BÉZOUT BOUNDS AND RATIONAL POINTS ON QUASIANALYTIC GRAPHS

ARMIN RAINER

Abstract. For functions $f : [0,1] \to [0,1]$ in quasianalytic Denjoy–Carleman classes $\mathcal{C}_M$, we establish explicit bounds for the number of intersection points of the graph $\Gamma(f)$ with algebraic curves of degree at most d. The bounds are given in terms of the Bang degree determined by the weight sequence M, the degree d, and the norming constant of $\Gamma(f)$ with respect to the space of real bivariate polynomials of degree at most d. We give instances, where the bounds are at most polynomial in d and, consequently, the graph $\Gamma(f)$ contains few rational points: the number of rational points of height at most H is bounded by a polynomial in $\log H$.

1. Introduction

Bounding the maximal number of intersection points between the graph $\Gamma(f)$ of a transcendental complex analytic function f on a bounded disk and complex algebraic curves of degree at most d, has attracted considerable interest. The case $f(z) = e^z$ was treated by Tijdeman [Tij71], and the general case was studied in [CP03], [CP07], and [Bru08]. In [CY18], such bounds were called *Bézout bounds for f*, and sufficient conditions were given, in the complex analytic setting, for the existence of Bézout bounds that are polynomial in d, together with Diophantine applications.

In this paper, we establish Bézout bounds for functions in quasianalytic Denjoy–Carleman classes $\mathcal{C}_M$, i.e., regularity classes of $\mathcal{C}^\infty$ functions with the unique continuation property, determined by bounds on the infinite sequence of derivatives in terms of a weight sequence $M = (M_j)$. For $M = (j!)$, we recover the real analytic class, in general, $\mathcal{C}_M$ is strictly larger. Let $f : [0,1] \to [0,1]$ be a $\mathcal{C}_M$ function and $p \in \mathbb{R}[X,Y] \setminus \{0\}$ a real polynomial in two variables. If $p(x, f(x))$ is not identically zero, then the number of zeros of $p(x, f(x))$, $x \in [0,1]$, (counted with multiplicities) is finite; we denote this number by $\mathbf{z}(p, f)$. We will see that also

$$\mathbf{z}_d(f) := \sup_{p \in \mathcal{P}_2^d \setminus \{0\}} \mathbf{z}(p, f)$$

Date: October 15, 2026.

2020 *Mathematics Subject Classification.* Primary 14G05, 26E10; Secondary 26D05, 41A17.

Key words and phrases. Quasianalytic functions, Denjoy–Carleman classes, Bézout bounds, intersection with algebraic curves, rational points.

This research was funded in whole or in part by the Austrian Science Fund (FWF) DOI 10.55776/PAT1381823. For open access purposes, the author has applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

0000-0003-3825-3313.

is a finite integer, provided that f is transcendental, i.e., $p(x, f(x))$ is not identically zero for all nonzero $p \in \mathbb{R}[X, Y]$. We denote by $\mathcal{P}_n^d = \mathbb{R}[X_1, \dots, X_n]_d$ the space of real polynomials in n variables of maximal degree d.

Our Bézout bounds will be given in terms of the *Bang degree*, a natural complexity parameter for functions in quasianalytic Denjoy–Carleman classes determined by the weight sequence M and the size of the function. It was introduced by Bang [Ban53] and used e.g. in [NSV04], [Rai23], and [Rai26]. We obtain an explicit bound for $\mathbf{z}(p, f)$ which depends only on the class $\mathcal{C}_M$ of f, the degree d of p, and the ratio $\|p\|_{[0,1]^2}/\|p\|_{\Gamma(f)}$ (Theorem 3.1), where $\|p\|_K := \sup_{x\in K} |p(x)|$. The proof easily generalizes to give a similar bound for the number of zeros of $p(x, f_1(x), \dots, f_n(x))$, $x \in [0, 1]$, for $\mathcal{C}_M$ functions $f_i : [0, 1] \to [0, 1]$, $1 \le i \le n$, and $p \in \mathcal{P}_{n+1}^d$, but this is left to the reader.

The number

$$\mathbf{e}_d(f) := \sup_{p\in\mathcal{P}_2^d\setminus\{0\}} \log\Big(\frac{\|p\|_{[0,1]^2}}{\|p\|_{\Gamma(f)}}\Big)$$

is another interesting transcendence measure of f; $\mathbf{e}_d(f)$ is a finite nonnegative number, provided that f is transcendental. In the terminology of [BY16], $\mathbf{e}_d(f)$ is the logarithm of the $\mathcal{P}_2^d$-norming constant of $\Gamma(f)$. Note that $\mathbf{e}_d(f)$ encodes the best constant of a Remez inequality for $p \in \mathcal{P}_2^d$ on a subset $\Gamma(f)$ of $[0, 1]^2$ of zero measure.

We conclude that

$$\mathbf{z}_d(f) \le F_M(2d \log d + \mathbf{e}_d(f)), \quad d \ge 1,$$

where F_M is an explicit function depending only on the weight sequence M (Theorem 4.2). For instance, for the analytic class $M = (j!)$ the function F_M grows linearly, and if $f : [0, 1] \to [0, 1]$ is transcendental and the restriction of an entire function, then we find

$$\mathbf{z}_d(f) \le 12\, d \log d + 6\, \mathbf{e}_d(f) + C, \quad d \ge 1,$$

for a constant $C = C(f)$ (Corollary 4.4). In the framework of entire functions, similar connections have been established in [CP03], [CP07], and [Bru08].

We estimate F_M for a one-parameter family of quasianalytic weights, the so-called *Denjoy weights*, and identify precisely the range of parameters for which F_M is bounded by a polynomial (Proposition 4.9). In particular, F_{M_s} is bounded by a polynomial for all $M_s = (j!\,(\prod_{i=1}^j \log(ei))^s)_j$ with $s \in [0, 1]$. Further away from the analytic class, F_M grows like higher and higher towers of exponentials.

Polynomial Bézout bounds for f have a striking Diophantine consequence (see [Pil06]): the number of rational points of height at most H in $\Gamma(f)$ is bounded by a polynomial in $\log H$. Thus, it is interesting to ask when $\mathbf{e}_d(f)$ is bounded by a polynomial in d, especially, for transcendental $f : [0, 1] \to [0, 1]$ of class $\mathcal{C}_{M_s}$, where $s \in [0, 1]$. Even in the real analytic class ($s = 0$), $\mathbf{e}_d(f)$ can grow superpolynomially, see e.g. [CY18, Remark 5.3] or Remark 4.7, while $\mathbf{e}_d(f)$ is bounded by a polynomial for exponential polynomials f, see [BY16, Section 2.7].

It seems to be difficult to identify non-analytic quasianalytic functions f such that $\mathbf{e}_d(f)$ is at most polynomial in d. (Note that $\Gamma(f)$ is always pluripolar, see [CLP05].)

However, we provide a family $\{f_\xi\}$ of functions $f_\xi : [0,1] \to [0,1]$ of class $\mathcal{C}_M$, parameterized by a sequence $\xi = (\xi_n)_{n\geq 1}$ of independent random variables, each uniformly distributed in $[-1,1]$, such that f_ξ is almost surely not real analytic and transcendental, and $\mathbf{e}_d(f)$ is almost surely at most polynomial in d (Proposition 5.4). Interestingly, the proof of this result does not use quasianalyticity, but clearly it requires that the class $\mathcal{C}_M$ contains the real analytic functions as a proper subclass.

As a consequence, specializing to $M = M_s$ for $s \in [0,1]$, the number of rational points of height at most H in $\Gamma(f_\xi)$ is almost surely bounded by a polynomial in $\log H$ (Theorem 5.5). In contrast, for a general function $f : [0,1] \to [0,1]$ of class $\mathcal{C}_{M_s}$, the number of rational points of height at most H in $\Gamma(f)$ is bounded by $C_\epsilon H^\epsilon$ for every $\epsilon > 0$, by the Pila–Wilkie theorem [PW06], and this bound is sharp for the class of real analytic functions. Indeed, the expansion of the real field by restricted functions of class $\mathcal{C}_{M_s}$ is o-minimal, see [RSW03].

The paper is organized as follows. In Section 2, we recall basics on quasianalytic Denjoy–Carleman classes and the Bang degree. In Section 3, we prove our bound for $\mathbf{z}(p,f)$. In Section 4, we investigate the relation of this Bézout bound for f with $\mathbf{e}_d(f)$, the (logarithm of the) $\mathcal{P}_2^d$-norming constant of $\Gamma(f)$. Finally, in Section 5, we find instances of polynomial Bézout bounds for quasianalytic functions and discuss the consequences for the number of rational points on corresponding function graphs.

