1(\delta) &:= \max\left\{\delta^{-1} |\tilde{a}_2(x_0)|^{1/2}, |\tilde{a}_2'|^{1/2}_{C^{0,1}(\overline{I}(x_0,\delta))}\right\},\\ A_2(\delta) &:= \max_{2 \le j \le d} \left\{ |\tilde{a}_j^{(d-1)}|_{C^{0,1}(\overline{I}(x_0,\delta))} \cdot \|\tilde{a}_2\|^{(d-j)/2}_{L^\infty(I(x_0,\delta))} \right\}^{1/d}. \end{aligned}$$

The proof follows the general strategy of the proof of Theorem 4.1 in [7] and [12], but some modifications are required. Before we prove Theorem 4.5 let us recall two important tools.

4.4. **Local Glaeser inequality.** Glaeser's inequality [5] gives Bronshtein's theorem in the simplest nontrivial case: for nonnegative C^1 functions f on $\mathbb{R}$ with $f'' \in L^\infty(\mathbb{R})$ we have

$$f'(x)^2 \le 2\, f(x) \|f''\|_{L^\infty(\mathbb{R})}, \quad x \in \mathbb{R}.$$

We need a local version.

Lemma 4.6 ([12, Lemma 3.14])**.** *Let $I \subseteq \mathbb{R}$ be an open bounded interval. Let $f \in C^{1,1}(\overline{I})$ satisfy $f \ge 0$ or $f \le 0$ on I. Let $M > 0$ and assume that $x_0 \in I$, $f(x_0) \ne 0$, and $I_0 := I(x_0, M^{-1}|f(x_0)|^{1/2}) \subseteq I$. Then*

$$|f'(x_0)| \le (M + M^{-1}|f'|_{C^{0,1}(\overline{I}_0)})|f(x_0)|^{1/2}.$$

If additionally $|f'|_{C^{0,1}(\overline{I}_0)} \le M^2$, then

$$|f'(x_0)| \le 2M\, |f(x_0)|^{1/2}.$$

Note that if $f(x_0) = 0$ also $f'(x_0) = 0$.

4.5. **Interpolation.** Let us recall an interpolation inequality for intermediate derivatives.

Lemma 4.7 ([12, Lemma 3.16])**.** *Let $f \in C^{k,1}(\overline{I})$, where $I \subseteq \mathbb{R}$ is a bounded open interval. Then, for $1 \le j \le k$,*

$$|f^{(j)}(x)| \le C(k)\, |I|^{-j}\big(\|f\|_{L^\infty(I)} + |f^{(k)}|_{C^{0,1}(\overline{I})}|I|^{k+1}\big), \quad x \in I.$$

4.6. **Proof of Theorem 4.5.** The rest of the section is devoted to the proof of Theorem 4.5.

Lemma 4.8. *Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{a} \in C^{d-1,1}(\overline{I}(x_0, \delta), \mathrm{Hyp}_T(d))$ be such that $\tilde{a}_2(x_0) \ne 0$. Let $A(\delta)$ be defined by (4.3) and, for any $A > 0$, set*

$$I_A(x_0) := I(x_0, A^{-1}|\tilde{a}_2(x_0)|^{1/2})$$

and let $\overline{I}_A(x_0)$ denote its closure. Then the following holds:

(1) *$I_{A(\delta)}(x_0) \subseteq I(x_0, \delta)$.*

(2) *For all $x \in I_{A(\delta)}(x_0)$,*

$$\frac{1}{2} \le \frac{\tilde{a}_2(x)}{\tilde{a}_2(x_0)} \le 2. \tag{4.4}$$

(3) *For all $2 \le j \le d$, $1 \le k \le d-1$, and $x \in I_{A(\delta)}(x_0)$,*

$$|\tilde{a}_j^{(k)}(x)| \le C(d) A(\delta)^k\, |\tilde{a}_2(x_0)|^{(j-k)/2}. \tag{4.5}$$

(4) *For all $2 \le j \le d$,*

$$|\tilde{a}_j^{(d-1)}|_{C^{0,1}(\overline{I}_{A(\delta)}(x_0))} \le A(\delta)^d\, |\tilde{a}_2(x_0)|^{(j-d)/2}. \tag{4.6}$$

Proof. (1) By definition, $A(\delta) \ge A_1(\delta) \ge \delta^{-1}|\tilde{a}_2(x_0)|^{1/2}$ and thus

$$I_{A(\delta)}(x_0) \subseteq I_{A_1(\delta)}(x_0) \subseteq I(x_0, \delta).$$

(2) By Lemma 4.6 and the definition of $A_1(\delta)$,

$$|\tilde{a}_2'(x_0)| \le 2A_1(\delta)\, |\tilde{a}_2(x_0)|^{1/2}.$$

Then, for $x \in I_{6A_1(\delta)}(x_0)$,

$$|\tilde{a}_2(x) - \tilde{a}_2(x_0)| \le |\tilde{a}_2'(x_0)||x - x_0| + |\tilde{a}_2'|_{C^{0,1}(\overline{I}_{A_1(\delta)}(x_0))}|x - x_0|^2$$

$$\leq \frac{1}{3}|\tilde{a}_2(x_0)| + \frac{1}{36}|\tilde{a}_2(x_0)| \leq \frac{1}{2}|\tilde{a}_2(x_0)|$$

which implies (4.4).

(4) By the definition of $A_2(\delta)$, (4.6) is clear.

(3) By Lemma 3.1 and (4.4), we have $|\tilde{a}_j(x)| \leq (\sqrt{2}\,|\tilde{a}_2(x)|^{1/2})^j \leq 2^j\,|\tilde{a}_2(x_0)|^{j/2}$, for $x \in I_{A(\delta)}(x_0)$. In conjuction with (4.6), it implies (4.5), by Lemma 4.7. □

Definition 4.9. Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{a} \in C^{d-1,1}(\overline{I}(x_0,\delta), \mathrm{Hyp}_T(d))$ be such that $\tilde{a}_2(x_0) \neq 0$. Let $A > 0$ be a constant. Let $A > 0$ be a constant. We say that $(\tilde{a}, x_0, \delta, A)$ is $C^{d-1,1}$-*admissible* if the following holds:

(1) $I_A(x_0) \subseteq I(x_0,\delta)$.
(2) For all $x \in I_A(x_0)$,
$$\frac{1}{2} \leq \frac{\tilde{a}_2(x)}{\tilde{a}_2(x_0)} \leq 2. \tag{4.7}$$
(3) For all $2 \leq j \leq d$, $1 \leq k \leq d-1$, and $x \in I_A(x_0)$,
$$|\tilde{a}_j^{(k)}(x)| \leq A^k\,|\tilde{a}_2(x_0)|^{(j-k)/2}. \tag{4.8}$$
(4) For all $2 \leq j \leq d$,
$$|\tilde{a}_j^{(d-1)}|_{C^{0,1}(\overline{I}_A(x_0))} \leq A^d\,|\tilde{a}_2(x_0)|^{(j-d)/2}. \tag{4.9}$$

Note that Lemma 4.8 shows that $(\tilde{a}, x_0, \delta, C(d)A(\delta))$ is $C^{d-1,1}$-admissible with $A(\delta)$ defined by (4.3) and $C(d) \geq 1$.

