AN ALGORITHM DETECTING DEHN PRESENTATIONS.

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Abstract. An algorithm is given detecting whether or not a finite presentation of a group is a Dehn presentation (i.e. admitting Dehn's algorithm for the word problem) with a certain condition.

Because being word hyperbolic is a Markov property of groups there cannot exist an effective procedure for determining if a finitely presented group admits a Dehn presentation (see, for example, [9]). However, there may exist an algorithm to decide whether a finite presentation of a group is a Dehn presentation. In this article we prove a result in this direction.

Definition 1. Let $\frac{1}{2} \leq \alpha < 1$. We call a presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$ of a group G an *α-Dehn presentation* if any non-empty freely reduced word w representing the identity in G contains, as a subword, a word u which is also a subword of a cyclic shift of some $r \in \mathcal{R}^{\pm 1}$ with $|u| > \alpha|r|$.

A Dehn presentation in the traditional sense is an α-Dehn presentation with $\alpha = 1/2$. Observe also that any α-Dehn presentation is a Dehn presentation.

A Dehn presentation of a group G leads to a known Dehn's algorithm solving the word problem for G. We refer to [5] for an interesting discussion and problems on other Dehn's algorithms and different notions of Dehn presentations.

Our main result is the following.

Theorem. *There exists an algorithm determining whether or not a finite presentation of a group is an α-Dehn presentation for some $\frac{3}{4} \leq \alpha < 1$.*

Note that a group G given by a finite Dehn presentation is word hyperbolic [6, 2.3], the word problem for G is solvable in linear time by Dehn's algorithm [4], and the hyperbolicity constant for G can be calculated, see for example subsection 1.2.

1. Dehn's algorithm

Given a group G generated by a finite set $\mathcal{X}$ and a word w in the alphabet $\mathcal{X}^{\pm 1}$ we denote by $|w|$ the length of w and by $||w||$ the length of a shortest word in $\mathcal{X}^{\pm 1}$ representing the same element of G.

Definition 2. Let $\langle \mathcal{X} \mid \mathcal{R} \rangle$ be a presentation of a group. A word w in the alphabet $\mathcal{X}^{\pm 1}$ is said to be *Dehn irreducible* with respect to this presentation if w has no subword u such that u is also a subword of a cyclic shift of some $r \in \mathcal{R}^{\pm 1}$ with $|u| > \frac{1}{2}|r|$.

The following lemma shows that Dehn irreducible words with respect to certain Dehn presentations are quasigeodesics. Let $\lambda \geq 1$ and $c \geq 0$. Recall that a word w in the alphabet $\mathcal{X}^{\pm 1}$ is called *(λ, c)-quasigeodesic* in G if $|u| \leq \lambda||u|| + c$ for any subword u of w. A word w is *(k, λ, c)-local quasigeodesic* for $k > 0$ if any its subword

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of length at most k is (λ, c)-quasigeodesic. Similarly, a word w is *k-local geodesic* if any subword u of w of length $|u| \leq k$ is geodesic, that is, $|u| = ||u||$.

Lemma 1. *Let $\langle \mathcal{X} \mid \mathcal{R} \rangle$ be a finite α-Dehn presentation of a group G with $\frac{3}{4} \leq \alpha < 1$. Let $M = \max_{r \in \mathcal{R}} |r|$. Then for any Dehn irreducible word w and any word z both representing the same element of G, one has*

$$|z| \geq \frac{1}{M}|w|$$

In particular, $||w|| \geq \frac{1}{M}|w|$, i.e. Dehn irreducible words are $(M, 0)$-quasigeodesics.

Proof. Assume that a presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$ and words w and z satisfy the hypotheses of the lemma. In particular, $wz^{-1} =_G 1$. Without loss of generality we assume that wz^{-1} is freely reduced.

We proceed by induction on $|w| + |z|$.

If $|w| = |z| = 0$ the statement is trivial. Let $|w| + |z| > 0$. By definition of an α-Dehn presentation, the word wz^{-1} has a subword u which is also a subword of a cyclis shift of some $r \in \mathcal{R}^{\pm 1}$ with $|u| > \frac{3}{4}|r|$. Let $wz^{-1} = AuB$ and $r = uv^{-1}$ for some words A, B and v. The word u cannot be a subword of w as w is Dehn irreducible. So there are two different cases.