2. Weights, quasianalytic functions, and the Bang degree

We recall basics on quasianalytic Denjoy–Carleman classes and introduce our terminology.

2.1. **Quasianalytic Denjoy–Carleman classes.** Let $M = (M_j)_{j\geq 0}$ be a sequence of positive reals. The *Denjoy–Carleman class* $\mathcal{C}_M$ is defined as follows. For open $U \subseteq \mathbb{R}^n$, $\mathcal{C}_M(U)$ consists of all $f \in \mathcal{C}^\infty(U)$ such that for all compact $K \subseteq U$ there are constants $C, \rho > 0$ such that

$$\|\partial^\alpha f\|_K \leq C\rho^{|\alpha|} M_{|\alpha|}, \quad \alpha \in \mathbb{N}^n. \tag{2.1}$$

It will be convenient for us to write

$$M_j = M_0 \prod_{i=1}^{j} \mu_i, \quad j \geq 1,$$

where $\mu_j = \frac{M_j}{M_{j-1}}$. We will require that the μ_j are increasing.

Definition 2.1 (Admissible weights)**.** An *admissible weight* is an infinite increasing sequence $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ of elements in $\mathbb{R}_{>0}$, i.e., $0 < \mu_1 \leq \mu_2 \leq \cdots$.

A pair $(\mu^{|\infty}, M_0)$, where $\mu^{|\infty}$ is an admissible weight and $M_0 > 0$, uniquely determines $M = (M_j)_{j\geq 0}$ which satisfies the logarithmic convexity property

$$M_j^2 \leq M_{j-1} M_{j+1}, \quad j \geq 1,$$

and vice versa.

Definition 2.2 (Quasianalytic weights)**.** An admissible weight $\mu^{|\infty} = (\mu_j)_{j\ge1}$ is called *quasianalytic* if

$$\sum_{j\ge1} \frac{1}{\mu_j} = \infty. \tag{2.2}$$

By the Denjoy–Carleman theorem, the class $\mathcal{C}_M$ is quasianalytic (i.e., the infinite Taylor series homomorphism $\mathcal{C}_M(U) \to \mathbb{R}[\![X_1,\dots,X_n]\!]$ is injective at any point of U, provided that U is connected) if and only if (2.2) holds. See [Rai22] for a comprehensive survey of Denjoy–Carleman classes.

We will work with the following definition.

Definition 2.3 ($(\mu^{|N}, M_0)$-smooth functions)**.** Let $\mu^{|\infty} = (\mu_j)_{j\ge1}$ be an admissible weight and $M_0 > 0$. Let $N \in \mathbb{N}_{\ge1} \cup \{\infty\}$. A function $f : I \to \mathbb{R}$, defined on a compact interval $I \subseteq \mathbb{R}$, is called $(\mu^{|N}, M_0)$-*smooth* if f is of class $\mathcal{C}^N$ and

$$\|f^{(j)}\|_I \le M_j, \quad 0 \le j < N+1.$$

Note that, compared to (2.1), this is no restriction because the constants C, ρ can be absorbed by redefining M.

2.2. **The Bang degree N.** We define an integer $\mathbf{N}$ that plays the role of a complexity parameter for $(\mu^{|N}, M_0)$-smooth functions. It is a variant of an invariant introduced by [Ban53], later called *Bang degree* by [NSV04].

In the following, $\lceil r\rceil_{\mathbb{N}}$ denotes the smallest nonnegative integer j satisfying $j \ge r$.

Definition 2.4. Let $\mu^{|\infty} = (\mu_j)_{j\ge1}$ be an admissible weight and $M_0, a, b > 0$. We say that $(\mu^{|\infty}, M_0, a, b)$ is *admissible*, if

$$\sum_{j=j_0+1}^{\infty} \frac{1}{\mu_j} > a\,e, \qquad \text{where } j_0 = \lceil \log(\tfrac{M_0}{b})\rceil_{\mathbb{N}}. \tag{2.3}$$

In that case, we define the integer

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a, b) := 2\,\sup\Big\{n \in \mathbb{N} : \sum_{j=j_0+1}^{n} \frac{1}{\mu_j} < a\,e\Big\}$$

and say that $\mathbf{N}$ is the *degree associated with* $(\mu^{|\infty}, M_0, a, b)$.

If $\mu^{|\infty}$ is quasianalytic, then (2.3) is always satisfied, for all choices of $M_0, a, b > 0$. But some of our results (e.g. Theorem 3.1) hold also for non-quasianalytic weights, provided that (2.3) is satisfied.

Remark 2.5. (1) By definition, for $r, s > 0$, $(r\,\mu^{|\infty}, s\,M_0, a, b)$ is admissible if and only if $(\mu^{|\infty}, M_0, r\,a, \frac{b}{s})$ is admissible, and

$$\mathbf{N}(r\,\mu^{|\infty}, s\,M_0, a, b) = \mathbf{N}(\mu^{|\infty}, M_0, r\,a, \tfrac{b}{s}).$$

(2) Assuming admissibility, the integer $\mathbf{N}(\mu^{|\infty}, M_0, a, b)$ is increasing in a and decreasing in b.

3. On the number of intersection points with algebraic curves

The goal of this section is Theorem 3.1 which provides a bound for the number of intersection points of the graph of $\Gamma(f)$ of an $(\mu^{|N}, M_0)$-smooth function $f : [0,1] \to [0,1]$ with an algebraic curve in terms of the Bang degree and certain data of the algebraic curve.

We need an additional assumption on the admissible weight $\mu^{|\infty} = (\mu_j)_{j\geq 1}$: we will assume that the associated sequence

$$a_j := \Big(\frac{\mu_1 \cdots \mu_j}{j!}\Big)^{1/j} \tag{3.1}$$

is almost increasing with constant $H \geq 1$, i.e.,

$$a_i \leq H\, a_j \quad \text{ if } i \leq j. \tag{3.2}$$

For convenience, given $M_0 > 0$ and hence $M_1 = M_0\mu_1$, we put

$$\overline{m} := M_0\sqrt{1 + \tfrac{1}{M_1^2}}.$$

Theorem 3.1. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying (3.2) with constant $H \geq 1$. Let $M_0 > 0$, $B_0 \geq b_0 > 0$, and $d \in \mathbb{N}_{\geq 1}$. Assume that $(\mu^{|\infty}, \frac{1}{2}e^{4H\overline{m}}\, d^{2d}B_0, 2, b_0)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, \tfrac{1}{2}e^{4H\overline{m}}d^{2d}B_0, 2, b_0).$$

Let $f : [0,1] \to [0,1]$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth and $p \in \mathcal{P}_2^d$ satisfy $\|p\|_{\Gamma(f)} \geq b_0$ and $\|p\|_{[0,1]^2} \leq B_0$. Then

$$\mathbf{z}(p, f) \leq \mathbf{N},$$

where $\mathbf{z}(p, f)$ denotes the total number of zeros of $p(x, f(x))$, for $x \in [0,1]$, counted with multiplicities.

Observe that in the theorem f is only assumed to be of class $\mathcal{C}^{\mathbf{N}}$ and to satisfy $\|f^{(j)}\|_{[0,1]} \leq M_j$ for $j \leq \mathbf{N}$. The proof of Theorem 3.1 is based on the following proposition which is a variant of a result of Bang [Ban53].

Proposition 3.2 ([Rai23, Corollary 3.6 and Remark 3.7])**.** *Let $I \subseteq \mathbb{R}$ be a compact interval. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > b_0 > 0$. Assume that $(\mu^{|\infty}, M_0, |I|, b_0)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, |I|, b_0).$$

Let $f : I \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}}, M_0)$-smooth function satisfying $\|f\|_I \geq b_0$. Then the total number of zeros of f in I (counted with multiplicities) is finite and bounded by $\mathbf{N}$.

In view of Proposition 3.2, it suffices to show that the composite function $h(x) = p(x, f(x))$, $x \in [0,1]$, is $(2\,\mu^{|\mathbf{N}}, \frac{1}{2}e^{4H\overline{m}}d^{2d}B_0)$-smooth. To this end, we will prove Proposition 3.3.

