Viscoelasticity and accretive phase-change at finite strains

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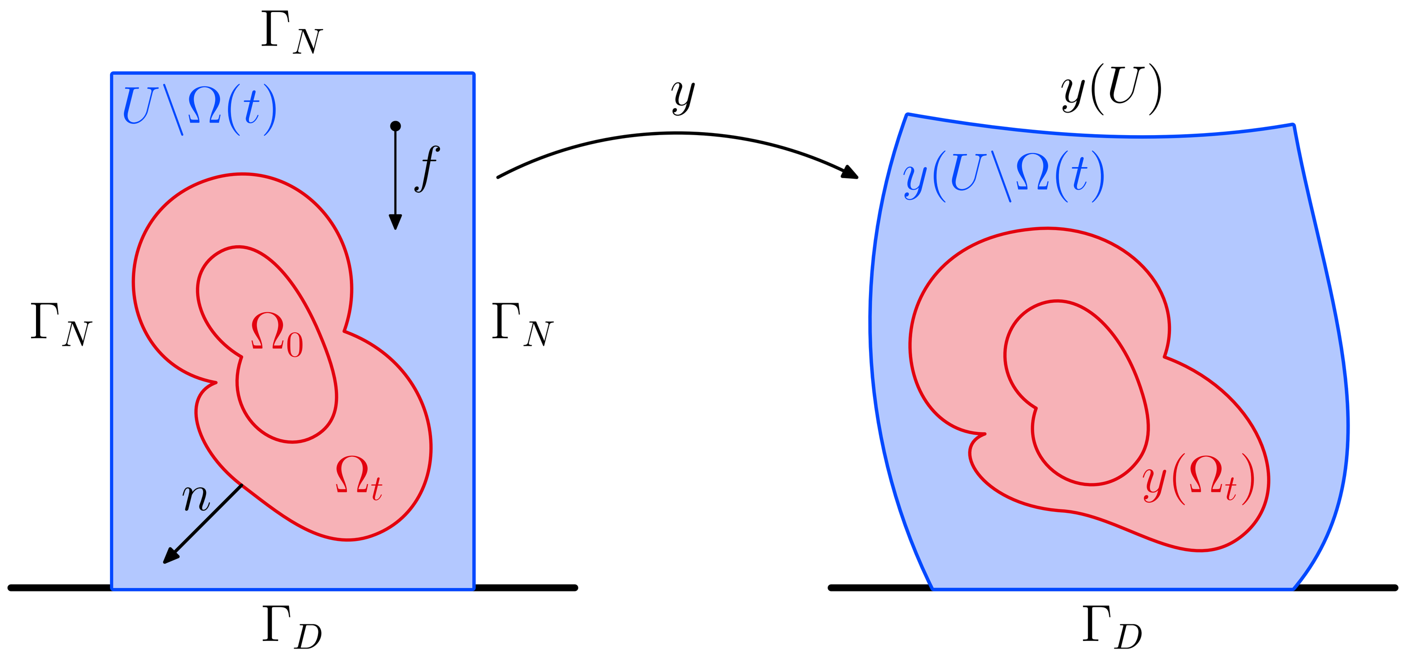
The model

Evolution of a viscoelastic solid + phase change (accretive growth)