Lemma 4.10. *Let $(\tilde{a}, x_0, \delta, A)$ be $C^{d-1,1}$-admissible. Then the functions $\underline{a}_j := |\tilde{a}_2|^{-j/2}\tilde{a}_j$, $2 \leq j \leq d$, are well-defined on $I_A(x_0)$ and satisfy*

$$|\underline{a}_j^{(k)}(x)| \leq C(d)\,A^k\,|\tilde{a}_2(x_0)|^{-k/2}, \quad 2 \leq j \leq d,\ 1 \leq k \leq d-1,\ x \in I_A(x_0), \tag{4.10}$$
$$|\underline{a}_j^{(d-1)}|_{C^{0,1}(\overline{I}_A(x_0))} \leq C(d)\,A^d\,|\tilde{a}_2(x_0)|^{-d/2}, \quad 2 \leq j \leq d. \tag{4.11}$$

Proof. This follows easily; see [7] for details. □

Lemma 4.11. *Let $x_0 \in \mathbb{R}$ and $A, \delta > 0$. Let $\tilde{a} \in C^{d-1,1}(\overline{I}(x_0,\delta), \mathrm{Hyp}_T(d))$ be such that $\tilde{a}_2(x_0) \neq 0$. Assume that $I_A(x_0) \subseteq I(x_0,\delta)$, (4.8) for $k \geq j$, $2 \leq j \leq d$, and (4.9) hold. Then there is a constant $C(d) \geq 1$ such that $(\tilde{a}, x_0, \delta, C(d)A)$ is $C^{d-1,1}$-admissible.*

Proof. By (4.8) for $j = k = 2$, we have $|\tilde{a}_2'|_{C^{0,1}(\overline{I}_A(x_0))} \leq A^2$. By Lemma 4.6,
$$|\tilde{a}_2'(x_0)| \leq 2A\,|\tilde{a}_2(x_0)|^{1/2}.$$
Thus we get (4.7), for $x \in I_{6A}(x_0)$, as in the proof of Lemma 4.8. That (4.8) also holds for $k < j$ (up to multiplying the right-hand side with a suitable constant $C(d) \geq 1$) follows from Lemma 4.7 applied to $f = \tilde{a}_j$ and $k = j-1$, together with (4.7) and Lemma 3.1. □

Proposition 4.12. *Let $(\tilde{a}, x_0, \delta, A)$ be $C^{d-1,1}$-admissible. There exists $\delta_1 > 0$ and a constant $C(d) > 1$ such that the following holds. There is a splitting*
$$P_{\tilde{a}} = P_b P_c, \quad \text{on } I(x_0,\delta_1),$$
where P_b and P_c are monic hyperbolic polynomials of degree $< d$ with coefficients in $C^{d-1,1}(\overline{I}(x_0,\delta_1))$. We have, for all $1 \leq i \leq \deg P_b$,
$$|b_i^{(k)}(x)| \leq C(d)A^k\,|\tilde{a}_2(x_0)|^{(i-k)/2}, \quad 1 \leq k \leq d-1,\ x \in I(x_0,\delta_1), \tag{4.12}$$

$$|b_i^{(d-1)}|_{C^{0,1}(\overline{I}(x_0,\delta_1))} \le C(d) A^d \, |\tilde a_2(x_0)|^{(i-d)/2}. \tag{4.13}$$

If, after Tschirnhausen transformation, $\tilde b_2(x_0) \ne 0$*, then* $(\tilde b, x_0, \delta_1, C(d)A)$ *is* $C^{d-1,1}$*-admissible. (Analogously, for* $\tilde c$*.)*

Proof. Consider the continuous bounded (cf. Lemma 3.1) curve

$$\underline{a} := (0, -1, \underline{a}_3, \ldots, \underline{a}_d) : I_A(x_0) \to \mathbb{R}^d,$$

where $\underline{a}_j := |\tilde a_2|^{-j/2} \tilde a_j$. Then, by (4.10), there exists $C_1 = C_1(d) > 1$ such that

$$\|\underline{a}'(x)\|_2 \le C_1 A \, |\tilde a_2(x_0)|^{-1/2}, \quad x \in I_A(x_0).$$

Let $0 < r < 1$ be the radius of the splitting (see Definition 3.5) and define

$$\delta_1 := \frac{|\tilde a_2(x_0)|^{1/2} \, r}{C_1 A}.$$

Then $I(x_0, \delta_1) \subseteq I_A(x_0)$ and $\underline{a}(I(x_0, \delta_1)) \subseteq B(\underline{a}(x_0), r)$. Consequently, we have a splitting

$$P_{\tilde a} = P_b P_c, \quad \text{ on } I(x_0, \delta_1);$$

cf. Definition 3.5. The estimates (4.12) and (4.13) follow from (3.2) and Lemma 4.10; for details see [12, Proposition 3.20].

Suppose that $\tilde b_2(x_0) \ne 0$ and let us show that $(\tilde b, x_0, \delta_1, C(d)A)$, for suitable $C(d) > 1$, is $C^{d-1,1}$-admissible. Set

$$B := \frac{2 C_1 A}{r}.$$

Then, by Lemma 3.4,

$$B^{-1} |\tilde b_2(x_0)|^{1/2} \le \frac{|\tilde a_2(x_0)|^{1/2} \, r}{C_1 A} = \delta_1,$$

whence $J_B(x_0) := I(x_0, B^{-1}|\tilde b_2(x_0)|^{1/2}) \subseteq I(x_0, \delta_1)$. From (4.12) and (4.13), we easily get the same bounds for $\tilde b_i$ instead of b_i (by means of (3.1)). Since $|\tilde a_2(x_0)|^{-1} \le 4 \, |\tilde b_2(x_0)|^{-1}$, by Lemma 3.4, we may replace $\tilde a_2(x_0)$ by $\tilde b_2(x_0)$ in these estimates if $k \ge i$. Now it suffices to invoke Lemma 4.11. □

Proposition 4.13. *Let* $(\tilde a, x_0, \delta, A)$ *be* $C^{d-1,1}$*-admissible and assume that* x_0 *is not reclusive for* $\tilde a$*. If* $\lambda \in C^0(I(x_0, \delta))$ *is a root of* $P_{\tilde a}$ *and* $\lambda'(x_0)$ *exists, then*

$$|\lambda'(x_0)| \le C(d) A.$$

Proof. By assumption, $\tilde a_2(x_0) \ne 0$ and hence $d \ge 2$. By Proposition 4.12, there exists $\delta_1 > 0$ such that there is a splitting $P_{\tilde a} = P_b P_c$ on $I(x_0, \delta_1)$. We may assume that λ is a root of P_b and hence

$$\lambda(x) = -\frac{b_1(x)}{\deg P_b} + \mu(x), \quad x \in I(x_0, \delta_1), \tag{4.14}$$

where μ is a continuous root of $P_{\tilde b}$ and $\mu'(x_0)$ exists (since we assumed that $\lambda'(x_0)$ exists). By (4.12) for $i = k = 1$, we have

$$|b_1'(x_0)| \le C(d) A. \tag{4.15}$$

By Lemma 4.3, x_0 is not reclusive for $\tilde b$.

Let us now proceed by induction on d.

In the case $d = 2$, we have $\tilde b \equiv 0$ and $\lambda(x) = -b_1(x)$ for $x \in I(x_0, \delta_1)$ so that (4.15) gives the assertion.

Assume that $d \ge 3$. If $\tilde b_2(x_0) \ne 0$, then $(\tilde b, x_0, \delta_1, C(d)A)$ is $C^{d-1,1}$-admissible, by Proposition 4.12. By the induction hypothesis,

$$|\mu'(x_0)| \le C(d)A.$$

Thus the statement for $\lambda'(x_0)$ follows from (4.14) and (4.15).

If $\tilde b_2(x_0) = 0$, then x_0 (being not reclusive for $\tilde b$) is an accumulation point of $Z_{\tilde b_2}$. It follows that $\mu'(x_0) = 0$ and the assertion again follows. □

Proof of Theorem 4.5. Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde a \in C^{d-1,1}(\overline{I}(x_0,\delta), \mathrm{Hyp}_T(d))$. Assume that x_0 is not reclusive for $\tilde a$. Let $\lambda \in C^0(I(x_0,\delta))$ be a continuous root of $P_{\tilde a}$ and assume that $\lambda'(x_0)$ exists.