Proposition 3.3. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying (3.2) with constant $H \geq 1$. Let $M_0 > 0$. If $f : [0,1] \to [0,1]$ is $(\mu^{|N}, M_0)$-smooth and $p \in \mathcal{P}_2^d$, then $h := p \circ \overline{f} : [0,1] \to \mathbb{R}$, where $\overline{f} : [0,1] \to [0,1]^2$, $\overline{f}(x) = (x, f(x))$, is $(2\,\mu^{|N}, \frac{1}{2}e^{4H\overline{m}}\, d^{2d}\, \|p\|_{[0,1]^2})$-smooth.*

We begin with a few preparatory results.

Lemma 3.4. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying (3.2) with constant $H \geq 1$. Then the sequence $b_j := a_j^j$ satisfies*

$$b_{j_1} b_{j_2} \cdots b_{j_\ell} \leq H^\ell \, b_k$$

whenever $j_1 + \cdots + j_\ell = k$ and $j_i \geq 1$ for all $1 \leq i \leq \ell$.

Proof. By (3.2), we have $b_i \leq H \, b_j^{i/j}$ if $i \leq j$. Thus, if $j_1 + \cdots + j_\ell = k$ and $j_i \geq 1$ for all $1 \leq i \leq \ell$, then

$$b_{j_1} b_{j_2} \cdots b_{j_\ell} \leq H \, b_k^{j_1/k} \, H \, b_k^{j_2/k} \cdots H \, b_k^{j_\ell/k} = H^\ell \, b_k$$

as claimed. □

Lemma 3.5. *For all $p \in \mathcal{P}_n^d$ and $j \in \mathbb{N}$,*

$$\sup_{x\in[0,1]^n} \|d^j p(x)\|_{L^j(\mathbb{R}^n,\mathbb{R})} \leq (2nd^2)^j \|p\|_{[0,1]^n}.$$

Proof. Markov's inequality implies

$$\|\partial^\alpha p\|_{[0,1]^n} \leq (2d^2)^{|\alpha|} \, \|p\|_{[0,1]^n},$$

for all $\alpha \in \mathbb{N}^n$. Then, for $v^i = (v_1^i, \dots, v_n^i) \in \mathbb{R}^n$ with $|v^i| \leq 1$ and $x \in [0,1]^n$,

$$|(d^j p)(x)(v^1, \dots, v^j)| = \Big| \sum_{i_1=1,\dots,i_j=1}^{n} \partial_{i_1} \cdots \partial_{i_j} p(x) \, v_{i_1}^1 \cdots v_{i_j}^j \Big| \leq (2nd^2)^j \, \|p\|_{[0,1]^n}.$$

The lemma is proved. □

Proposition 3.6 (Faà di Bruno's formula; cf. [Yam89, Theorem 1])**.** *Let $f : U \to V$ and $g : V \to W$ be $\mathcal{C}^k$ maps, where U, V, W are open subsets of normed spaces. Then, for $x \in U$,*

$$\frac{d^k(g \circ f)(x)}{k!} = \operatorname{sym} \sum \frac{d^j g(f(x))}{j_1! \cdots j_k!} \Big(\frac{d^1 f(x)}{1!}\Big)^{j_1} \cdots \Big(\frac{d^k f(x)}{k!}\Big)^{j_k},$$

where $j = j_1 + \cdots + j_k$, the sum is taken over all $j_1, \dots, j_k$ such that $j_1 + 2j_2 + \cdots + kj_k = k$, and sym *denotes symmetrization of a multilinear operator.*

Let us now prove Proposition 3.3. Clearly, $\|h\|_{[0,1]} \leq \|p\|_{[0,1]^2}$. By Proposition 3.6, for $1 \leq k \leq N$ and $x \in [0,1]$,

$$\frac{|h^{(k)}(x)|}{k!} \leq \sum \frac{\|d^j p(x, f(x))\|_{L^j(\mathbb{R}^2,\mathbb{R})}}{j_1! \cdots j_k!} \Big(\frac{|\partial^1 \overline{f}(x)|}{1!}\Big)^{j_1} \cdots \Big(\frac{|\partial^k \overline{f}(x)|}{k!}\Big)^{j_k},$$

where $j = j_1 + \cdots + j_k$ and the sum is taken over all $j_1, \dots, j_k$ such that $j_1 + 2j_2 + \cdots + kj_k = k$. Since $\deg p \leq d$, we have $d^j p = 0$ for $j \geq d+1$. Thus it is enough to take the sum on the right-hand side over all $j_1, \dots, j_k$ satisfying $j_1 + 2j_2 + \cdots + kj_k = k$ and $j_1 + \cdots + j_k = j \leq d$; let us write $\sum_{j\leq d}$ for this restricted sum.

Since f is $(\mu^{|N}, M_0)$-smooth and

$$\partial^j \overline{f}(x) = \begin{cases} (1, f'(x)) & \text{if } j = 1, \\ (0, f^{(j)}(x)) & \text{if } j \geq 2, \end{cases}$$

we have, for $x \in [0,1]$,

$$|\partial^1 \overline{f}(x)| = \sqrt{1+|f'(x)|^2} \le \sqrt{1+M_1^2} = M_1\sqrt{1+\tfrac{1}{M_1^2}}$$

and, if $j \ge 2$,

$$\frac{|\partial^j \overline{f}(x)|}{j!} = \frac{|f^{(j)}(x)|}{j!} \le \frac{M_j}{j!}.$$

Writing $m_j := M_j/j!$ we find, by Lemma 3.4, using its notation,

$$m_1^{j_1}\cdots m_k^{j_k} = M_0^j b_1^{j_1}\cdots b_k^{j_k} \le M_0^j H^j b_k,$$

since $j_1+\cdots+j_k = j$ and $j_1+2j_2+\cdots+kj_k = k$. By Lemma 3.5, we conclude

$$\begin{aligned}
\frac{|h^{(k)}(x)|}{k!} &\le \|p\|_{[0,1]^2} \sum_{j\le d} \frac{(4d^2)^j}{j_1!\cdots j_k!}\Big(\sqrt{1+\tfrac{1}{M_1^2}}\Big)^{j_1} m_1^{j_1}\cdots m_k^{j_k} \qquad (3.3)\\
&\le \|p\|_{[0,1]^2}\, b_k \sum_{j\le d} \frac{\Big(4\,d^2 H M_0\sqrt{1+\frac{1}{M_1^2}}\Big)^j}{j_1!\cdots j_k!}\\
&\le e^{4H\overline{m}}\, d^{2d}\, \|p\|_{[0,1]^2}\, b_k \sum_{j\le d} \frac{j!}{j_1!\cdots j_k!} \qquad (\text{using } (4H\overline{m})^j \le e^{4H\overline{m}} j!)\\
&\le e^{4H\overline{m}}\, d^{2d}\, \|p\|_{[0,1]^2}\, 2^{k-1}\, b_k,
\end{aligned}$$

using $\sum \frac{j!}{j_1!\cdots j_k!} = 2^{k-1}$ in the last inequality; see e.g. [KP02, Lemma 1.4.1]. This implies

$$\|h^{(k)}\|_{[0,1]} \le \tfrac{1}{2} e^{4H\overline{m}}\, d^{2d}\, \|p\|_{[0,1]^2}\, 2^k \mu_1\cdots\mu_k.$$

The proof of Proposition 3.3 and thus of Theorem 3.1 is complete.

The following is a variant of Proposition 3.3 which gives a slightly better bound in Theorem 3.1 for small d.

Proposition 3.7. *In the setting of Proposition 3.3, the composite $h = p\circ\overline{f} : [0,1]\to\mathbb{R}$ is $(2\,\mu^{|N}, \frac{1}{2}(4d^2H\hat{m})^d\,\|p\|_{[0,1]^2})$-smooth, where*

$$\hat{m} := \max\Big\{M_0\sqrt{1+\tfrac{1}{M_1^2}}, 1\Big\}.$$

Proof. It suffices to modify the estimate (3.3) in the following way:

$$\frac{|h^{(k)}(x)|}{k!} \le \|p\|_{[0,1]^2}\, b_k \sum_{j\le d} \frac{(4\,d^2 H\hat{m})^j}{j_1!\cdots j_k!} \le (4\,d^2H\hat{m})^d\,\|p\|_{[0,1]^2}\, b_k \sum \frac{j!}{j_1!\cdots j_k!}.$$

Then the proof of Proposition 3.3 implies the statement. □

Remark 3.8. An example, where d is small, is the number of fixed points of f. Indeed, the polynomial $p(X,Y) = X - Y$ satisfies $\|p\|_{[0,1]^2} = 1$ and $\deg p = 1$. Assume that $(\mu^{|\infty}, M_0(1+\frac{1}{M_1}), 1, b_0)$ is admissible and let

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0(1+\tfrac{1}{M_1}), 1, b_0).$$

Let $f : [0,1]\to[0,1]$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth with $\sup_{x\in[0,1]}|x-f(x)| \ge b_0$. By a direct computation, the composite function $h(x) = p(x, f(x)) = x - f(x)$, $x\in[0,1]$, is $(\mu^{|\mathbf{N}}, M_0(1+\frac{1}{M_1}))$-smooth. (Alternatively, one can use Proposition 3.7 which however gives weaker bounds in this case.) Moreover, $\|h\|_{[0,1]} \ge b_0$. By

Proposition 3.2, the total number of fixed points of f (counted with multiplicities) is finite and bounded by $\mathbf{N}$.

