Convergence of thresholding energies for anisotropic MCF on inhomogeneous obstacle

Andrea Chiesa

Joint work with Karel Švadlenka
(Tokyo Metropolitan University)

Applied Analysis Seminar - Universität Heidelberg

universität wien

Vienna School of Mathematics

SFB PDE Taming Complexity in Partial Differential Systems

京都大学 KYOTO UNIVERSITY

Outline

- Setting and model derivation
- The MBO scheme and the approximate energies
- The Γ-convergence result

Evolution by mean curvature flow

MCF of $(d-1)$-dimensional manifolds $(M_t)_t \subset \mathbb{R}^d$

$$\underbrace{V}_{\text{normal velocity}} = - \underbrace{H}_{\text{mean curvature}}$$

MCF of $(d-1)$-dimensional manifolds $(M_t)_t \subset \mathbb{R}^d$

$$\underbrace{V}_{\text{normal velocity}} = -\underbrace{H}_{\text{mean curvature}}$$

L^2-gradient flow of the area functional

$$\langle \mathrm{d}E, V\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{M_t} \mathrm{d}\mathcal{H}^{d-1}\right) = -\int_{M_t} HV \,\mathrm{d}\mathcal{H}^{d-1}$$

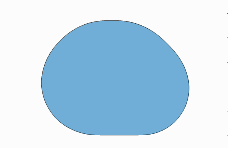
MCF of $(d-1)$-dimensional manifolds $(M_t)_t \subset \mathbb{R}^d$

$$\underbrace{V}_{\text{normal velocity}} = - \underbrace{H}_{\text{mean curvature}}$$

L^2-gradient flow of the area functional

$$\langle \mathrm{d}E, V\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{M_t} |\nu| \,\mathrm{d}\mathcal{H}^{d-1}\right) = -\int_{M_t} HV \,\mathrm{d}\mathcal{H}^{d-1}$$

isotropic
homogeneous



$\psi = |\nu|$

Evolution by mean curvature flow

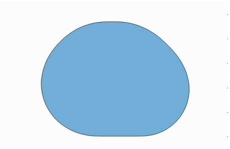
MCF of $(d-1)$-dimensional manifolds $(M_t)_t \subset \mathbb{R}^d$

$$\underbrace{V}_{\text{normal velocity}} = - \underbrace{H}_{\text{mean curvature}}$$

L^2-gradient flow of the area functional

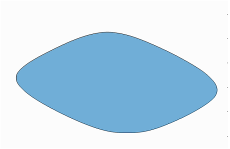
$$\langle \mathrm{d}E, V \rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{M_t} \psi(\nu)\, \mathrm{d}\mathcal{H}^{d-1} \right) = -\int_{M_t} H_\psi V \, \mathrm{d}\mathcal{H}^{d-1}$$

isotropic
homogeneous



$\psi = |\nu|$

anisotropic
homogeneous



$\psi = \psi_\nu = \max\{2|\nu_x|, |\nu_y|\}$

Evolution by mean curvature flow

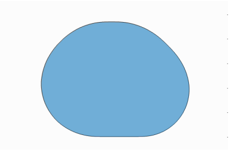
MCF of $(d-1)$-dimensional manifolds $(M_t)_t \subset \mathbb{R}^d$

$$\underbrace{V}_{\text{normal velocity}} = - \underbrace{H}_{\text{mean curvature}}$$

L^2-gradient flow of the area functional

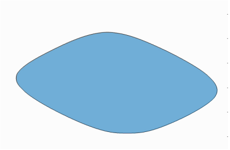
$$\langle \mathrm{d}E, V\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{M_t} \psi(x,\nu)\,\mathrm{d}\mathcal{H}^{d-1}\right) = -\int_{M_t} H_\psi V\,\mathrm{d}\mathcal{H}^{d-1}$$

isotropic homogeneous



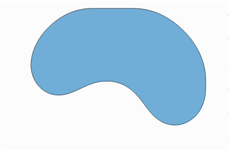
$\psi = |\nu|$

anisotropic homogeneous



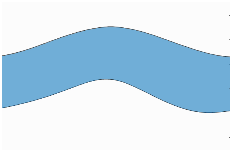
$\psi = \psi_\nu = \max\{2|\nu_x|, |\nu_y|\}$