If $\tilde a_2(x_0) \ne 0$, then $(\tilde a, x_0, \delta, C(d)A(\delta))$ is $C^{d-1,1}$-admissible with $A(\delta)$ defined by (4.3) and $C(d) \ge 1$, in view of Lemma 4.8. Then Proposition 4.13 yields

$$|\lambda'(x_0)| \le C(d)A(\delta).$$

If $\tilde a_2(x_0) = 0$, then x_0 (being not reclusive for $\tilde a$) is an accumulation point of $Z_{\tilde a_2}$. Hence $\lambda'(x_0) = 0$ and the assertion is trivially true. □

5. Proof of Theorem 1.3

Let $I \subseteq \mathbb{R}$ be an open interval. Let $a_n \to a$ in $C^d(I, \mathrm{Hyp}(d))$, i.e., for each relatively compact open interval $I_1 \Subset I$,

$$\|a - a_n\|_{C^d(\overline{I}_1,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

It follows from Theorem 4.1 that the set $\{\lambda^\uparrow \circ a_n : n \ge 1\}$ is bounded in $C^{0,1}(I, \mathbb{R}^d)$. We must show that, for each relatively compact open interval $I_0 \Subset I$ and each $1 \le q < \infty$,

$$\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{W^{1,q}(I_0,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

5.1. **Strategy of the proof.** The proof that

$$\|(\lambda^\uparrow \circ a)' - (\lambda^\uparrow \circ a_n)'\|_{L^q(I_0,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty \tag{5.1}$$

is based on the dominated convergence theorem. We check its assumptions in two steps:

Step 1: The sequence $(\lambda^\uparrow \circ a_n)'$ is dominated almost everywhere on I_0 by a nonnegative L^q function. This part follows easily from the uniform bound (4.1).

Step 2: The sequence $(\lambda^\uparrow \circ a_n)'$ converges to $(\lambda^\uparrow \circ a)'$ pointwise almost everywhere in I_0 as $n \to \infty$. This is the new part of the proof.

Finally, the proof is completed by

Step 3: The sequence $\lambda^\uparrow \circ a_n$ converges to $\lambda^\uparrow \circ a$ with respect to the L^∞ norm on I_0 as $n \to \infty$.

Step 1. Fix $I_0 \Subset I_1 \Subset I$. By assumption of Theorem 1.3, $\{a_n|_{I_1} : n \ge 1\}$ is a bounded subset of $C^{d-1,1}(\overline{I}_1, \mathbb{R}^d)$. By Theorem 4.1, the derivative of $\lambda^\uparrow \circ a_n$ exists almost everywhere in I_0 and satisfies

$$\|(\lambda^\uparrow \circ a_n)'\|_{L^\infty(I_0,\mathbb{R}^d)} \le C \sup_{n\ge 1} \max_{1\le j\le d} \|a_{n,j}\|^{1/j}_{C^{d-1,1}(\overline{I}_1)} =: B < \infty.$$

In particular, the sequence $(\lambda^\uparrow \circ a_n)'$ is dominated almost everywhere on I_0 by the constant B, which evidently is a L^q function on I_0 for every $1 \le q < \infty$.

Step 2. We will prove the following result.

Theorem 5.1. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $a_n \to a$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for almost every $x \in I$,*

$$(\lambda^\uparrow \circ a_n)'(x) \to (\lambda^\uparrow \circ a)'(x) \quad \text{as } n \to \infty.$$

As a consequence, the dominated convergence theorem yields that (5.1) holds, for each relatively compact open interval $I_0 \Subset I$ and each $1 \le q < \infty$.

Step 3. The map $\lambda^\uparrow : \mathrm{Hyp}(d) \to \mathbb{R}^d$ is continuous (cf. [1, Lemma 4.1]). Hence $\lambda^\uparrow(a_n(x)) \to \lambda^\uparrow(a(x))$ for each $x \in I_0$ as $n \to \infty$. Fix $x_0 \in I_0$ and $1 \le j \le d$. Then, for all $x \in I_0$,

$$\begin{aligned}
&|\lambda_j^\uparrow(a(x)) - \lambda_j^\uparrow(a_n(x))| \\
&\quad = \Big|\lambda_j^\uparrow(a(x_0)) - \lambda_j^\uparrow(a_n(x_0)) + \int_{x_0}^x (\lambda_j^\uparrow \circ a)'(t) - (\lambda_j^\uparrow \circ a_n)'(t)\, dt\Big| \\
&\quad \le |\lambda_j^\uparrow(a(x_0)) - \lambda_j^\uparrow(a_n(x_0))| + \|(\lambda_j^\uparrow \circ a)' - (\lambda_j^\uparrow \circ a_n)'\|_{L^1(I_0)}.
\end{aligned}$$

Thus, in view of (5.1), we find

$$\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{L^\infty(I_0, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{5.2}$$

This ends the proof of Theorem 1.3, assuming the validity of Theorem 5.1.

The rest of the section is devoted to the proof of Theorem 5.1.

5.2. **On the zero set of $\tilde a_2$.** For a function $f : I \to \mathbb{R}$, we denote by $Z_f := \{x \in I : f(x) = 0\}$ its zero set and by $\mathrm{acc}(Z_f)$ the set of accumulation points of Z_f.

Lemma 5.2. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $\tilde a_n \to \tilde a$ in $C^d(I, \mathrm{Hyp}_T(d))$ as $n \to \infty$. Then, for almost every $x_0 \in Z_{\tilde a_2}$,*

$$(\lambda^\uparrow \circ \tilde a_n)'(x_0) \to 0 \quad \text{as } n \to \infty.$$

Proof. Fix $x_0 \in \mathrm{acc}(Z_{\tilde a_2})$. By Lemma 3.1, $x_0 \in \mathrm{acc}(Z_{\tilde a_j})$, for all $2 \le j \le d$. Thus $\tilde a_j^{(k)}(x_0) = 0$ for all $2 \le j \le d$ and $0 \le k \le d$, by Rolle's theorem. Let $\epsilon > 0$ be fixed. By continuity, there exists $\delta > 0$ such that $I(x_0, \delta) \Subset I$ and

$$\|\tilde a_j\|_{C^d(\overline I(x_0,\delta))} \le \frac{\epsilon^j}{2}, \quad 2 \le j \le d.$$

By the assumption, there exists $n_0 \ge 1$ such that, for $n \ge n_0$,

$$\|\tilde a_j - \tilde a_{n,j}\|_{C^d(\overline I(x_0,\delta))} \le \frac{\epsilon^j}{2}, \quad 2 \le j \le d,$$

and

$$|\tilde a_{n,2}(x_0)| \le \delta^2 \epsilon^2.$$

In particular, for $n \ge n_0$ and $2 \le j \le d$,

$$\|\tilde a_{n,j}\|_{C^d(\overline I(x_0,\delta))} \le \|\tilde a_j\|_{C^d(\overline I(x_0,\delta))} + \|\tilde a_j - \tilde a_{n,j}\|_{C^d(\overline I(x_0,\delta))} \le \epsilon^j.$$

If x_0 is not reclusive for $\tilde a_n$ and $(\lambda^\uparrow \circ \tilde a_n)'(x_0)$ exists, then we may apply Theorem 4.5 to $\tilde a_n$ and conclude

$$|(\lambda^\uparrow \circ a_n)'(x_0)| \le C(d)\, \epsilon.$$

By Lemma 4.4, the set $\{x_0 \in I : \ \exists n \ge 1 \text{ such that } x_0 \text{ is reclusive for } \tilde a_n\}$ has measure zero. Since also $Z_{\tilde a_2} \setminus \mathrm{acc}(Z_{\tilde a_2})$ has measure zero, the assertion follows. $\square$

5.3. **Admissible data.** At points x_0 with $\tilde{a}_2(x_0) \neq 0$ we have a splitting of $P_{\tilde{a}}$ and we may use induction on the degree. The following definition is a preparation for the induction argument.