4. Relation to the norming constant

4.1. **Norming sets and the norming constant.** We recall a definition of [BY16].

Definition 4.1 ([BY16, Definition 1.3])**.** Let $\mathcal{V} \subseteq C^0([0,1]^n, \mathbb{R})$ be a finite dimensional subspace. A compact subset $Z \subseteq [0,1]^n$ is said to be $\mathcal{V}$*-norming* if there is a constant $C > 0$ such that

$$\|v\|_{[0,1]^n} \le C \, \|v\|_Z, \quad v \in \mathcal{V}.$$

The infimum of all such constants C is denoted by $N_{\mathcal{V}}(Z)$ and called the $\mathcal{V}$*-norming constant of* Z. This is an inequality of Remez type on a set of zero measure.

Let $f : [0,1] \to [0,1]$ be a transcendental function, i.e., $p(x, f(x)) \not\equiv 0$ for all nonzero $p \in \mathbb{R}[X,Y]$. Then the graph $\Gamma(f)$ of f is $\mathcal{P}_2^d$-norming for every d; see [BY16, Proposition 1.4]. Let us define

$$\mathbf{e}(p,f) := \log\Big(\frac{\|p\|_{[0,1]^2}}{\|p\|_{\Gamma(f)}}\Big),$$

for $p \in \mathcal{P}_2^d \setminus \{0\}$, and

$$\mathbf{e}_d(f) := \sup_{p \in \mathcal{P}_2^d \setminus \{0\}} \mathbf{e}(p,f) = \log N_{\mathcal{P}_2^d}(\Gamma(f)).$$

Then $\mathbf{e}(p,f)$ and $\mathbf{e}_d(f)$ are nonnegative numbers.

4.2. **Bézout bounds in terms of the norming constant.** Let $f : [0,1] \to [0,1]$ be transcendental and $p \in \mathcal{P}_2^d \setminus \{0\}$. Recall that $\mathbf{z}(p,f)$ is the total number of zeros of $p(x, f(x))$, for $x \in [0,1]$, (counted with multiplicities). Define

$$\mathbf{z}_d(f) := \sup_{p \in \mathcal{P}_2^d \setminus \{0\}} \mathbf{z}(p,f).$$

Let $\mu^{|\infty} = (\mu_j)_{j \ge 1}$ be an admissible quasianalytic weight and $M_0 > 0$. Let $f : [0,1] \to [0,1]$ be a transcendental $(\mu^{|\infty}, M_0)$-smooth function. We will see that $\mathbf{z}_d(f)$ is finite and can be bounded by an explicit function of d and the logarithm of the norming constant $\mathbf{e}_d(f)$. This explicit function only depends on $(\mu^{|\infty}, M_0)$.

To this end, we define $\eta_{\mu^{|\infty}} : [0,\infty) \to \mathbb{N}$ by

$$\eta_{\mu^{|\infty}}(t) := \mathbf{N}(\mu^{|\infty}, e^t, 2, 1).$$

By the definition of $\mathbf{N}$, we have $\eta_{\mu^{|\infty}}(t) = \eta_{\mu^{|\infty}}(\lceil t \rceil)$. Note that the function $\eta_{\mu^{|\infty}}$ is increasing (see Remark 2.5).

Theorem 4.2. *Let $\mu^{|\infty} = (\mu_j)_{j \ge 1}$ be an admissible quasianalytic weight satisfying (3.2) with constant $H \ge 1$. Let $M_0 > 0$. If $f : [0,1] \to [0,1]$ is a transcendental $(\mu^{|\infty}, M_0)$-smooth function, then, for each $p \in \mathcal{P}_2^d \setminus \{0\}$,*

$$\mathbf{z}(p,f) \le \eta_{\mu^{|\infty}}\big(2d \log d + \mathbf{e}(p,f) + 4H\overline{m} - \log 2\big). \tag{4.1}$$

Consequently,

$$\mathbf{z}_d(f) \le \eta_{\mu^{|\infty}}\big(2d \log d + \mathbf{e}_d(f) + 4H\overline{m} - \log 2\big), \quad d \ge 1. \tag{4.2}$$

Proof. By (the proof of) Theorem 3.1,

$$\mathbf{z}(p,f) \le \mathbf{N}(\mu^{|\infty}, \tfrac{1}{2}e^{4H\overline{m}}d^{2d}\,\|p\|_{[0,1]^2}, 2, \|p\|_{\Gamma(f)}).$$

Now, by the properties of $\mathbf{N}$ (see Remark 2.5),

$$\begin{aligned}\mathbf{N}(\mu^{|\infty}, \tfrac{1}{2}e^{4H\overline{m}}d^{2d}\,\|p\|_{[0,1]^2}, 2, \|p\|_{\Gamma(f)}) &= \mathbf{N}\Big(\mu^{|\infty}, \tfrac{1}{2}e^{4H\overline{m}}d^{2d}\,\frac{\|p\|_{[0,1]^2}}{\|p\|_{\Gamma(f)}}, 2, 1\Big)\\ &= \eta_{\mu^{|\infty}}\big(2d\log d + \mathbf{e}(p,f) + 4H\overline{m} - \log 2\big)\end{aligned}$$

implying (4.1). Since $\eta_{\mu^{|\infty}}$ is increasing, (4.2) is immediate from (4.1). □

Relations similar to (4.1) and (4.2) have been established in [CP03], [CP07], and [Bru08] in the framework of entire functions. In the following, we analyze $\eta_{\mu^{|\infty}}$ for particular weights.

4.3. **Real analytic and entire transcendental functions.** Let $f : [0,1] \to [0,1]$ be real analytic. Then there exist constants $C, \rho > 0$ such that $\|f^{(j)}\|_{[0,1]} \le C\rho^j j!$ for all $j \ge 0$. So f is $(\rho\,\mu_{\mathrm{an}}^{|\infty}, C)$-smooth, where $\mu_{\mathrm{an}}^{|\infty} := (j)_{j\ge 1}$. By Proposition 4.9,

$$e^{2e\rho-1}(m+1) - 1 \le \tfrac{1}{2}\eta_{\rho\,\mu_{\mathrm{an}}^{|\infty}}(m) \le e^{2e\rho+1}(m+1) - 1, \quad m \in \mathbb{N}_{\ge 1}. \tag{4.3}$$

Thus, Theorem 4.2 implies the following corollary.

Corollary 4.3. *Let $f : [0,1] \to [0,1]$ be real analytic and transcendental. There exist $C, \rho > 0$ (depending on f) such that*

$$\mathbf{z}_d(f) \le 2\,e^{2e\rho+1}\Big(2d\log d + \mathbf{e}_d(f) + 4C\sqrt{1 + \tfrac{1}{C^2\rho^2}} + 1 - \log 2\Big), \quad d \ge 1. \tag{4.4}$$

Proof. In this case, $H = 1$, $M_0 = C$, and $M_1 = C\rho$. The bound follows from (4.2) and (4.3). □

If $f : [0,1] \to [0,1]$ is the restriction of an entire function, then for each $\tau > 0$ there exists $C_\tau > 0$ such that $\|f^{(j)}\|_{[0,1]} \le C_\tau \tau^j j!$ for all $j \ge 0$.

Corollary 4.4. *Let $f : [0,1] \to [0,1]$ be transcendental and the restriction of an entire function. There exists $C = C(f) > 0$ such that*

$$\mathbf{z}_d(f) \le 12\,d\log d + 6\,\mathbf{e}_d(f) + C, \quad d \ge 1.$$

Proof. Choose $\tau > 0$ such that $2e\tau + 1 \le \log 3$ and apply (4.4) for $\rho = \tau$. □

We can combine this with a result which is a consequence of [Bru08, Theorem 2.5]; see also [CP07, Theorem 1.1] for the special case of $n = 1$ and f being of finite order.