Case 1. u is a subword of z^{-1}, i.e. $z^{-1} = CuD$ for some words C and D.

Since $uv^{-1} =_G 1$ we have $wz_1^{-1} =_G 1$ with $z_1^{-1} = CvD$. But $|w| + |z_1| < |w| + |z|$ as $|u| > |v|$. So, using the inductive hypothesis, we get $M|z| > M|z_1| \geq |w|$.

Case 2. $u = u_1 u_2$ where u_1 is a subword of w and u_2 is a subword of z^{-1}.

There are two subcases. First suppose that $|v| < |u_2|$. Then $|B| + |v| < |B| + |u_2| = |z|$. Since any subword of a Dehn irreducible word is again Dehn irreducible, we can use the inductive hypothesis for the words $w_1 = A$ and $z_1 = vB$. Hence, $M(|B| + |v|) \geq |A|$. Since $|v| \leq |u_2| - 1$, it follows that $M(|B| + |u_2|) - M \geq |A|$. This implies $M(|B| + |u_2|) \geq |A| + M \geq |A| + |u_1| = |w|$ as desired.

Now suppose that $|v| \geq |u_2|$. Then $|u_2| \leq \frac{|r|}{4}$ as $|v| \leq \frac{|r|}{4}$. But $|u_1| > \frac{3}{4}|r| - |u_2| \geq \frac{|r|}{2}$. This contradicts the assumption that w is Dehn irreducible word. □

1.1. **Word hyperbolic groups.** Let $F = F(\mathcal{X})$ be a group freely generated by $\mathcal{X}$ and

$$G = \langle \mathcal{X} \mid \mathcal{R} \rangle \tag{1}$$

be a finitely presented group. For any word w in the generators $\mathcal{X}^{\pm 1}$ representing the identity in G, there is an expression

$$w =_F \prod_{i=1}^{N} z_i r_i^{\pm 1} z_i^{-1}, \tag{2}$$

where $r_i \in \mathcal{R}$, $z_i \in F$ for $i = \{1, \ldots, N\}$. The minimal possible N in (2) for a given w is called the area of w and denoted by Area (w). A finitely presented group defined by (1) is said to be *word hyperbolic* if there is a constant $C > 0$ such that Area $(w) \leq C|w|$ for any w representing the identity in G. In other words, G satisfies a linear isoperimetric inequality.

Proceeding by induction on the number of terms in the expression (2), it is not hard to see that Dehn's algorithm implies a linear isoperimetric inequality (with multiplicative constant 1). The converse is also true. The proof uses a more delicate analysis of local quasigeodesics in the Caley graph of a word hyperbolic group and shows essentially that to find a Dehn presentation (with a parameter) it suffices to take all long enough relations in the group as the defining relations [2], [11, Th 2.12], [8], and [1, Ch. III. Γ, Th. 2.6].

Proposition 1. *([6, 2.3], [8], [11, Th 2.12]) The following conditions on a finitely generated group G are equivalent:*
(a) G is word hyperbolic;
(b) G has a Dehn presentation;
(c) G has an α-Dehn presentation for any α, $\frac{1}{2} \leq \alpha < 1$.

The following reformulation of the word hyperbolicity condition is equivalent to the definition given above. A finitely generated group is *δ-hyperbolic* if there is a constant $\delta \geq 0$ such that any geodesic triangle in the Cayley graph of the group with respect to some (and therefore to any) generating set is δ-slim. A geodesic triangle is δ-slim if each of its side is contained in the δ-neighbourhood of the union of the other two sides [6, 6.3], [7, Ch.2, §3], and [3, Ch.1, §3].

It is known that for $k > 8\delta$ any k-local geodesic word in the generators of a δ-hyperbolic group is quasigeodesic [6, 7.2.B], [1, Ch.III.H, Th.1.13]. In particular, there are no nontrivial k-local geodesic words representing the identity in a word hyperbolic group. We will need a similar result for *k-local quasigeodesics* words defined in a natural way.