Definition 5.3. Let $I_1 \subseteq \mathbb{R}$ be an open bounded interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{a} \in C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$. Let $A > 0$ be a constant. Set

$$I_{A,\tilde{a}}(x_0) := I(x_0, A^{-1}|\tilde{a}_2(x_0)|^{1/2}).$$

We say that $(\tilde{a}, I_1, I_0, A)$ is C^d-*admissible* if, for every $x_0 \in I_0 \setminus \{x : \tilde{a}_2(x) = 0\}$, the following holds:

(1) $I_{A,\tilde{a}}(x_0) \subseteq I_1$.
(2) For all $x \in I_{A,\tilde{a}}(x_0)$,

$$\frac{1}{2} \le \frac{\tilde{a}_2(x)}{\tilde{a}_2(x_0)} \le 2. \tag{5.3}$$

(3) For all $2 \le j \le d$, $1 \le k \le d$, and $x \in I_{A,\tilde{a}}(x_0)$,

$$|\tilde{a}_j^{(k)}(x)| \le A^k\, |\tilde{a}_2(x_0)|^{(j-k)/2}. \tag{5.4}$$

Note that if we take $I_1 := I(x_0, \delta)$, let I_0 shrink to the point x_0, assume $\tilde{a}_2(x_0) \neq 0$, and use $C^{d-1,1}$ regularity instead of C^d, we recover the notion from Definition 4.9.

Lemma 5.4. *Let $I_1 \subseteq \mathbb{R}$ be a bounded open interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{a}_n \to \tilde{a}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ as $n \to \infty$. Set*

$$A := 6 \max\{A_1, A_2\}, \tag{5.5}$$

where, using $\tilde{a}_{0,j} = \tilde{a}_j$ for convenience and $\delta := \mathrm{dist}(I_0, \mathbb{R} \setminus I_1)$,

$$A_1 := \sup_{n\ge 0} \max\big\{\delta^{-1}\|\tilde{a}_{n,2}\|^{1/2}_{L^\infty(I_1)}, |\tilde{a}'_{n,2}|^{1/2}_{C^{0,1}(\overline{I}_1)}\big\},$$

$$A_2 := \sup_{n\ge 0} \max_{2\le j\le d} \big\{|\tilde{a}^{(d-1)}_{n,j}|_{C^{0,1}(\overline{I}_1)} \cdot \|\tilde{a}_{n,2}\|^{(d-j)/2}_{L^\infty(I_1)}\big\}^{1/d}.$$

Then $(\tilde{a}, I_1, I_0, A)$ and $(\tilde{a}_n, I_1, I_0, A)$, for $n \ge 1$, are C^d-admissible.

Proof. This is an easy consequence of [12, Lemma 3.23]. □

5.4. **Towards a simultaneous splitting.** Our next goal is to show that, if $(\tilde{a}, I_1, I_0, A)$ and $(\tilde{a}_n, I_1, I_0, A)$, for $n \ge 1$, are C^d-admissible and (5.6) holds, then $P_{\tilde{a}}$ and $P_{\tilde{a}_n}$, for n large enough, admit a simultaneous splitting; see Definition 5.5.

Let $I_1 \subseteq \mathbb{R}$ be a bounded open interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{a}_n \to \tilde{a}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$, i.e.,

$$\|\tilde{a} - \tilde{a}_n\|_{C^d(\overline{I}_1, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{5.6}$$

Assume that $(\tilde{a}, I_1, I_0, A)$ and $(\tilde{a}_n, I_1, I_0, A)$, for $n \ge 1$, are C^d-admissible for some $A > 0$.

Fix $x_0 \in I_0 \setminus \{x : \tilde{a}_2(x) = 0\}$. By (5.6), there is $n_0 \ge 1$ such that

$$\big||\tilde{a}_2(x_0)|^{1/2} - |\tilde{a}_{n,2}(x_0)|^{1/2}\big| < \frac{1}{2}\, |\tilde{a}_2(x_0)|^{1/2}, \quad n \ge n_0,$$

and hence

$$\frac{1}{2} < \frac{|\tilde{a}_{n,2}(x_0)|^{1/2}}{|\tilde{a}_2(x_0)|^{1/2}} < 2, \quad n \ge n_0. \tag{5.7}$$

So, for $n \ge n_0$,

$$I_{2A,\tilde{a}}(x_0) \subseteq I_{A,\tilde{a}_n}(x_0) \subseteq I_{2A/3,\tilde{a}}(x_0). \tag{5.8}$$

For simplicity, we henceforth write

$$I_C(x_0) := I_{C,\tilde a}(x_0) = I(x_0, C^{-1}|\tilde a_2(x_0)|^{1/2}),$$

for $C > 0$. Since $(\tilde a_n, I_1, I_0, A)$, for $n \ge 1$, is C^d-admissible and thanks to (5.8) we see that, for $n \ge n_0$,

$$I_{2A}(x_0) \subseteq I_1, \tag{5.9}$$

$$\frac{1}{2} \le \frac{\tilde a_{n,2}(x)}{\tilde a_{n,2}(x_0)} \le 2, \quad t \in I_{2A}(x_0), \tag{5.10}$$

$$|\tilde a^{(k)}_{n,j}(x)| \le A^k\, |\tilde a_{n,2}(x_0)|^{(j-k)/2}, \quad 2 \le j \le d,\ 1 \le k \le d,\ x \in I_{2A}(x_0). \tag{5.11}$$

Consider the C^d curves

$$\underline{a} := (0, -1, \underline{a}_3, \ldots, \underline{a}_d) : I_{2A}(x_0) \to \mathbb{R}^d,$$
$$\underline{a}_n := (0, -1, \underline{a}_{n,3}, \ldots, \underline{a}_{n,d}) : I_{2A}(x_0) \to \mathbb{R}^d, \quad n \ge n_0,$$

where $\underline{a}_j := |\tilde a_2|^{-j/2}\tilde a_j$ and $\underline{a}_{n,j} := |\tilde a_{n,2}|^{-j/2}\tilde a_{n,j}$. Then, by [12, Lemma 3.18] which is a variant of Lemma 4.10, there is a constant $C_1 = C_1(d) > 1$ such that, for $x \in I_{2A}(x_0)$,

$$\|\underline{a}'(x)\|_2 \le C_1 A\, |\tilde a_2(x_0)|^{-1/2} \quad \text{and} \quad \|\underline{a}_n'(x)\|_2 \le C_1 A\, |\tilde a_{n,2}(x_0)|^{-1/2}.$$

Let $0 < r < 1$ be the radius of the splitting (see Definition 3.5) and define

$$J_1 := I_{4C_1A/r}(x_0) = I(x_0, \tfrac{r}{4C_1A}|\tilde a_2(x_0)|^{1/2}).$$

Then $\underline{a}(J_1) \subseteq B(\underline{a}(x_0), r/4)$ and $\underline{a}_n(J_1) \subseteq B(\underline{a}_n(x_0), r/2)$, using (5.7). By (5.6), there is $n_1 \ge n_0$ such that

$$\|\underline{a}(x_0) - \underline{a}_n(x_0)\|_2 < \frac{r}{4}, \quad n \ge n_1. \tag{5.12}$$

Consequently, $B(\underline{a}_n(x_0), r/2)$ is contained in $B(\underline{a}(x_0), 3r/4)$, for $n \ge n_1$.

In view of Definition 3.5, we have splittings on J_1,

$$P_{\tilde a} = P_b P_c \quad \text{and} \quad P_{\tilde a_n} = P_{b_n} P_{c_n}, \quad n \ge n_1, \tag{5.13}$$

with the following properties:

1. $d_b := \deg P_b = \deg P_{b_n}$, for all $n \ge n_1$, and $d_b < d$.
2. There exist bounded analytic functions $\psi_1, \ldots, \psi_{d_b}$ with bounded partial derivatives of all orders such that, for $x \in J_1$ and $1 \le i \le d_b$,

$$b_i(x) = |\tilde a_2(x)|^{i/2}\, \psi_i(\underline{a}(x)),$$
$$b_{n,i}(x) = |\tilde a_{n,2}(x)|^{i/2}\, \psi_i(\underline{a}_n(x)), \quad n \ge n_1.$$

The same is true for the second factors P_c and P_{c_n}.