isotropic inhomogeneous



$\psi = \psi_x = 2 - e^{-x^2-y^2}$

anisotropic inhomogeneous



$\psi = \psi_x \psi_\nu$

Evolution by mean curvature flow

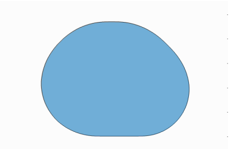
MCF of $(d-1)$-dimensional manifolds $(M_t)_t \subset \mathbb{R}^d$

$$\underbrace{V}_{\text{normal velocity}} = -\underbrace{H}_{\text{mean curvature}} + \underbrace{\Lambda}_{\text{volume preservation}}$$

L^2-gradient flow of the area functional

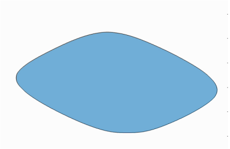
$$\langle \mathrm{d}E, V\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{M_t} \psi(x,\nu)\,\mathrm{d}\mathcal{H}^{d-1}\right) = -\int_{M_t} H_\psi V\,\mathrm{d}\mathcal{H}^{d-1}$$

isotropic homogeneous



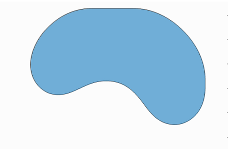
$\psi = |\nu|$

anisotropic homogeneous



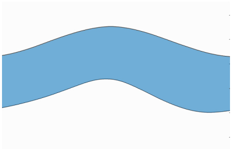
$\psi = \psi_\nu = \max\{2|\nu_x|, |\nu_y|\}$

isotropic inhomogeneous



$\psi = \psi_x = 2 - e^{-x^2-y^2}$

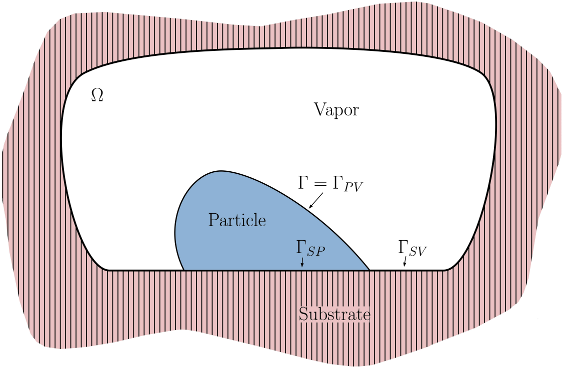
anisotropic inhomogeneous



$\psi = \psi_x \psi_\nu$

Setting and model derivation

What's our problem?



2 phases with obstacle:

- Particle P
- Vapor V

- Fixed substrate S
- $\Omega := \mathbb{R}^d \setminus S$ bounded

$$E(P) = \int_\Gamma \psi \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial P \cap \partial S} \gamma_{SP} \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial V \cap \partial S} \gamma_{SV} \, \mathrm{d}\mathcal{H}^{d-1}$$

$$E(P) = \int_\Gamma \psi \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial P \cap \partial S} \sigma \, \mathrm{d}\mathcal{H}^{d-1} \qquad \sigma := \gamma_{SP} - \gamma_{SV}$$

First variation of the energy

- $\psi = \psi(x,\nu) \in C^2(\overline{\Omega}\times\mathbb{R}^d\setminus\{0\})$, $\psi^2(x,\cdot)$ is convex, 1-homogeneous, and such that $c_\psi|\nu| \le \psi(x,\nu) \le C_\psi|\nu|$ [Bellettini-Paolini, ‘96]
- $\sigma = \sigma(x) \in C^1(\mathbb{R}^d)$

$$E(P) = \int_\Gamma \psi \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial P\cap\partial S} \sigma \,\mathrm{d}\mathcal{H}^{d-1}$$