Lemma 2. *([6, 7.2.D], [3, Ch.3, Th.1.4]) Let G be a δ-hyperbolic group and $M > 0$ be a constant. There exists a constant $A = A(\delta, M) > 0$ such that if for any subword y of a word w in the generators of G of length $|y| \leq A$ we have $||y|| \geq \frac{1}{M}|y|$ then $w \neq_G 1$.*

We notice that the constant $A = A(\delta, M)$ can be effectively calculated in terms of the complexity of a group presentation for G, see Section 1.2.

The next result is due to Gromov [6, 2.3.F]. He used it to give a partial algorithm which, given a finite presentation of a group, stops if and only if the group is word hyperbolic. We mention that Gromov gave an analytical idea how to prove the lemma. The combinatorial proof was given by Papasoglu in [10].

Recall that given a finite presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$ of a group G one can use Tietze's transformations to obtain a triangular presentation, i.e. the presentation of the same group with all defining relations of length at most 3. Namely, suppose that $r \in \mathcal{R}$ has length greater than 3, that is, $r = r_1 r_2$ for some words r_1 and r_2 of length greater than 1. One introduces a new generator t and observes that $\langle \mathcal{X} \cup \{t\} \mid (\mathcal{R} \setminus \{r\}) \cup \{t r_1^{-1}, t r_2\} \rangle$ is also a presentation of G. Repeating this step finitely many times one gets a triangular presentation of G.

Lemma 3. *([10]) Let G be a group given by a finite triangular presentation. Assume that for some integer $K > 0$ every word w representing the identity in G with area*

$$\frac{K^2}{2} \leq Area(w) \leq 240K^2$$

satisfies $Area(w) \leq \frac{1}{2 \cdot 10^4}|w|^2$. Then for any word w representing the identity in G we have $Area(w) \leq K^2|w|$. In particular, G is word hyperbolic and the hyperbolicity constant δ can be effectively calculated.

1.2. **Remarks on the calculations.** Now we recall some estimates on different constants from the above given definitions of a word hyperbolic group.

Let $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$ be a word hyperbolic group. Hence for some $C > 0$ the linear isoperimetric inequality $\mathrm{Area}\,(w) \leq C|w|$ holds for any w representing the identity in G.

Remark 1. Suppose that $\langle \mathcal{X}' \mid \mathcal{R}' \rangle$ is any other finite presentation of G. We are going to estimate multiplicative constant C' in the linear isoperimetric inequality with respect to this presentation. For each generator $x' \in \mathcal{X}'$ we fix some its expression $S(x')$ as a word in the alphabet $\mathcal{X}$. Similarly we define a word $S'(x)$ for each generator $x \in \mathcal{X}$. For a word w' in the alphabet $\mathcal{X}'$ we denote by $S(w')$ the word that we obtain replacing any generator $x' \in \mathcal{X}'$ in w' by $S(x')$. In the same way we define $S'(w)$ for a word w in the alphabet $\mathcal{X}$. Let us set

$$l_1 = \max_{x' \in \mathcal{X}'} |S(x')| \quad \text{and} \quad l_2 = \max_{r \in \mathcal{R}} \mathrm{Area}'\,(S'(r)), \tag{3}$$

where Area' denotes the area with respect to the presentation $\langle \mathcal{X}' \mid \mathcal{R}' \rangle$. Then

$$\mathrm{Area}'\,(w') \leq l_2 \mathrm{Area}\,(S(w')) \leq C l_2 |S(w')| \leq C l_1 l_2 |w'|.$$

Thus we can take $C' = C l_1 l_2$.

Remark 2. Given C as above and $M = \max_{r \in \mathcal{R}} |r|$, we can find the hyperbolicity constant $\delta = \delta(C, M)$ as well as the hyperbolicity constant δ' with respect to $\langle \mathcal{X}' \mid \mathcal{R}' \rangle$, see [11, Th.2.5], [8].

Remark 3. To get an estimate on $A = A(\delta, M)$ in Lemma 2, we first find δ using the previous remark. Then we choose A satisfying the inequality

$$A = A(\delta, M) > \max(A_1, 8\varepsilon M^2 + 16\varepsilon M)$$

where $\varepsilon = 4M(16\delta \log_2(128M^2 + 16M) + 2) + 32M\delta$ and $A_1 > 2\varepsilon$, see [3, Th1.2, Th.1.4].