Definition 5.5. We say that the family $\{P_{\tilde a}\} \cup \{P_{\tilde a_n}\}_{n \ge n_1}$ has a *simultaneous splitting on* J_1 if (5.13) and the properties (1) and (2) are satisfied.

Note that, applying the Tschirnhausen transformation to P_b and P_{b_n}, we find bounded analytic functions $\tilde\psi_1, \ldots, \tilde\psi_{d_b}$ with bounded partial derivatives of all orders such that, for $x \in J_1$ and $1 \le i \le d_b$,

$$\tilde b_i(x) = |\tilde a_2(x)|^{i/2}\, \tilde\psi_i(\underline{a}(x)),$$
$$\tilde b_{n,i}(x) = |\tilde a_{n,2}(x)|^{i/2}\, \tilde\psi_i(\underline{a}_n(x)), \quad n \ge n_1.$$

That follows from (3.1).

Lemma 5.6. *We have* $b_n \to b$ *and* $\tilde b_n \to \tilde b$ *in* $C^d(\overline{J}_1, \mathbb{R}^{d_b})$ *as* $n \to \infty$.

Proof. By (5.3) and (5.10), $|\tilde a_2|^{1/2}, |\tilde a_{n,2}|^{1/2} \in C^d(\overline{J}_1)$ and $\underline{a}, \underline{a}_n \in C^d(\overline{J}_1, \mathbb{R}^d)$, for $n \ge n_0$, and the assertion follows from Proposition 2.1. □

Summarizing, we have the following proposition.

Proposition 5.7. *Let* $I_1 \subseteq \mathbb{R}$ *be a bounded open interval and* $I_0 \Subset I_1$ *a relatively compact open subinterval. Let* $\tilde a_n \to \tilde a$ *in* $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ *as* $n \to \infty$. *Assume that* $(\tilde a, I_1, I_0, A)$ *and* $(\tilde a_n, I_1, I_0, A)$, *for* $n \ge 1$, *are* C^d*-admissible for some* $A > 0$. *Let* $x_0 \in I_0 \setminus \{x : \tilde a_2(x) = 0\}$. *Then the following holds:*

1. *There exist an interval* J_1 *containing* x_0 *and* $n_0 \ge 1$ *such that the family* $\{P_{\tilde a}\} \cup \{P_{\tilde a_n}\}_{n \ge n_0}$ *has a simultaneous splitting* (5.13) *on* J_1.
2. *For the factors in the simultaneous splitting* (5.13), $b_n \to b$ *and* $\tilde b_n \to \tilde b$ *in* $C^d(\overline{J}_1, \mathbb{R}^{d_b})$ *as* $n \to \infty$.
3. *There exist a relatively compact open subinterval* $J_0 \Subset J_1$ *containing* x_0 *and* $C = C(d) > 1$ *such that* $(\tilde b, J_1, J_0, CA)$ *and* $(\tilde b_n, J_1, J_0, CA)$, *for* $n \ge n_0$, *are* C^d*-admissible.*

(2) *and* (3) *also hold for* $b, b_n, \tilde b, \tilde b_n$ *replaced by* $c, c_n, \tilde c, \tilde c_n$.

Proof. (1) This was proved above.
(2) Lemma 5.6.
(3) follows from [12, Proposition 3.20]; one may take $J_0 := I_{8C_1A/r}(x_0)$. □

5.5. **The induction argument.**

Proposition 5.8. *Let* $I_1 \subseteq \mathbb{R}$ *be a bounded open interval and* $I_0 \Subset I_1$ *a relatively compact open subinterval. Let* $\tilde a_n \to \tilde a$ *in* $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ *as* $n \to \infty$. *Assume that* $(\tilde a, I_1, I_0, A)$ *and* $(\tilde a_n, I_1, I_0, A)$, *for* $n \ge 1$, *are* C^d*-admissible for some* $A > 0$. *Then, for almost every* $x \in I_0$,

$$(\lambda^\uparrow \circ \tilde a_n)'(x) \to (\lambda^\uparrow \circ \tilde a)'(x) \qquad \text{as } n \to \infty. \tag{5.14}$$

Proof. We proceed by induction on d. The base case is trivial, since Z is the unique polynomial in Tschirnhausen form of degree 1. Let us assume that $d \ge 2$ and the statement is true for monic hyperbolic polynomials of degree $\le d-1$.

By Lemma 5.2, it is enough to show that (5.14) holds for almost every $x \in I_0$ such that $\tilde a_2(x) \ne 0$. Fix $x_0 \in I_0 \setminus \{x : \tilde a_2(x) = 0\}$. By Proposition 5.7, there exist intervals $J_1 \Supset J_0 \ni x_0$, $n_0 \ge 1$, and $C = C(d) > 1$ such that the family $\{P_{\tilde a}\} \cup \{P_{\tilde a_n}\}_{n \ge n_0}$ has a simultaneous splitting (5.13) on J_1, $(\tilde b, J_1, J_0, CA)$ and $(\tilde b_n, J_1, J_0, CA)$, for $n \ge n_0$, are C^d-admissible, and $b_n \to b$ and $\tilde b_n \to \tilde b$ in $C^d(\overline{J}_1, \mathbb{R}^{d_b})$ as $n \to \infty$.

We may assume that, for $x \in J_1$,

$$\mu(x) := (\lambda_1^\uparrow(\tilde a(x)), \lambda_2^\uparrow(\tilde a(x)), \ldots, \lambda_{d_b}^\uparrow(\tilde a(x)))$$

is the increasingly ordered root vector of $P_{b(x)}$ and, for $n \ge n_0$,

$$\mu_n(x) := (\lambda_1^\uparrow(\tilde a_n(x)), \lambda_2^\uparrow(\tilde a_n(x)), \ldots, \lambda_{d_b}^\uparrow(\tilde a_n(x)))$$

is the increasingly ordered root vector of $P_{b_n(x)}$; see Definition 3.5. Then

$$\mu(x) + \tfrac{1}{b_c}(b_1(x), \ldots, b_1(x)) \quad \text{and} \quad \mu_n(x) + \tfrac{1}{b_c}(b_{n,1}(x), \ldots, b_{n,1}(x))$$

are the corresponding root vectors for $P_{\tilde b(x)}$ and $P_{\tilde b_n(x)}$, respectively. By induction hypothesis and since $b'_{n,1}(x) \to b'_1(x)$ as $n \to \infty$, we have

$$\mu'_n(x) \to \mu'(x) \quad \text{as } n \to \infty,$$

for almost every $x \in J_0$.

Treating the second factors P_c and P_{c_n} analogously, we conclude that (5.14) holds for almost every $x \in J_0$.

The set $I_0 \setminus \{x : \tilde a_2(x) = 0\}$ can be covered by the open intervals J_0 and this cover admits a countable subcover. This ends the proof. □

5.6. **Proof of Theorem 5.1.** Let $I \subseteq \mathbb{R}$ be an open interval. Let $a_n \to a$ in $C^d(I, \operatorname{Hyp}(d))$ as $n \to \infty$. The Tschirnhausen transformation effects a shift of $\lambda^\uparrow \circ a$ by $\frac{1}{d}(a_1, \ldots, a_1)$ and $\lambda^\uparrow \circ a_n$ by $\frac{1}{d}(a_{n,1}, \ldots, a_{n,1})$. The new coefficients are polynomials in the old ones, see (3.1). Hence we may assume that the polynomials are all in Tschirnhausen form (by Proposition 2.1). Then Theorem 5.1 follows from Lemma 5.4 and Proposition 5.8.

This also completes the proof of Theorem 1.3.

Remark 5.9. We need C^d convergence in Lemma 5.2. For all other arguments, it would be enough to work in the class $C^{d-1,1}$.