Theorem 4.5 ([BY16, Theorem 2.9])**.** *Let f be a non-polynomial entire function on $\mathbb{C}^n$ that is real on $\mathbb{R}^n$. Assume that the graph $\Gamma(f|_{\mathbb{R}^n})$ of $f|_{\mathbb{R}^n}$ intersects $[0,1]^{n+1}$ in a subset Z of real dimension n. Then there exists a sequence of positive integers $d_j \to \infty$ and a sequence of positive reals $\epsilon_j \to 0$ such that*

$$\log N_{\mathcal{P}_{n+1}^{d_j}}(Z) \le d_j^{2+\epsilon_j}, \quad \textit{for all } j.$$

The sequence (d_j) depends only on f and the sequence (ϵ_j) only on f, n, and the n-dimensional Hausdorff measure of Z.

Corollary 4.6. *Let $f : [0,1] \to [0,1]$ be transcendental and the restriction of an entire function that is real on $\mathbb{R}$. There exists a sequence of positive integers $d_j \to \infty$, a sequence of positive reals $\epsilon_j \to 0$, and a constant $C > 0$ such that*

$$\mathbf{z}_{d_j}(f) \le 6\, d_j^{2+\epsilon_j} + 12\, d_j \log d_j + C, \qquad \text{for all } j.$$

Proof. Combine Corollary 4.4 with Theorem 4.5. □

In general, one cannot expect that $\mathbf{z}_d(f)$ is bounded by a polynomial in d, for all degrees $d \ge 1$.

Remark 4.7. For each $\zeta \in (0,1)$, there exists an analytic function f defined in a neighborhood of $[0,1]$ and a sequence of positive integers $d_j \to \infty$ satisfying

$$\mathbf{z}_{d_j}(f) \ge e^{d_j^\zeta}, \qquad \text{for all } j.$$

See [CY18, Remark 5.3]. In view of (4.4), we conclude that

$$\mathbf{e}_{d_j}(f) \ge c_1\, e^{d_j^\zeta} - 2d_j \log d_j - C_2, \qquad \text{for all } j,$$

for constants $c_1 = c_1(f) \in (0,1)$ and $C_2 = C_2(f) > 0$.

4.4. **Denjoy weights.** We define the towers of exponentials and logarithms,

$$E_0(x) = L_0(x) := x, \quad E_k(x) := \exp(E_{k-1}(x)), \quad L_k(x) := \log(L_{k-1}(x)),$$

for $k \ge 1$, and set $e_k := E_k(1)$.

Definition 4.8 (Denjoy weights)**.** Let $s \in [0,\infty)$ and set $k = \lceil s \rceil \in \mathbb{N}$ and $\sigma = k - s \in [0,1)$. We define

$$(\mu_s)_j := jL_1(e_k j)L_2(e_k j)\cdots L_{k-1}(e_k j)(L_k(e_k j))^{1-\sigma}, \quad j \ge 1.$$

Then $\mu_s^{|\infty} := ((\mu_s)_j)_{j\ge1}$ is an admissible quasianalytic weight for all $s \in [0,\infty)$. (Quasianalyticity follows from the integral test, in view of (4.8).) Since $(\mu_s)_j/j$ is increasing, so is the sequence a_j defined in (3.1), i.e., (3.2) holds with $H = 1$.

The sequences $\mu_s^{|\infty}$ are called *Denjoy weights*. Note that $\mu_0^{|\infty}$ coincides with the analytic weight $\mu_{\mathrm{an}}^{|\infty}$.

Let us investigate the associated function $\eta_{\mu_s^{|\infty}}$. The following proposition extends [Rai26, Proposition A.1].

Proposition 4.9. *Let $s \ge 0$ and $a \ge 1$. Then $\eta_{a\,\mu_s^{|\infty}} : \mathbb{N}_{\ge1} \to \mathbb{N}$ is bounded by a polynomial if and only if $s \in [0,1]$.*

For $s = k \in \mathbb{N}$ and $m \in \mathbb{N}_{\ge1}$,

$$\frac{E_{k+1}(f_k(a,m)-1)}{e_k} - 1 \le \tfrac12 \eta_{a\,\mu_k^{|\infty}}(m) \le \frac{E_{k+1}(f_k(a,m)+1)}{e_k} - 1, \tag{4.5}$$

where $f_k(a,m) := 2ae + L_{k+1}(e_k(m+1))$. In particular,

$$e^{2ae-1}(m+1) - 1 \le \tfrac12 \eta_{a\,\mu_0^{|\infty}}(m) \le e^{2ae+1}(m+1) - 1, \tag{4.6}$$

$$e^{e^{2ae-1}-1}(m+1)^{e^{2ae-1}} - 1 \le \tfrac12 \eta_{a\,\mu_1^{|\infty}}(m) \le e^{e^{2ae+1}-1}(m+1)^{e^{2ae+1}} - 1. \tag{4.7}$$

Proof. Let us first prove (4.5). Set $A_n^k := \sum_{j=1}^n \frac{1}{(\mu_k)_j}$, for $n \geq 1$, and $A_0^k := 0$. Consider $g(t) = L_{k+1}(e_k t)$, for $t \geq 1$. Then

$$g'(t) = \frac{1}{tL_1(e_k t)L_2(e_k t)\cdots L_k(e_k t)} \tag{4.8}$$

is decreasing, i.e., g is concave. Thus

$$g(j+1) - g(j) \leq g'(j) \leq g(j) - g(j-1),$$

where the left inequality holds for $j \geq 1$ and the right for $j \geq 2$. Summing over j, we find

$$g(n+1) - g(1) \leq A_n^k \quad \text{and} \quad A_n^k - A_1^k \leq g(n) - g(1)$$

which implies

$$L_{k+1}(e_k(n+1)) \leq A_n^k \leq L_{k+1}(e_k n) + 1, \quad n \geq 1. \tag{4.9}$$

For $x > 1$ define the positive integer $n(x)$ by

$$A_{n(x)}^k < x \leq A_{n(x)+1}^k. \tag{4.10}$$

By (4.9) and (4.10),

$$\frac{E_{k+1}(x-1)}{e_k} - 1 \leq n(x) < \frac{E_{k+1}(x)}{e_k} - 1. \tag{4.11}$$

By (4.10), we have

$$\sum_{j=m+1}^{n(x)} \frac{1}{(\mu_k)_j} < x - A_m^k \leq \sum_{j=m+1}^{n(x)+1} \frac{1}{(\mu_k)_j}$$

so that, setting $x = 2ae + A_m^k$, we get

$$\sum_{j=m+1}^{n} \frac{1}{(\mu_k)_j} < 2ae \leq \sum_{j=m+1}^{n+1} \frac{1}{(\mu_k)_j}$$

with $n = n(2ae + A_m^k)$, since $2ae + A_m^k > 1$. Thus, by (4.9) and (4.11),

$$\frac{E_{k+1}(2ae - 1 + L_{k+1}(e_k(m+1)))}{e_k} - 1 \leq n \leq \frac{E_{k+1}(2ae + 1 + L_{k+1}(e_k(m+1)))}{e_k} - 1.$$

This implies (4.5), since $n = \frac{1}{2}\eta_{a\,\mu_k^{|\infty}}(m)$.

Now (4.6) and (4.7) easily follow from (4.5). Since $(\mu_s)_j \leq (\mu_t)_j$ for all j if $s \leq t$ and thus $\eta_{a\,\mu_s^{|\infty}} \leq \eta_{a\,\mu_t^{|\infty}}$, we conclude that $\eta_{a\,\mu_s^{|\infty}}$ is bounded by a polynomial, for all $s \in [0,1]$.