First variation of the energy

- $\psi = \psi(x,\nu) \in C^2(\overline{\Omega}\times\mathbb{R}^d\setminus\{0\})$, $\psi^2(x,\cdot)$ is convex, 1-homogeneous, and such that $c_\psi|\nu| \le \psi(x,\nu) \le C_\psi|\nu|$ [Bellettini-Paolini, '96]
- $\sigma = \sigma(x) \in C^1(\mathbb{R}^d)$

$$E(P) = \int_\Gamma \psi \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial P\cap\partial S} \sigma \,\mathrm{d}\mathcal{H}^{d-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E(P_t)\Big|_{t=0} = \int_\Gamma H_\psi \,\xi\cdot\nu \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Gamma} \psi\,\xi_\tau\cdot b_P \,\mathrm{d}\mathcal{H}^{d-2} + \int_{\partial\Gamma} \sigma\,\xi_\tau\cdot b_S \,\mathrm{d}\mathcal{H}^{d-2}$$

where H_ψ is the anisotropic mean curvature

$$H_\psi = \operatorname{div}(\nabla_\nu \psi)$$

and ξ_τ is the tangential component of ξ, i.e., $\xi_\tau\cdot\nu = 0$

First variation of the energy

- $\psi = |\nu| \in C^2(\overline{\Omega}\times\mathbb{R}^d\setminus\{0\})$, $\psi^2(x,\cdot)$ is convex, 1-homogeneous, and such that $c_\psi|\nu| \le \psi(x,\nu) \le C_\psi|\nu|$ [Bellettini-Paolini, '96]
- $\sigma = \sigma(x) \in C^1(\mathbb{R}^d)$

$$\boxed{E(P) = \int_\Gamma \psi \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial P\cap\partial S} \sigma \,\mathrm{d}\mathcal{H}^{d-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E(P_t)\Big|_{t=0} = \int_\Gamma H_\psi\, \xi\cdot\nu \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Gamma} \psi\, \xi_\tau\cdot b_P \,\mathrm{d}\mathcal{H}^{d-2} + \int_{\partial\Gamma} \sigma\, \xi_\tau\cdot b_S \,\mathrm{d}\mathcal{H}^{d-2}$$

where H_ψ is the isotropic mean curvature

$$H_\psi = \mathrm{div}(\nabla_\nu\psi) = \mathrm{div}(\nu/|\nu|) = H$$

and ξ_τ is the tangential component of ξ, i.e., $\xi_\tau\cdot\nu = 0$

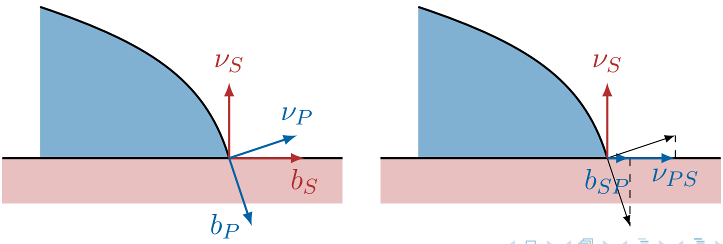
Gradient flow equations

Corresponding (volume preserving) gradient flow equation

$$V = \begin{cases} -(H_\psi - \Lambda) & \text{on } \Gamma \\ 0 & \text{on } \partial P \cap \partial S \end{cases}$$

Herring angle condition

$$\psi\, b_{PS} - (\nabla_\nu \psi \cdot b_P)\nu_{PS} + \sigma\, b_S = 0 \quad \text{on } \partial\Gamma$$



The Herring angle condition

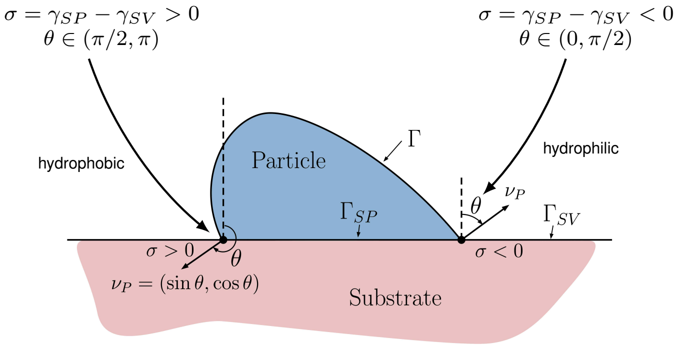
$$\psi\, b_{PS} - (\nabla_\nu \psi \cdot b_P)\nu_{PS} + \sigma\, b_S = 0 \quad \xRightarrow{\psi(x,\nu)=\psi(x,\theta)} \quad \psi \cos\theta - \psi_\theta \sin\theta + \sigma = 0$$