6. Proofs of Theorem 1.1 and Corollary 1.2

6.1. **A multiparameter version.** The following theorem is a multiparameter version of Theorem 1.3.

Theorem 6.1. *Let $U \subseteq \mathbb{R}^m$ be open. Let $a_n \to a$ in $C^d(U, \operatorname{Hyp}(d))$, i.e., for each relatively compact open subset $U_1 \Subset U$,*

$$\|a - a_n\|_{C^d(\overline{U}_1, \mathbb{R}^d)} \to 0 \quad \textit{as } n \to \infty.$$

Then $\{\lambda^\uparrow \circ a_n : n \ge 1\}$ is a bounded set in $C^{0,1}(U, \mathbb{R}^d)$ and, for each relatively compact open subset $U_0 \Subset U$ and each $1 \le q < \infty$,

$$\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{W^{1,q}(U_0, \mathbb{R}^d)} \to 0 \quad \textit{as } n \to \infty.$$

Proof. Let us assume that U_0 is an open box $U_0 = I_1 \times \cdots \times I_m$ parallel to the coordinate axes. Set $\lambda := \lambda^\uparrow \circ a$ and $\lambda_n := \lambda^\uparrow \circ a_n$. Let $x = (x_1, x')$ and for $x' \in U'_0 = I_2 \times \cdots \times I_m$ consider

$$A_n(x') := \int_{I_1} \|\partial_1 \lambda(x_1, x') - \partial_1 \lambda_n(x_1, x')\|_2^q \, dx_1.$$

Then $A_n(x') \to 0$ as $n \to \infty$, by Theorem 1.3. The boundedness of $\{\lambda_n : n \ge 1\}$ in $C^{0,1}(U, \mathbb{R}^d)$ is a consequence of Bronshtein's theorem. It implies that $|\partial_1 \lambda - \partial_1 \lambda_n|$ is dominated on U_0 by an integrable function. By Fubini's theorem,

$$\int_{U_0} \|\partial_1 \lambda(x) - \partial_1 \lambda_n(x)\|_2^q \, dx = \int_{U'_0} A_n(x') \, dx'.$$

By the dominated convergence theorem, we conclude that

$$\int_{U_0} \|\partial_1 \lambda(x) - \partial_1 \lambda_n(x)\|_2^q \, dx \to 0 \quad \text{as } n \to \infty.$$

In an analogous way, one sees that $\|\partial_j \lambda - \partial_j \lambda_n\|_{L^q(U_0, \mathbb{R}^d)} \to 0$ as $n \to \infty$, for each $1 \le j \le m$.

We may conclude that $\|\lambda - \lambda_n\|_{L^\infty(U_0,\mathbb{R}^d)} \to 0$ as $n \to \infty$ from the fact that this is true component-wise (see Step 3 in Section 5.1).

For general U_0, we observe that there are finitely many open boxes, relatively compact in U, that cover U_0. □

6.2. **Proof of Theorem 1.1.** It is clear that Theorem 6.1 implies Theorem 1.1 because $C^d(U, \mathrm{Hyp}(d))$ is first-countable.

6.3. **Proof of Corollary 1.2.** Corollary 1.2 is an immediate consequence of the following corollary of Theorem 6.1.

Corollary 6.2. *Let $U \subseteq \mathbb{R}^m$ be open. Let $a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for each relatively compact open set $U_0 \Subset U$ and each $0 < \alpha < 1$,*

$$\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{C^{0,\alpha}(\overline{U}_0,\mathbb{R}^d)} \to 0 \qquad \text{as } n \to \infty.$$

Proof. Again we may assume that U_0 is a box (and hence has Lipschitz boundary). Then the assertion follows from Theorem 6.1 and Morrey's inequality,

$$\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{C^{0,\alpha}(\overline{U}_0,\mathbb{R}^d)} \le C\,\|\lambda^\uparrow \circ a - \lambda^\uparrow \circ a_n\|_{W^{1,q}(U_0,\mathbb{R}^d)},$$

where $\alpha = 1 - m/q$, $q > m$, and $C = C(m, q, U_0)$. □

7. Applications

7.1. **Relation to the results for general polynomials.** The case of general complex (not necessarily hyperbolic) polynomials is treated in [11] which builds on the results of [8, 9]. The crucial difference is that in general there is no canonical choice of a continuous ordered d-tuple of the complex roots. Even worse, if the parameter space is at least two-dimensional, then a parameterization of the roots by continuous functions might not exist; but see [10]. Therefore the continuity results in [11] are formulated in terms of the *unordered* d-tuple of the roots.

Let us compare the results obtained in this paper with the ones of [11]. To this end, we investigate the metric space $\mathcal{A}_d(\mathbb{R})$ of unordered d-tuples of real numbers.

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $[x] = [x_1, \ldots, x_d]$ be the corresponding unordered d-tuple, i.e., the orbit through x of the action of the symmetric group S_d on $\mathbb{R}^d$ by permutation of the coordinates.

The set $\mathcal{A}_d(\mathbb{R}) := \{[x] : x \in \mathbb{R}^d\}$ with the distance

$$\mathbf{d}([x],[y]) := \min_{\sigma \in \mathrm{S}_d} \frac{1}{\sqrt{d}}\,\|x - \sigma y\|_2$$

is a complete metric space.

For $x \in \mathbb{R}^d$, let $x^\uparrow \in \mathbb{R}^d$ be the representative of the equivalence class $[x]$ with increasingly ordered coordinates. Clearly, $x^\uparrow$ only depends on $[x]$ and thus we have an injective map $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$. It is a right-inverse of $[\ ] : \mathbb{R}^d \to \mathcal{A}_d(\mathbb{R})$.

Lemma 7.1. *We have*

$$\mathbf{d}([x],[y]) = \frac{1}{\sqrt{d}}\,\|x^\uparrow - y^\uparrow\|_2, \quad x, y \in \mathbb{R}^d.$$

In particular, $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ and $[\ ] : \mathbb{R}^d \to \mathcal{A}_d(\mathbb{R})$ are Lipschitz maps.

Proof. Evidently,

$$\mathbf{d}([x],[y]) = \mathbf{d}([x^\uparrow],[y^\uparrow]) = \min_{\sigma \in \mathrm{S}_d} \frac{1}{\sqrt{d}} \|x^\uparrow - \sigma y^\uparrow\|_2 \le \frac{1}{\sqrt{d}} \|x^\uparrow - y^\uparrow\|_2.$$

Thus the assertion will follow from $\|x^\uparrow - y^\uparrow\|_2 \le \|x^\uparrow - y\|_2$, for $x, y \in \mathbb{R}^d$. For $d = 2$, this is equivalent to $(x_1 - y_1)^2 + (x_2 - y_2)^2 \le (x_1 - y_2)^2 + (x_2 - y_1)^2$ whenever $x_1 \le x_2$ and $y_1 \le y_2$. By a simple computation, it is further equivalent to the true statement $(x_2 - x_1)(y_2 - y_1) \ge 0$. The general case follows from the fact that any permutation is a finite composite of transpositions. □

By Lemma 7.1, the map $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ satisfies the conditions of an Almgren embedding (as defined in [11] following [2] and [4]). Thus Theorem 1.3 can be interpreted as a special version of the general theorem [11, Theorem 1.1] with the important difference that (1.3) holds for each $1 \le q < \infty$, while in the general result the corresponding fact is valid only for $1 \le q < d/(d-1)$.

In this spirit, also other results of [11] have stronger versions in the hyperbolic case. With regard to [11, Theorem 1.3], consider the locally absolutely continuous curves $\Lambda := [\lambda^\uparrow \circ a]$ and $\Lambda_n := [\lambda^\uparrow \circ a_n]$ in $\mathcal{A}_d(\mathbb{R})$. Let $|\dot\Lambda|$, $|\dot\Lambda_n|$ denote their metric speed and $\mathcal{E}_{q,I_0}(\Lambda)$, $\mathcal{E}_{q,I_0}(\Lambda_n)$ their q-energy on I_0, respectively. (See [11] for definitions.)