- Swelling in polymer gels
- Solidification processes
- Early tumor development

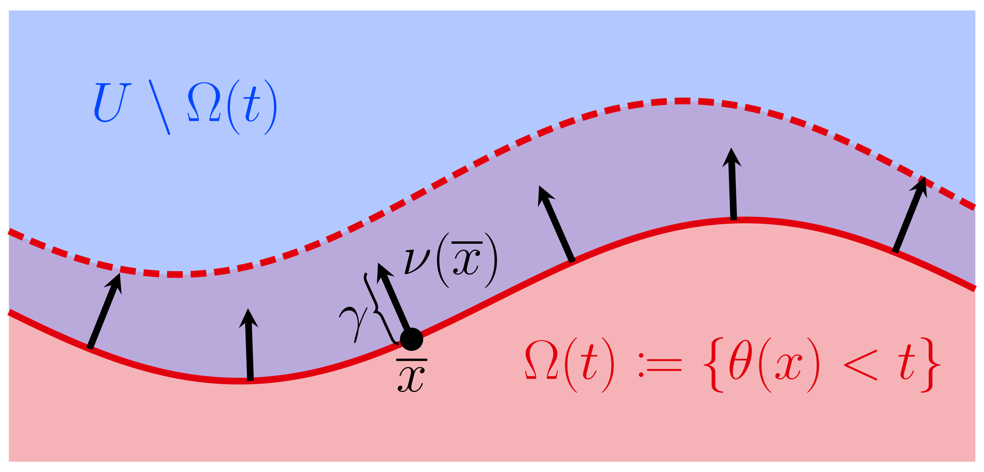
Viscoelastic equilibrium

$$-\operatorname{div}\left(\partial_{\nabla y}W_\varepsilon(\theta(x)-t,\nabla y)+\partial_{\nabla\dot y}R_\varepsilon(\theta(x)-t,\nabla y,\nabla\dot y)-\operatorname{div}DH(\nabla^2 y)\right) = f(\theta(x)-t,x) \quad \text{in } [0,T]\times U,$$

Boundary and initial conditions

Accretive growth

$$\gamma\big(y(\theta(x),x),\nabla y(\theta(x),x)\big)|\nabla(-\theta)(x)|=1 \quad \text{in } U\setminus\overline{\Omega_0},$$
$$\theta=0 \quad \text{on } \Omega_0.$$



Definition (Weak/viscosity solution)

$$(y,\theta)\in\left(L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d))\right)\times C^{0,1}(\overline{U})$$

is a *weak/viscosity solution* to the initial-boundary-value problem if $\det\nabla y(t,\cdot)>0$ for all $t\in(0,T)$, $y(0,\cdot)=y_0$, and

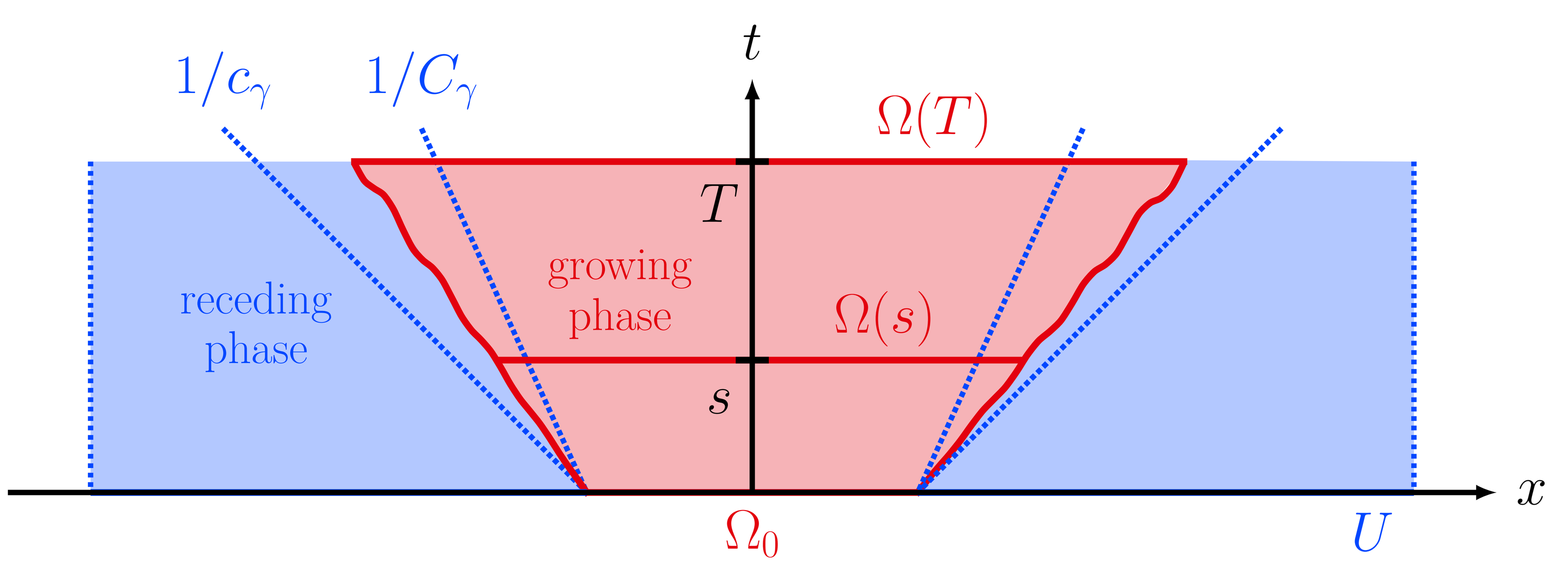
$$\int_0^T\!\!\int_U\left(\partial_F W_\varepsilon(\theta-t,\nabla y):\nabla z+\partial_{\dot F}R_\varepsilon(\theta-t,\nabla y,\nabla\dot y):\nabla z+\mathrm{D}H\left(\nabla^2 y\right)\vdots\nabla^2 z\right)\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\!\!\int_U f(\theta-t)\cdot z\,\mathrm{d}x\mathrm{d}t \quad \forall z\in C^\infty(\overline{Q};\mathbb{R}^d) \text{ with } z=0 \text{ on } \Sigma_D,$$