To see that $\eta_{a\,\mu_s^{|\infty}}$ cannot be bounded by a polynomial if $s > 1$ it is enough to show this for $s \in (1,2)$ and $a = 1$. Fix such s, let $\sigma = 2 - s \in (0,1)$, and consider $g(t) = (\log\log(e^e t))^\sigma$, for $t \geq 1$. Since

$$g'(t) = \frac{\sigma}{t\log(e^e t)(\log\log(e^e t))^{1-\sigma}}$$

and g is concave, we find similarly to (4.9),

$$(\log\log(e^e(n+1)))^\sigma - 1 \leq \sigma A_n^s \leq (\log\log(e^e n))^\sigma - 1 + \tfrac{\sigma}{e}, \quad n \geq 1, \tag{4.12}$$

where $A_n^s = \sum_{j=1}^n \frac{1}{(\mu_s)_j}$ and noting that $A_1^s = \frac{1}{e}$. As above, let $n(x)$, for $x > 1$, be the positive integer satisfying $A_{n(x)}^s < x \le A_{n(x)+1}^s$. Then, by (4.12),

$$e_2(n(x)+1) \ge E_2((\sigma x + 1 - \tfrac{\sigma}{e})^{\frac{1}{\sigma}})$$

and, setting $n = n(2e + A_m^s) = \frac{1}{2}\eta_{\mu_s^{|\infty}}(m)$, we find

$$e_2(n+1) \ge E_2((2\sigma e + \sigma A_m^s + 1 - \tfrac{\sigma}{e})^{\frac{1}{\sigma}}) \ge E_2(((2e - \tfrac{1}{e})\sigma + (L_2(e_2(m+1)))^\sigma)^{\frac{1}{\sigma}}).$$

Suppose for contradiction that n can be bounded by a polynomial in m. Then there exist $C \ge 1$ and $N \in \mathbb{N}_{\ge 1}$ such that, for all m,

$$E_2(((2e - \tfrac{1}{e})\sigma + (L_2(e_2(m+1)))^\sigma)^{\frac{1}{\sigma}}) \le C(e_2(m+1))^N,$$

or equivalently,

$$(2e - \tfrac{1}{e})\sigma \le (L_2(C(e_2(m+1))^N))^\sigma - (L_2(e_2(m+1)))^\sigma. \tag{4.13}$$

Setting $z = e_2(m+1)$, we have

$$\begin{aligned} L_2(C(e_2(m+1))^N) &= L_2(Cz^N) = \log(\log C + N\log z) \\ &= \log\Big(N\log z\Big(1 + \frac{\log C}{N\log z}\Big)\Big) \\ &= \log\log z + \log N + o(1) \end{aligned}$$

as $z \to \infty$. By the mean value theorem, the right-hand side of (4.13) equals

$$(\log\log z + \log N + o(1))^\sigma - (\log\log z)^\sigma = \sigma\zeta^{\sigma-1}(\log N + o(1)), \tag{4.14}$$

for some ζ between $\log\log z + \log N + o(1)$ and $\log\log z$. Since $\sigma \in (0,1)$, (4.14) tends to zero as $z \to \infty$ which contradicts (4.13). □

Corollary 4.10. *Let $s \in [0,1]$. If $f : [0,1] \to [0,1]$ is a transcendental $(\mu_s^{|\infty}, M_0)$-smooth function, then $\mathbf{z}_d(f)$ is bounded by a polynomial in d and $\mathbf{e}_d(f)$.*

Proof. This is a consequence of Theorem 4.2 and Proposition 4.9. □

5. On the number of rational points on quasianalytic graphs

It is well-known that polynomial Bézout bounds give good bounds on the number of rational points (see [Pil06]). In view of Corollary 4.10, it is thus desirable to detect instances where $\mathbf{e}_d(f)$ is at most polynomial in d.

5.1. Polynomial bounds for $\mathbf{e}_d(f)$ and rational points on $\Gamma(f)$.

In Corollary 5.1, we adapt the arguments of [Pil06] to our setting. Recall that the *height* of a rational number $x \in \mathbb{Q}$ is $\max\{|p|, |q|\}$, where $p, q \in \mathbb{Z}$ are coprime such that $x = p/q$. The height of $(x, y) \in \mathbb{Q}^2$ is the maximum of the heights of x and y.

Corollary 5.1. *Let $s \in [0,1]$ and let $f : [0,1] \to [0,1]$ be a transcendental function of class $\mathcal{C}_{M_s}$ associated with the Denjoy weight $\mu_s^{|\infty}$ (with $(M_s)_0 = 1$) such that $\mathbf{e}_d(f)$ is bounded by a polynomial in d. Then the number of rational points of height at most H on the graph of f is bounded by a polynomial in $\log H$.*

Proof. By [Pil06, Proposition 2.4], the set $\Gamma(f)(\mathbb{Q}; H)$ of rational points of height at most H in $\Gamma(f)$ is contained in the union of at most $6B(\frac{(d+1)(d+2)}{2})^C\, H^{8/(d+3)}$ algebraic curves of degree d, where $d \geq 5$ and B, C are constants depending on the class of f. (The Denjoy–Carleman class $\mathcal{C}_{M_s}$ is contained in the class $M(B, C)$ of mild functions used in [Pil06].)

Let $d = \lfloor \log H \rfloor$, assuming that $H \geq e$. Then $\Gamma(f)(\mathbb{Q}; H)$ is contained in the union of at most

$$6B\, 3^C\, e^8 (\log H)^{2C}$$

algebraic curves of degree d, using that $\frac{(d+1)(d+2)}{2} \leq 3(\log H)^2$ and $H^{8/(d+3)} \leq e^8$. Thus, the number of points in $\Gamma(f)(\mathbb{Q}; H)$ is bounded by

$$6B\, 3^C\, e^8 (\log H)^{2C} \cdot \mathbf{z}_{\lfloor \log H \rfloor}(f)$$

which is bounded by a polynomial in $\log H$, by Corollary 4.10, provided that $\mathbf{e}_d(f)$ is bounded by a polynomial in d. Thus Corollary 5.1 is proved. □

When $\mathbf{e}_d(f)$ is bounded by a polynomial in d? By Remark 4.7, there are real analytic f for which this is not the case. On the other hand, it holds for real analytic functions of the form $f = \sum_{j=1}^{k} p_j e^{q_j}$, where p_j, q_j are real polynomials; see [BY16, Section 2.7]. It seems to be difficult to identify *concrete* non-analytic quasianalytic functions with this property.

We will provide a family of functions $f_\xi : [0, 1] \to [0, 1]$ in $\mathcal{C}_{M_s}$, for $s \in (0, 1]$, parameterized by a sequence of independent random variables, each uniformly distributed in $[-1, 1]$, such that f_ξ is almost surely non-analytic and transcendental, and $\mathbf{e}_d(f)$ is almost surely bounded by a polynomial in d.

Definition 5.2 (The functions f_ξ)**.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be any admissible weight (not necessarily quasianalytic) and $M_0 = 1$ such that the associated function

$$\omega_M(t) := \sup_{k\in\mathbb{N}} \log \frac{t^k}{M_k}, \quad t > 0,$$

satisfies $\omega_M(t) = o(t)$ and $\omega_M(2t) = O(\omega_M(t))$ as $t \to \infty$. Let $\xi = (\xi_n)_{n\geq 1}$ be a sequence of independent random variables, each uniformly distributed on $[-1, 1]$. Define

$$f_\xi(x) := \tfrac{1}{2} + \sum_{n\geq 1} \xi_n c_n \cos(2\pi n x), \qquad \text{where } c_n := \frac{1}{4n^2 e^{\omega_M(2\pi n)}}.$$

Remark 5.3. Note that $\omega_M(t) = o(t)$ as $t \to \infty$ means that $\mathcal{C}_M$ contains the real analytic functions as a proper subclass. This condition is satisfied if the associated sequence a_j (see (3.1)) tends to ∞. Moreover, $\omega_M(2t) = O(\omega_M(t))$ as $t \to \infty$ is a technical condition which holds provided that $\liminf_{j\to\infty} \frac{\mu_{\ell j}}{\mu_j} > 1$ for some positive integer ℓ; see [BMM07, Lemma 12]. Clearly, these conditions are satisfied by all Denjoy weights. Moreover, $\log t = o(\omega_M(t))$ as $t \to \infty$ by definition.

Proposition 5.4. *The family $\{f_\xi\}$ has the following properties:*

(1) *$f_\xi : [0, 1] \to [0, 1]$ belongs to $\mathcal{C}_M$.*

(2) *Almost surely, f_ξ is not real analytic. More generally, almost surely f_ξ does not belong to any class $\mathcal{C}_L$ with $\omega_M(t) = o(\omega_L(t))$ as $t \to \infty$, where L is any weight sequence as in Definition 5.2.*

(3) *Almost surely,* $\mathbf{e}_d(f_\xi)$ *is bounded by a polynomial in* d *of degree* 7 *(whose coefficients depend on* ξ*).*

Proposition 5.4 will be proved in Section 5.2. In the proof, we will see that the condition $\omega_M(t) = o(t)$ as $t \to \infty$ is only used for (2). For (1) and (3), the weaker condition $\omega_M(t) = O(t)$ as $t \to \infty$ suffices. Thus (1) and (3) also hold for the analytic class $M = (j!)$.