The Herring angle condition

$$\psi\, b_{PS} - (\nabla_\nu \psi \cdot b_P)\nu_{PS} + \sigma\, b_S = 0 \quad \xRightarrow{\psi(x,\nu)=\psi(x,\theta)} \quad \psi \cos\theta - \psi_\theta \sin\theta + \sigma = 0$$

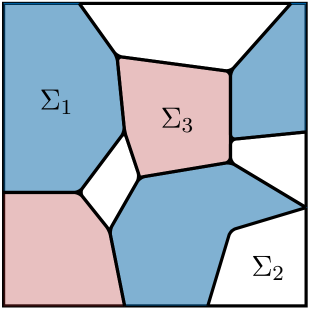
Isotropic cases (classical Young's equation)

$$\psi \cos\theta + \sigma = 0$$



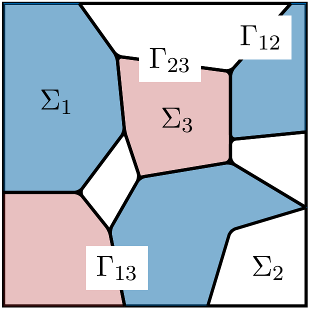
The MBO scheme and the approximate energies

The classical MBO scheme

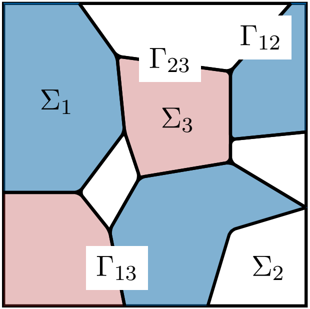


- 3 phases $(\Sigma_i)_{i=1}^3$ with $\mathbb{T}^d = \cup_{i=1}^3 \overline{\Sigma_i}$

The classical MBO scheme



- 3 phases $(\Sigma_i)_{i=1}^3$ with $\mathbb{T}^d = \cup_{i=1}^3 \overline{\Sigma_i}$
- Interfaces $\Gamma_{ij} := \partial\Sigma_i \cap \partial\Sigma_j$
- Surface densities $(\gamma_{ij})_{i,j=1}^3 \in \mathbb{R}^{3\times 3}$ with $\gamma_{ii} = 0$ and $\gamma_{ij} = \gamma_{ji}$

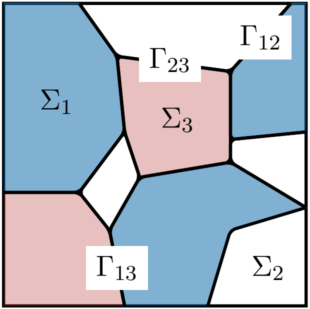


- 3 phases $(\Sigma_i)_{i=1}^3$ with $\mathbb{T}^d = \cup_{i=1}^3 \overline{\Sigma_i}$
- Interfaces $\Gamma_{ij} := \partial\Sigma_i \cap \partial\Sigma_j$
- Surface densities $(\gamma_{ij})_{i,j=1}^3 \in \mathbb{R}^{3\times 3}$ with $\gamma_{ii} = 0$ and $\gamma_{ij} = \gamma_{ji}$

$$E(\mathbb{1}_{\Sigma_1}, \mathbb{1}_{\Sigma_2}, \mathbb{1}_{\Sigma_3}) := \frac{1}{2}\sum_{i,j=1}^{3}\int_{\Gamma_{ij}} \gamma_{ij}\, \mathrm{d}\mathcal{H}^{d-1}$$

$$V_{ij} = -H_{ij} \quad \text{on } \Gamma_{ij}$$

The classical MBO scheme



- 3 phases $(\Sigma_i)_{i=1}^3$ with $\mathbb{T}^d = \cup_{i=1}^3 \overline{\Sigma_i}$
- Interfaces $\Gamma_{ij} := \partial\Sigma_i \cap \partial\Sigma_j$
- Surface densities $(\gamma_{ij})_{i,j=1}^3 \in \mathbb{R}^{3\times 3}$ with $\gamma_{ii} = 0$ and $\gamma_{ij} = \gamma_{ji}$

$$E(\mathbb{1}_{\Sigma_1}, \mathbb{1}_{\Sigma_2}, \mathbb{1}_{\Sigma_3}) := \frac{1}{2}\sum_{i,j=1}^{3} \int_{\Gamma_{ij}} \gamma_{ij}\, \mathrm{d}\mathcal{H}^{d-1}$$