and θ is a viscosity solution to

$$\begin{cases}\gamma\big(y(\theta(x),x),\nabla y(\theta(x),x)\big)|\nabla(-\theta)(x)|=1 & \text{in } U\setminus\overline{\Omega_0},\\ \theta=0 & \text{in } \Omega_0.\end{cases}$$

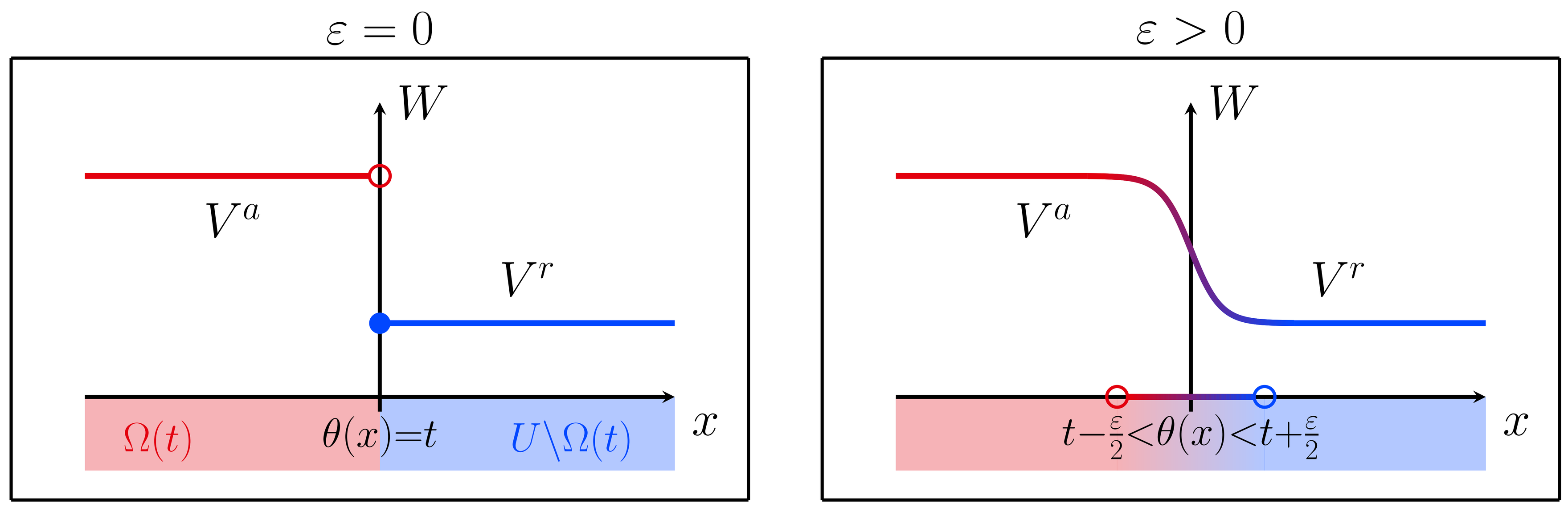
Assumptions: growth

- $\gamma\in C^{0,1}(\mathbb{R}^d\times \mathrm{GL}_+(d))$ with $0<c_\gamma\le\gamma(\cdot)\le C_\gamma$