The algorithm

Here we describe our algorithm. We use the previous remarks.

Step 1. Given a finite presentation $\pi = \langle \mathcal{X} \mid \mathcal{R} \rangle$ of a group G, we triangulate π getting a finite presentation $\pi' = \langle \mathcal{X}' \mid \mathcal{R}' \rangle$ whose any defining relation is of length at most 3.

Step 2. Using Remark 1 we find l_1 and l_2 for the presentations π and π'. Let $T = l_1 l_2$.

Step 3. Take $K > 2 \cdot 10^2 T$. We check whether or not $\mathrm{Area}'\,(w) \leq \frac{1}{2 \cdot 10^4}|w|^2$ for all words w in the generators of π' representing the identity in G whenever $\frac{K^2}{2} \leq \mathrm{Area}'\,(w) \leq 240K^2$.

If the answer is "no" then π is not a Dehn presentation. Indeed, assume that π is a Dehn presentation. Then $\mathrm{Area}\,(w) \leq |w|$ for any w representing the identity in G. By the choice of T, π' is a presentation of a word hyperbolic group with $\mathrm{Area}'\,(w) \leq T|w|$. Hence we have $\mathrm{Area}'\,(w) \leq \frac{1}{2 \cdot 10^4}|w|^2$ for $K > 2 \cdot 10^2 T$ and there is a word w such that $\frac{K^2}{2} \leq \mathrm{Area}'\,(w) \leq 240K^2$. But this contradicts the assumed negative answer. In this case, the algorithm stops.

If the answer is "yes", by Lemma 3 G is word hyperbolic and the hyperbolicity constant δ' with respect to π' can be effectively calculated from K, see Remark 2. Using also Remark 1, we calculate the hyperbolicity constant δ with respect to π. We can effectively find a new presentation $\pi'' = \langle \mathcal{X} \mid \mathcal{R}'' \rangle$ for G which is a

α-Dehn presentation with $\frac{3}{4} \leq \alpha < 1$ (see for example [8]). We also calculate the hyperbolicity constant δ'' with respect to π'' using Remarks 1 and 2.

Step 4. Let π_1 be the presentation of G obtained from π'' by removing one defining relation which is a consequence of the others. To complete the algorithm we have to give a way to determine whether or not π_1 remains a Dehn presentation.

Let δ_1 be the hyperbolicity constant with respect to π_1. We know that δ_1 depends linearly on already calculated δ'', Remarks 1 and 2. We denote by M_1 the maximum of lengths of the defining relations of π_1. Using Lemma 2 and Remark 3, we calculate the constant $A = A(\delta_1, M_1)$. Next we consider all Dehn irreducible words with respect to π_1 of length at most A. We check for these words whether they satisfy the inequality from Lemma 1 with the constant M_1. If the answer is "no" then, by Lemma 1, we conclude that π_1 (and hence π) is not an α-Dehn presentation for $\frac{3}{4} \leq \alpha < 1$, and the algorithm stops. Otherwise, in view of the definition of α-Dehn presentation and Lemma 2, we see that π_1 is a Dehn presentation. If π_1 coincides with π then π is a Dehn presentation with a parameter and the algorithm stops. Otherwise, we go to the next step.

Step 5. We consider a new presentation π_2 of G obtained from π_1 by removing one defining relation which is a consequence of the others. We repeat everything for π_2 as in the previous step.

We repeat this final step as long as there are redundant relations. It is clear that proceeding in this way we determine whether or not π is a Dehn presentation with a parameter.

1.3. **Question.** The restriction $\frac{3}{4} \leq \alpha < 1$ were used the only time in the final part of the proof of Lemma 1. Everything else is true also in case when $\frac{1}{2} \leq \alpha < 1$. So, to prove our theorem in more general setting we need to show that Dehn irreducible words are quasigeodesic with respect to any Dehn's presentation with a parameter $\frac{1}{2} \leq \alpha < 1$. (It could give also another hyperbolicity condition.)

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