Theorem 5.5. *Let* $M = M_s$ *a Denjoy weight for* $s \in (0,1]$ *with* $(M_s)_0 = 1$*. Then, for the family* $\{f_\xi\}$*, the number of rational points of height at most* H *on the graph of* f_ξ *is almost surely bounded by a polynomial in* $\log H$*.*

Proof. This follows from Corollary 5.1 and Proposition 5.4. Note that f_ξ is almost surely transcendent, because if there exists a nonzero polynomial $p \in \mathbb{R}[X,Y]$ such that $p(x, f_\xi(x)) = 0$, for all $x \in [0,1]$, then f_ξ is real analytic. Indeed, the 1-periodic function f_ξ belongs to $\mathcal{C}_{M_s}(\mathbb{R})$ and, by quasianalyticity, $p(x, f_\xi(x)) = 0$, for all $x \in \mathbb{R}$. By [Sic70, Theorem 2] or [Mal66, Proposition VI.3.11], f_ξ is real analytic. □

5.2. **Proof of Proposition 5.4.** (1) It is easy to check that f_ξ is of class $\mathcal{C}_M$ because (see e.g. [Kom73, Proposition 3.2])

$$\frac{(2\pi n)^j}{e^{\omega_M(2\pi n)}} \le \sup_{t>0} \frac{t^j}{e^{\omega_M(t)}} = M_j.$$

Since $\sum_{n\ge1} c_n \le \sum_{n\ge1} \frac{1}{4n^2} = \frac{\pi^2}{24} < \frac{1}{2}$, f_ξ maps $[0,1]$ into $[0,1]$.

(2) If the 1-periodic function f_ξ is of class $\mathcal{C}_L$, then there exist constants $C, c > 0$ such that the Fourier coefficients $\xi_n c_n$ of f_ξ satisfy

$$|\xi_n| c_n \le C e^{-\omega_L(cn)}, \quad n \ge 1. \tag{5.1}$$

Indeed,

$$|\hat{f}(n)| \le \inf_{j\in\mathbb{N}} \frac{\|f_\xi^{(j)}\|_{[0,1]}}{(2\pi n)^j} \le \inf_{j\in\mathbb{N}} \frac{C\rho^j L_j}{(2\pi n)^j} = C e^{-\omega_L(cn)}$$

with $c = 2\pi/\rho$. Since $\omega_M(t) = o(\omega_L(t))$, $\omega_L(2t) = O(\omega_L(t))$, and $\log t = o(\omega_L(t))$ as $t \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$c_n = \frac{1}{4n^2 e^{\omega_M(2\pi n)}} \ge e^{-\frac{1}{2}\omega_L(cn)}, \quad n \ge n_0.$$

By the Borel–Cantelli lemma, almost surely $|\xi_n| \ge 1/2$ for infinitely many n. Hence,

$$|\xi_n| c_n \ge \tfrac{1}{2} e^{-\frac{1}{2}\omega_L(cn)},$$

almost surely, for infinitely many $n \ge n_0$, contradicting (5.1). This proves (2); the case $\omega_L(t) = t$ shows that almost surely f_ξ is not real analytic.

(3) Set $N = \dim \mathcal{P}_2^d = \frac{(d+1)(d+2)}{2}$. Let $x_1, \dots, x_N$ be distinct points in $[0,1]$ and A the $N \times N$ matrix with entries $x_k^i f_\xi(x_k)^j$, for $1 \le k \le N$ and $0 \le i+j \le d$. For a polynomial $p(x,y) = \sum_{i+j\le d} c_{ij} x^i y^j \in \mathcal{P}_2^d$ with coefficient vector $c = (c_{ij})$, we have $\|p\|_{[0,1]^2} \le \sum_{i+j\le d} |c_{ij}| = \|c\|_1$ and

$$\|p\|_{\Gamma(f_\xi)} \ge \|Ac\|_\infty \ge N^{-1/2}\|Ac\|_2 \ge N^{-1/2}\sigma_N(A)\|c\|_2 \ge N^{-1}\sigma_N(A)\|c\|_1,$$

where $\sigma_1(A) \geq \cdots \geq \sigma_N(A)$ are the singular values of A. Using that $|\det A| = \sigma_1(A)\cdots\sigma_N(A) \leq \sigma_N(A)\|A\|_2^{N-1} \leq \sigma_N(A)N^{N-1}$, we conclude that

$$\frac{\|p\|_{[0,1]^2}}{\|p\|_{\Gamma(f_\xi)}} \leq \frac{N^N}{|\det A|}.$$

This implies

$$\mathbf{e}_d(f_\xi) \leq N\log N + \log\Big(\frac{1}{|\det A|}\Big). \tag{5.2}$$

Let $\xi_{\leq N} := (\xi_1, \ldots, \xi_N)$ and $\xi_{>N} := (\xi_{N+1}, \xi_{N+2}, \ldots)$. We keep $\xi_{>N}$ fixed. Then $\xi_{\leq N}$ is uniformly distributed in $[-1,1]^N$. We will estimate $|\det A|$ for the choice

$$x_k := \frac{2k-1}{4(N+1)}, \quad 1 \leq k \leq N. \tag{5.3}$$

Set $y_k := f_\xi(x_k)$ and $y = (y_1, \ldots, y_N)$. This gives a decomposition $y^\top = a^\top + B\xi_{\leq N}^\top$, where $B = C\,\mathrm{diag}(c_1, \ldots, c_N)$, $C = (\cos(2\pi n x_k))_{1\leq k,n\leq N}$, and $a = (a_1, \ldots, a_N)$ with $a_k = \frac{1}{2} + \sum_{n>N} \xi_n c_n \cos(2\pi n x_k)$. It follows that y is uniformly distributed in the parallelepiped $K = a + B[-1,1]^N$ contained in $[0,1]^N$.

Let us compute the volume $|K|$. To this end, let $\hat{C} = (\hat{C}_{k,n})$ be the $(N+1)\times(N+1)$ matrix, where $\hat{C}_{k,n} = (\frac{2}{N+1})^{1/2}\cos(2\pi n x_k)$ if $1 \leq k \leq N+1$ and $1 \leq n \leq N$ and $\hat{C}_{k,0} = (\frac{1}{N+1})^{1/2}$ if $1 \leq k \leq N+1$. Then $\hat{C}$ (being a discrete cosine transform of type IV) is orthogonal and $(\frac{2}{N+1})^{1/2}\,C$ is its minor with $(N+1)$-st row and 0-th column removed. Thus

$$|\det C| = \Big(\frac{N+1}{2}\Big)^{N/2}|\hat{C}_{N+1,0}| = \Big(\frac{N+1}{2}\Big)^{N/2}\Big(\frac{1}{N+1}\Big)^{1/2}.$$

It follows that

$$|K| = 2^N\Big(\frac{N+1}{2}\Big)^{N/2}\Big(\frac{1}{N+1}\Big)^{1/2} c_1\cdots c_N. \tag{5.4}$$

For x_k given by (5.3), consider the homogeneous polynomial in the variables $y = (y_1, \ldots, y_N)$,

$$D(y) = \det(x_k^i y_k^j)_{1\leq k\leq N, 0\leq i+j\leq d} = \sum_\sigma \operatorname{sgn}\sigma \prod_{k=1}^N x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}},$$

where σ ranges over all bijections from $\{1, \ldots, N\}$ to $\{(i,j) : 0 \leq i+j \leq d\}$. The degree of $D(y)$ is $m := \sum_{0\leq i+j\leq d} j = \sum_{j=0}^d j(d+1-j) = \frac{1}{6}d(d+1)(d+2)$. Let b_e for $e = (e_1, \ldots, e_N)$ denote the coefficient of the monomial $\prod_{k=1}^N y_k^{e_k}$ in $D(y)$. Then b_e is up to sign a product of Vandermonde determinants

$$\prod_{j=0}^d \det(x_k^i)_{k\in R_j, 0\leq i\leq d-j},$$

where $R_j = \{k : e_k = j\}$. Thus, for the choice of e which corresponds to R_0 consisting of the first $d+1$ points, R_1 of the next d points, and so on, we find

$$|b_e| \geq (2(N+1))^{-m}.$$

The coefficients of a univariate real polynomial p of degree d are bounded by $(8e)^d\|p\|_{[0,1]}$; indeed, by comparison with the Chebyshev polynomial (see [RS02, Theorem 16.3.3]) and the Remez inequality, each coefficient is bounded by