Assumptions: viscoelasticity

- $W_\varepsilon(\sigma,F):=(1-h_\varepsilon(\sigma))V^a(F)+h_\varepsilon(\sigma)V^r(F)+V^J(F)$
  - $V^a,\ V^r,\ V^J\in C^1(\mathrm{GL}_+(d);[0,\infty))$
  - $V^a(F),\ V^r(F)\ge c_W|F|^p-\frac{1}{c_W}$
  - $V^a(F)-V^r(F)\le\frac{1}{c_W}(1+|F|^p)$
  - $\exists q>\frac{pd}{p-d}$: $V^J(F)\ge\frac{c_W}{(\det F)^q}$



- $R_\varepsilon(\sigma,F,\dot F):=(1-h_\varepsilon(\sigma))R^a(F,\dot F)+h_\varepsilon(\sigma)R^r(F,\dot F)$
  - $R^i(F,\dot F):=\frac12\dot C{:}\mathbb{D}^i(C){:}\dot C,\quad i=a,r,\ C:=F^\top F,\ \dot C:=\dot F^\top F+F^\top\dot F$
- $H(G)=|G|^p,\quad p>d$
- $f\in W^{1,\infty}(\mathbb{R};L^2(U;\mathbb{R}^d))$

Theorem (Existence and energy equality) [1]

For all given $\varepsilon\ge0$ there exists a weak/viscosity solution $(y_\varepsilon,\theta_\varepsilon)$ and $(y_\varepsilon,\theta_\varepsilon)\to(y_0,\theta_0)$ uniformly as $\varepsilon\to0$.

Moreover, in the diffused-interface case $\varepsilon>0$, (y,θ) fulfills for all $t\in[0,T]$

$$\int_U\big(W_\varepsilon(\theta-t,\nabla y)+H(\nabla^2 y)-f(\theta-t)\cdot y\big)-\big(W_\varepsilon(\theta,\nabla y_0)+H(\nabla^2 y_0)-f(\theta)\cdot y_0\big)\mathrm{d}x$$
$$=-\int_0^t\!\!\int_U 2R_\varepsilon(\theta-s,\nabla y,\nabla\dot y)+\dot f(\theta-s)\cdot y\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\!\!\int_U\partial_\sigma W_\varepsilon(\theta-s,\nabla y)\,\mathrm{d}x\,\mathrm{d}s.$$

In the sharp-interface case $\varepsilon=0$, for all $t\in[0,T]$,

$$\int_U\big(W_0(\theta-t,\nabla y)+H(\nabla^2 y)-f(\theta-t)\cdot y\big)-\big(W_0(\theta,\nabla y_0)+H(\nabla^2 y_0)-f(\theta)\cdot y_0\big)\mathrm{d}x$$
$$=-\int_0^t\!\!\int_U 2R_0(\theta-s,\nabla y,\nabla\dot y)+\dot f(\theta-s)\cdot y\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\!\!\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathscr{H}^{d-1}\mathrm{d}s.$$

What comes next?

- Backstrain
- One phase problem
- Temperature
- Different growth models

References

[1] **A. Chiesa, U. Stefanelli. Viscoelasticity and accretive phase-change at finite strains.** ***Z. Angew. Math. Phys.*** **76, 53 (2025)**

[2] E. Davoli, K. Nik, U. Stefanelli, G. Tomassetti. An existence result for accretive growth in elastic solids. *Math. Models Methods Appl.* 34 (2024), no. 11, 2169–2190.

[3] G. Zurlo, L. Truskinovsky. Inelastic surface growth. *Mech. Res. Comm.* 93 (2018), 174–179.

Acknowledgements and contact information

This research was funded in whole or in part by the Austrian Science Fund (FWF) projects 10.55776/F65, 10.55776/I5149, 10.55776/P32788, and 10.55776/I4354, as well as by the OeAD-WTZ project CZ 09/2023.

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FWF Austrian Science Fund

SFB PDE Taming Complexity in Partial Differential Systems