Theorem 7.2. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $a_n \to a$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for each relatively compact open interval $I_0 \Subset I$,*

$$\|\mathbf{d}(\Lambda, \Lambda_n)\|_{L^\infty(I_0)} \to 0 \quad \text{as } n \to \infty, \tag{7.1}$$

$$\big\||\dot\Lambda| - |\dot\Lambda_n|\big\|_{L^q(I_0)} \to 0 \quad \text{as } n \to \infty, \tag{7.2}$$

$$\big|\mathcal{E}_{q,I_0}(\Lambda) - \mathcal{E}_{q,I_0}(\Lambda_n)\big| \to 0 \quad \text{as } n \to \infty, \tag{7.3}$$

for each $1 \le q < \infty$.

Proof. First, (7.1) is a consequence of (5.2) and Lemma 7.1. By [11, Lemma 11.1],

$$|\dot\Lambda|(x) = \frac{1}{\sqrt{d}} \|(\lambda^\uparrow \circ a)'(x)\|_2 \quad \text{and} \quad |\dot\Lambda_n|(x) = \frac{1}{\sqrt{d}} \|(\lambda^\uparrow \circ a_n)'(x)\|_2$$

for almost every $x \in I$. Thus, (7.2) and (7.3) follow from Corollary 1.4. □

7.2. **Continuity of the area of the solution map.** Let us first expand Corollary 1.4.

Corollary 7.3. *Let $U \subseteq \mathbb{R}^m$ be open. Let $a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Let $R \in \mathbb{R}[X_1, \ldots, X_{dm}]$ be any real polynomial in the $d \cdot m$ variables $X_1, \ldots, X_{dm}$. Set $\lambda := \lambda^\uparrow \circ a$ and $\lambda_n := \lambda^\uparrow \circ a_n$, for $n \ge 1$. Then, for each relatively compact open subset $U_0 \Subset U$ and each $1 \le q < \infty$,*

$$\Big\| R\Big((\partial_i \lambda_j)_{\substack{1 \le i \le m \\ 1 \le j \le d}}\Big) - R\Big((\partial_i \lambda_{n,j})_{\substack{1 \le i \le m \\ 1 \le j \le d}}\Big) \Big\|_{L^q(U_0)} \to 0 \quad \text{as } n \to \infty,$$

and consequently,

$$\Big\| R\Big((\partial_i \lambda_{n,j})_{\substack{1 \le i \le m \\ 1 \le j \le d}}\Big) \Big\|_{L^q(U_0)} \to \Big\| R\Big((\partial_i \lambda_j)_{\substack{1 \le i \le m \\ 1 \le j \le d}}\Big) \Big\|_{L^q(U_0)} \quad \text{as } n \to \infty.$$

Proof. It is enough to show the assertion for monomials R. Let us proceed by induction on the degree ℓ of the monomial R. For $\ell = 1$, the assertion follows from Theorem 6.1:

$$\|\partial_i \lambda_j - \partial_i \lambda_{n,j}\|_{L^q(U_0)} \le \|\partial_i \lambda - \partial_i \lambda_n\|_{L^q(U_0,\mathbb{R}^d)}.$$

If $\ell > 1$, then, by Hölder's inequality,

$$\begin{aligned}
&\|\partial_{i_1}\lambda_{j_1}\cdots\partial_{i_\ell}\lambda_{j_\ell} - \partial_{i_1}\lambda_{n,j_1}\cdots\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\quad\le \|\partial_{i_1}\lambda_{j_1}\cdots\partial_{i_\ell}\lambda_{j_\ell} - \partial_{i_1}\lambda_{j_1}\cdots\partial_{i_{\ell-1}}\lambda_{j_{\ell-1}}\cdot\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\qquad + \|\partial_{i_1}\lambda_{j_1}\cdots\partial_{i_{\ell-1}}\lambda_{j_{\ell-1}}\cdot\partial_{i_\ell}\lambda_{n,j_\ell} - \partial_{i_1}\lambda_{n,j_1}\cdots\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\quad\le \|\partial_{i_1}\lambda_{j_1}\cdots\partial_{i_{\ell-1}}\lambda_{j_{\ell-1}}\|_{L^\infty(U_0)}\|\partial_{i_\ell}\lambda_{j_\ell} - \partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\qquad + \|\partial_{i_1}\lambda_{j_1}\cdots\partial_{i_{\ell-1}}\lambda_{j_{\ell-1}} - \partial_{i_1}\lambda_{n,j_1}\cdots\partial_{i_{\ell-1}}\lambda_{n,j_{\ell-1}}\|_{L^q(U_0)}\|\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^\infty(U_0)}
\end{aligned}$$

which tends to zero as $n \to 0$, by the induction hypothesis, because the set of numbers $\|\partial_{i_1}\lambda_{j_1}\cdots\partial_{i_{\ell-1}}\lambda_{j_{\ell-1}}\|_{L^\infty(U_0)}$ is finite and the sequence $\|\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^\infty(U_0)}$ is bounded, by Bronshtein's theorem (see Theorem 4.1). □

Let $f : U \to \mathbb{R}^d$ be a Lipschitz map, where $U \subseteq \mathbb{R}^m$ is open. We recall that $|Jf|$ denotes the square root of the sum of the squares of the determinants of the $k \times k$ minors with $k = \min\{m, d\}$ of the Jacobian matrix

$$(\partial_i f_j)_{\substack{1\le i\le m,\\ 1\le j\le d}}$$

which exists almost everywhere, by Rademacher's theorem.

Corollary 7.4. *Let $U \subseteq \mathbb{R}^m$ be open. Let $a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for each relatively compact open subset $U_0 \Subset U$ and each $1 \le q < \infty$,*

$$\big\||J(\lambda^\uparrow \circ a)| - |J(\lambda^\uparrow \circ a_n)|\big\|_{L^q(U_0)} \to 0 \qquad \text{as } n \to \infty,$$

and consequently,

$$\big\||J(\lambda^\uparrow \circ a_n)|\big\|_{L^q(U_0)} \to \big\||J(\lambda^\uparrow \circ a)|\big\|_{L^q(U_0)} \qquad \text{as } n \to \infty.$$

Proof. Let $M_1, \dots, M_p$ and $M_{n,1}, \dots, M_{n,p}$ denote the determinants of all the $k \times k$ minors with $k = \min\{m, d\}$ of the Jacobian matrix of $\lambda^\uparrow \circ a$ and $\lambda^\uparrow \circ a_n$, respectively. Fix $1 \le q < \infty$. Then

$$\big\||J(\lambda^\uparrow \circ a)| - |J(\lambda^\uparrow \circ a_n)|\big\|_{L^q(U_0)} \le |U_0|^{1/q-1/(2q)}\,\big\||J(\lambda^\uparrow \circ a)| - |J(\lambda^\uparrow \circ a_n)|\big\|_{L^{2q}(U_0)}$$

and

$$\begin{aligned}
\big\||J(\lambda^\uparrow \circ a)| - |J(\lambda^\uparrow \circ a_n)|\big\|^{2q}_{L^{2q}(U_0)} &= \big\|\big(\sum M_i^2\big)^{1/2} - \big(\sum M_{n,i}^2\big)^{1/2}\big\|^{2q}_{L^{2q}(U_0)} \\
&\le \big\|\big|\sum M_i^2 - \sum M_{n,i}^2\big|^{1/2}\big\|^{2q}_{L^{2q}(U_0)} = \big\|\sum M_i^2 - \sum M_{n,i}^2\big\|^q_{L^q(U_0)}.
\end{aligned}$$

Now it suffices to apply Corollary 7.3. □

In view of the area and the coarea formula, the following corollary is an immediate consequence of Corollary 7.4 (for $q = 1$).