$$V_{ij} = -H_{ij} \quad \text{on } \Gamma_{ij}$$

Minimizing movements scheme

Time-discretization $0 = t_0 < \cdots < t_k = kh < \cdots < N_h h = T$

$$u^k = (u_1^k, u_2^k, u_3^k) \in \operatorname*{arg\,min}_{u \in BV(\mathbb{T}^d;[0,1]^3)} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, u^{k-1}) \mid \sum_{i=1}^{3} u_i^k = 1 \right\}$$

Minimizing movements scheme

$$u^k = \underbrace{(u_1^k,u_2^k,u_3^k)}_{(\mathbb{1}^k_{\Sigma_1},\mathbb{1}^k_{\Sigma_2},\mathbb{1}^k_{\Sigma_3})} \in \operatorname*{arg\,min}_{u\in BV(\mathbb{T}^d;[0,1]^3)} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u,u^{k-1}) \;\middle|\; \sum_{i=1}^3 u_i^k = 1 \right\}$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^3 \int_{\mathbb{T}^d} \gamma_{ij}\, u_i G_h * u_j \,\mathrm{d}x \qquad \frac{1}{2h} d_h^2(u,u^{k-1}) := -E_h(u-u^{k-1})$$

$$G_h(x) = \sqrt{\pi}(4\pi h)^{-d/2} e^{-|x|^2/4h}$$

The classical MBO scheme

Minimizing movements scheme

$$u^k = \underbrace{(u_1^k, u_2^k, u_3^k)}_{(\mathbb{1}_{\Sigma_1}^k, \mathbb{1}_{\Sigma_2}^k, \mathbb{1}_{\Sigma_3}^k)} \in \underset{u \in BV(\mathbb{T}^d;[0,1]^3)}{\arg\min} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, u^{k-1}) \mid \sum_{i=1}^{3} u_i^k = 1 \right\}$$

MBO scheme

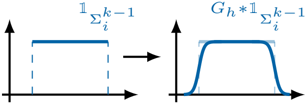
The classical MBO scheme

Minimizing movements scheme

$$u^k = \underbrace{(u_1^k, u_2^k, u_3^k)}_{(\mathbb{1}_{\Sigma_1}^k, \mathbb{1}_{\Sigma_2}^k, \mathbb{1}_{\Sigma_3}^k)} \in \underset{u \in BV(\mathbb{T}^d;[0,1]^3)}{\arg\min} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, u^{k-1}) \mid \sum_{i=1}^{3} u_i^k = 1 \right\}$$

MBO scheme

1. *convolution*: $\Psi_i^{k-1} := G_h * \left(\sum_{j=1}^{3} \gamma_{ij} \mathbb{1}_{\Sigma_j^{k-1}} \right)$



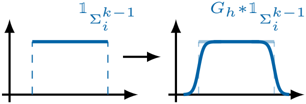
The classical MBO scheme

Minimizing movements scheme

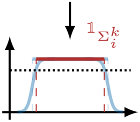
$$u^k = \underbrace{(u_1^k, u_2^k, u_3^k)}_{(\mathbb{1}_{\Sigma_1}^k, \mathbb{1}_{\Sigma_2}^k, \mathbb{1}_{\Sigma_3}^k)} \in \underset{u \in BV(\mathbb{T}^d;[0,1]^3)}{\arg\min} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, u^{k-1}) \mid \sum_{i=1}^{3} u_i^k = 1 \right\}$$

MBO scheme

1. *convolution*: $\Psi_i^{k-1} := G_h * \left(\sum_{j=1}^{3} \gamma_{ij} \mathbb{1}_{\Sigma_j^{k-1}} \right)$



2. *thresholding*: $\Sigma_i^k := \left\{ x \in \mathbb{T}^d \mid \Psi_i^{k-1}(x) < \min_{j \neq i} \Psi_j^{k-1}(x) \right\}$