$e^d\|p\|_{[-1,1]} \le (8e)^d\|p\|_{[0,1]}$. Since each y_k has degree at most d in $D(y)$, we find by iteration that $|b_e| \le (8e)^{dN}\|D\|_{[0,1]^N}$ and hence

$$\|D\|_{[0,1]^N} \ge (2(N+1))^{-m}(8e)^{-dN}. \tag{5.5}$$

By the Remez inequality (see [BG73]),

$$\|D\|_{[0,1]^N} \le \Big(\frac{4N}{|K|}\Big)^m \|D\|_K, \tag{5.6}$$

as well as, for $E := \{y \in K : |D(y)| \le (8N)^{-m}\|D\|_K\}$,

$$\|D\|_K \le \Big(\frac{4N|K|}{|E|}\Big)^m \|D\|_E \le \Big(\frac{|K|}{2\,|E|}\Big)^m \|D\|_K$$

implying that $|E| \le \frac{1}{2}|K|$. Thus

$$\int_K |D(y)|^2\,dy \ge \int_{K\setminus E} |D(y)|^2\,dy \ge \frac{1}{2}|K|\,(8N)^{-2m}\|D\|_K^2. \tag{5.7}$$

Combining (5.4), (5.5), (5.6), and (5.7), we find

$$\|D\|_{L^2(K;\frac{\mathcal{L}^N}{|K|})} = \Big(\int_K |D(y)|^2\,\frac{dy}{|K|}\Big)^{1/2} \ge \frac{1}{\sqrt{2}}(64N^2(N+1))^{-m}(8e)^{-dN}(c_1\cdots c_N)^m.$$

Since $m\sum_{k=1}^N \log(\frac{1}{c_k}) \le mN(\log(4N^2)+\omega_M(2\pi N)) = O(d^7)$, using that $\omega_M(t) = O(t)$ as $t\to\infty$, this implies that

$$\log\Big(\big(\|D\|_{L^2(K;\frac{\mathcal{L}^N}{|K|})}\big)^{-1}\Big) = O(d^7). \tag{5.8}$$

Note that (5.8) holds uniformly in a: the constants depend only on M through the use of $\omega_M(t) = O(t)$ as $t\to\infty$.

By [CW01, Theorem 8], there is an absolute constant $\alpha > 0$ such that, for all $t > 0$,

$$\frac{1}{|K|}\big|\big\{y \in K : |D(y)| \le t\,\|D\|_{L^{1/m}(K,\frac{\mathcal{L}^N}{|K|})}\big\}\big| \le \alpha\, t^{1/m}$$

and, by [CW01, Theorem 1], there is an absolute constant $\beta > 0$ such that

$$\|D\|_{L^2(K,\frac{\mathcal{L}^N}{|K|})} \le (\beta m)^m \|D\|_{L^{1/m}(K,\frac{\mathcal{L}^N}{|K|})}.$$

Thus

$$\frac{1}{|K|}\big|\big\{y \in K : |D(y)| \le t\,\|D\|_{L^2(K,\frac{\mathcal{L}^N}{|K|})}\big\}\big| \le \alpha\beta m\, t^{1/m}.$$

Taking $t = e^{-d^4}$, we have

$$\frac{1}{|K|}\big|\big\{y \in K : |D(y)| \le e^{-d^4}\|D\|_{L^2(K,\frac{\mathcal{L}^N}{|K|})}\big\}\big| \le \alpha\beta m\, e^{-d^4/m}. \tag{5.9}$$

Note that the bound on the right-hand side is independent of a.

Let now $Q_d(\xi) = D(f_\xi(x_1),\dots,f_\xi(x_N))$ denote the determinant of the $N\times N$ matrix with entries $x_k^i f_\xi(x_k)^j$, for $1 \le k \le N$ and $0 \le i+j \le d$, i.e., of A, where the x_k are give by (5.3). Consider the event

$$E_d = \{\xi : |Q_d(\xi)| \le e^{-d^4}\|Q_d\|_{L^2(K,\frac{\mathcal{L}^N}{|K|})}\}$$

and recall that K depends on a. By Tonelli's theorem and (5.9), the probability $P(E_d)$ of E_d is bounded by $\alpha\beta m\, e^{-d^4/m}$ so that $\sum_d P(E_d) < \infty$. By the Borel–Cantelli lemma, almost surely in ξ, the event E_d is failing for all $d \geq d_0(\xi)$. For such ξ and $d \geq d_0(\xi)$, (5.8) implies

$$\log\Big(\frac{1}{|Q_d(\xi)|}\Big) \leq d^4 + O(d^7)$$

In view of (5.2), this completes the proof.

Acknowledgement. This research was funded in whole or in part by the Austrian Science Fund (FWF) DOI 10.55776/PAT1381823. For open access purposes, the author has applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

The author acknowledges that Proposition 5.4 and its proof strategy were suggested by Claude Sonnet 5.5 (Anthropic). The author independently verified the proof and incorporated it; he takes full responsibility for the content of the paper.

References

[Ban53] T. Bang, *The theory of metric spaces applied to infinitely differentiable functions*, Math. Scand. **1** (1953), 137–152.

[BMM07] J. Bonet, R. Meise, and S. N. Melikhov, *A comparison of two different ways to define classes of ultradifferentiable functions*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 424–444.

[Bru08] A. Brudnyi, *On local behavior of holomorphic functions along complex submanifolds of* $\mathbb{C}^N$, Invent. Math. **173** (2008), no. 2, 315–363.

[BY16] A. Brudnyi and Y. Yomdin, *Norming sets and related Remez-type inequalities*, J. Aust. Math. Soc. **100** (2016), no. 2, 163–181.

[BG73] Yu. A. Brudnyi and M. I. Ganzburg, *A certain extremal problem for polynomials in n variables*, Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 344–355.

[CW01] A. Carbery and J. Wright, *Distributional and* L^q *norm inequalities for polynomials over convex bodies in* $\mathbb{R}^n$, Math. Res. Lett. **8** (2001), no. 3, 233–248.

[CLP05] D. Coman, N. Levenberg, and E. A. Poletsky, *Quasianalyticity and pluripolarity*, J. Am. Math. Soc. **18** (2005), no. 2, 239–252.

[CP03] D. Coman and E. A. Poletsky, *Measures of trancendency for entire functions*, Michigan Math. J. **51** (2003), no. 3, 575–591.

[CP07] ———, *Transcendence measures and algebraic growth of entire functions*, Invent. Math. **170** (2007), no. 1, 103–145.

[CY18] G. Comte and Y. Yomdin, *Zeroes and rational points of analytic functions*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 6, 2445–2476.

[Kom73] H. Komatsu, *Ultradistributions. I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.

[KP02] S. G. Krantz and H. R. Parks, *A primer of real analytic functions*, second ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston Inc., Boston, MA, 2002.

[Mal66] B. Malgrange, *Ideals of differentiable functions*, Tata Institute of Fundamental Research Studies in Mathematics, No. 3, Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1966.

[NSV04] F. Nazarov, M. Sodin, and A. Volberg, *Lower bounds for quasianalytic functions. I. How to control smooth functions*, Math. Scand. **95** (2004), no. 1, 59–79.

[Pil06] J. Pila, *Mild parameterization and the rational points of a Pfaff curve*, Comment. Math. Univ. St. Pauli **55** (2006), no. 1, 1–8.

[PW06] J. Pila and A. J. Wilkie, *The rational points of a definable sets*, Duke Math. J. **133** (2006), no. 3, 591–616.

[RS02] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press Oxford University Press, Oxford, 2002.

[Rai22] A. Rainer, *Ultradifferentiable extension theorems: a survey*, Expo. Math. **40** (2022), no. 3, 679–757.

[Rai23] ———, *Quantitative tame properties of differentiable functions with controlled derivatives*, Nonlinear Anal. **237** (2023), Paper No. 113372, 27 pp.

[Rai26] ———, *Effective quasianalytic Remez inequalities on tame sets*, https://arxiv.org/pdf/2606.08183.pdf.

[RSW03] J.-P. Rolin, P. Speissegger, and A. J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), no. 4, 751–777 (electronic).

[Sic70] J. Siciak, *A characterization of analytic functions of n real variables*, Studia Math. **35** (1970), 293–297.

[Tij71] R. Tijdeman, *On the number of zeros of general exponential polynomials*, Nederl. Akad. Wet., Proc., Ser. A **74** (1971), 1–7.

[Yam89] T. Yamanaka, *Inverse map theorem in the ultra-F-differentiable class*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), no. 7, 199–202.

Armin Rainer: Faculty of Mathematics and Geoinformation, Institute for Statistics and Mathematical Methods in Economics, E105-04, TU Wien, Wiedner Hauptstrasse 8, 1040 Vienna, Austria

Email address: armin.rainer@tuwien.ac.at