Corollary 7.5. *Let $U \subseteq \mathbb{R}^m$ be open. Let $a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Set $\lambda := \lambda^\uparrow \circ a$ and $\lambda_n := \lambda^\uparrow \circ a_n$, for $n \ge 1$.*

(1) *If $m \le d$, then for each relatively compact open subset $U_0 \Subset U$,*
$$\int_{\mathbb{R}^d} \mathcal{H}^0(U_0 \cap \lambda_n^{-1}(y))\, d\mathcal{H}^m(y) \to \int_{\mathbb{R}^d} \mathcal{H}^0(U_0 \cap \lambda^{-1}(y))\, d\mathcal{H}^m(y)$$
as $n \to \infty$.

(2) *If $m > d$, then for each relatively compact open subset $U_0 \Subset U$,*
$$\int_{\mathbb{R}^d} \mathcal{H}^{m-d}(U_0 \cap \lambda_n^{-1}(y))\, dy \to \int_{\mathbb{R}^d} \mathcal{H}^{m-d}(U_0 \cap \lambda^{-1}(y))\, dy$$
as $n \to \infty$.

We can also conclude that the surface area of the graphs of the single roots $\lambda_j^\uparrow \circ a_n$, for $1 \le j \le d$, is locally convergent as $n \to \infty$.

Corollary 7.6. *Let $U \subseteq \mathbb{R}^m$ be open. Let $a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. For each $1 \le j \le d$ and for each relatively compact open subset $U_0 \Subset U$, the surface area of the graph of $\lambda_{n,j} := \lambda_j^\uparrow \circ a_n$ converges to the surface area of the graph of $\lambda_j := \lambda_j^\uparrow \circ a$ as $n \to \infty$: if $\overline{\lambda}_{n,j}(x) := (x, \lambda_{n,j}(x))$ and $\overline{\lambda}_j(x) := (x, \lambda_j(x))$ denote the corresponding graph mappings, then*
$$\mathcal{H}^m(\overline{\lambda}_{n,j}(U_0)) \to \mathcal{H}^m(\overline{\lambda}_j(U_0)) \quad \text{as } n \to \infty.$$

Proof. We have
$$|J\overline{\lambda}_j| = \Big(1 + \sum_{i=1}^m (\partial_i \lambda_j)^2\Big)^{1/2} \quad \text{and} \quad |J\overline{\lambda}_{n,j}| = \Big(1 + \sum_{i=1}^m (\partial_i \lambda_{n,j})^2\Big)^{1/2}.$$
As in the proof of Corollary 7.4, we have
$$\begin{aligned}
\Big\| \Big(1 + \sum_{i=1}^m (\partial_i \lambda_j)^2\Big)^{1/2} - \Big(1 + \sum_{i=1}^m (\partial_i \lambda_{n,j})^2\Big)^{1/2} \Big\|_{L^2(U_0)}^2 \\
\le \Big\| \Big| \sum_{i=1}^m (\partial_i \lambda_j)^2 - \sum_{i=1}^m (\partial_i \lambda_{n,j})^2 \Big|^{1/2} \Big\|_{L^2(U_0)}^2 \\
= \Big\| \sum_{i=1}^m (\partial_i \lambda_j)^2 - \sum_{i=1}^m (\partial_i \lambda_{n,j})^2 \Big\|_{L^1(U_0)}.
\end{aligned}$$
So the assertion follows from Corollary 7.3 and the area formula. □

7.3. **Approximation by hyperbolic polynomials with all roots distinct.** We recall a lemma of Wakabayshi [13] which extends an observation of Nuij [6].

Lemma 7.7 ([13, Lemma 2.2]). *Let $P_a \in \mathrm{Hyp}(d)$ and set*
$$P_{a,s}(Z) := (1 + s\tfrac{\partial}{\partial Z})^{d-1} P_a(Z), \quad s \in \mathbb{R}. \tag{7.4}$$
Then $P_{a,s} \in \mathrm{Hyp}(d)$ for all $s \in \mathbb{R}$ and there are positive constants $c_i = c_i(d)$, $i = 1, 2$, such that, if $\lambda_1^\uparrow(a,s) \le \cdots \le \lambda_d^\uparrow(a,s)$ denote the increasingly ordered roots of $P_{a,s}$, then
$$\lambda_j^\uparrow(a,s) - \lambda_{j-1}^\uparrow(a,s) \ge c_1 |s|, \quad \text{for } s \in \mathbb{R} \text{ and } 2 \le j \le d, \tag{7.5}$$
and
$$0 < \pm(\lambda_j^\uparrow(a) - \lambda_j^\uparrow(a,s)) \le c_2 |s|, \quad \text{for } \pm s > 0 \text{ and } 1 \le j \le d. \tag{7.6}$$

In conjunction with our findings, Lemma 7.7 leads to the following approximation result.

Corollary 7.8. *Let $U \subseteq \mathbb{R}^m$ be open and $a \in C^d(U, \mathrm{Hyp}(d))$. There exists a sequence $(a_n)_{n\ge 1} \subseteq C^d(U, \mathrm{Hyp}(d))$ with the following properties:*

(1) *$a_n \to a$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$.*
(2) *$\lambda_1^\uparrow(a_n(x)) < \lambda_2^\uparrow(a_n(x)) < \cdots < \lambda_d^\uparrow(a_n(x))$ for all $x \in U$ and all $n \ge 1$,*
(3) *$\lambda^\uparrow \circ a_n \in C^d(U, \mathbb{R}^d)$ for all $n \ge 1$,*
(4) *$\lambda^\uparrow \circ a_n \to \lambda^\uparrow \circ a$ in $C^{0,1}_q(U, \mathbb{R}^d)$, for all $1 \le q < \infty$, as $n \to \infty$.*
(5) *For each relatively compact open $U_0 \Subset U$, consider the zero sets $Z = \{(x, y) \in U_0 \times \mathbb{R} : P_{a(x)}(y) = 0\}$ and $Z_n = \{(x, y) \in U_0 \times \mathbb{R} : P_{a_n(x)}(y) = 0\}$, for $n \ge 1$. Then $\lim_{n\to\infty} \mathcal{H}^m(Z_n)$ exists and*
$$\lim_{n\to\infty} \mathcal{H}^m(Z_n) \ge \mathcal{H}^m(Z).$$

Proof. Let $(s_n)_{n\ge 1}$ be any positive sequence of reals that tends to 0. Consider the polynomial $P_{a(x),s_n}$ (defined in (7.4)), where $x \in U$, and let $a_n(x)$ be its coefficient vector. Then, by Lemma 7.7, $a_n \in C^d(U, \mathrm{Hyp}(d))$, for $n \ge 1$.

(1) This is clear by the definition (7.4) and since $s_n \to 0$ as $n \to \infty$.

(2) follows from (7.5) and the fact that $s_n > 0$ for all $n \ge 1$.

(3) For fixed $x \in U$, $\frac{\partial}{\partial Z} P_{a_n(x)}(Z)$ does not vanish at any root of $P_{a_n(x)}$, by (2). So, by the implicit function theorem, the roots of $P_{a_n(x)}$ are of class C^d in a neighborhood of x. This implies (3).

(4) is a consequence of (1) and Theorem 1.1.

(5) For each $n \ge 1$, Z_n is the union of the graphs of $\lambda_j^\uparrow \circ a_n|_{U_0}$, for $1 \le j \le d$, and these graphs are pairwise disjoint, by (2). Thus, by Corollary 7.6 (using its notation),
$$\mathcal{H}^m(Z_n) = \sum_{j=1}^d \mathcal{H}^m(\overline{\lambda}_{n,j}(U_0)) \to \sum_{j=1}^d \mathcal{H}^m(\overline{\lambda}_j(U_0)) \ge \mathcal{H}^m(Z) \quad \text{as } n \to \infty.$$
Note that we have only an inequality at the end because there can be multiple roots. □

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