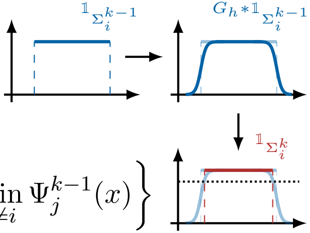
The classical MBO scheme

Minimizing movements scheme

$$u^k = \underbrace{(u_1^k, u_2^k, u_3^k)}_{(\mathbb{1}_{\Sigma_1}^k, \mathbb{1}_{\Sigma_2}^k, \mathbb{1}_{\Sigma_3}^k)} \in \operatorname*{arg\,min}_{u \in BV(\mathbb{T}^d;[0,1]^3)} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, u^{k-1}) \;\middle|\; \sum_{i=1}^{3} u_i^k = 1 \right\}$$

MBO scheme

1. *convolution*: $\Psi_i^{k-1} := G_h * \left(\sum_{j=1}^{3} \gamma_{ij} \mathbb{1}_{\Sigma_j^{k-1}} \right)$
2. *thresholding*: $\Sigma_i^k := \left\{ x \in \mathbb{T}^d \mid \Psi_i^{k-1}(x) < \min_{j \neq i} \Psi_j^{k-1}(x) \right\}$



$$u^k = (\mathbb{1}_{\Sigma_1^k}, \mathbb{1}_{\Sigma_2^k}, \mathbb{1}_{\Sigma_3^k}) \in BV(\mathbb{T}^d; \{0,1\}^3)$$

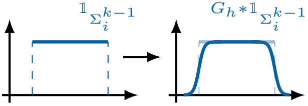
The classical MBO scheme

Minimizing movements scheme

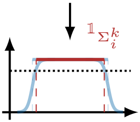
$$u^k = \underbrace{(u_1^k, u_2^k, u_3^k)}_{(\mathbb{1}_{\Sigma_1}^k, \mathbb{1}_{\Sigma_2}^k, \mathbb{1}_{\Sigma_3}^k)} \in \underset{u \in BV(\mathbb{T}^d;[0,1]^3)}{\arg\min} \left\{ E_h(u) + \frac{1}{2h} d_h^2(u, u^{k-1}) \mid \sum_{i=1}^{3} u_i^k = 1 \right\}$$

MBO scheme [Laux-Otto, '16]

1. *convolution*: $\Psi_i^{k-1} := G_h * \left(\sum_{j=1}^{3} \gamma_{ij} \mathbb{1}_{\Sigma_j^{k-1}} \right)$



2. *thresholding*: $\Sigma_i^k := \left\{ x \in \mathbb{T}^d \mid \Psi_i^{k-1}(x) < \min_{j \neq i} \Psi_j^{k-1}(x) \right\}$



$$u^k = (\mathbb{1}_{\Sigma_1^k}, \mathbb{1}_{\Sigma_2^k}, \mathbb{1}_{\Sigma_3^k}) \in BV(\mathbb{T}^d; \{0,1\}^3)$$

The classical approximate energy

$$E(u) \coloneqq \frac{1}{2}\sum_{i,j=1}^{3}\int_{\Gamma_{ij}} \gamma_{ij}\,\mathrm{d}\mathcal{H}^{d-1} = \frac{1}{2}\sum_{i,j=1}^{3}\int_{\mathbb{T}^d} \gamma_{ij}\left(|\nabla u_i|+|\nabla u_j|-|\nabla(u_i+u_j)|\right)$$

$$E_h(u) \coloneqq \frac{1}{2\sqrt{h}}\sum_{i,j=1}^{3}\int_{\mathbb{T}^d} \gamma_{ij}\, u_i G_h * u_j\,\mathrm{d}x$$

The classical approximate energy

$$E(u) := \frac{1}{2} \sum_{i,j=1}^{3} \int_{\Gamma_{ij}} \gamma_{ij} \, \mathrm{d}\mathcal{H}^{d-1}$$

$\Gamma(\mathrm{s}\text{-}L^1(\mathbb{T}^d))$ $\Big\uparrow$ [Esedoḡlu-Otto, '15]

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^{3} \int_{\mathbb{T}^d} \gamma_{ij} \, u_i G_h * u_j \, \mathrm{d}x$$

Triangular inequalities

$\gamma_{ij} \leq \gamma_{ik} + \gamma_{kj}$ for every $i, j, k \in \{1, 2, 3\}$

The approximate energy

$$E(u) := \frac{1}{2}\sum_{i,j=1}^{3}\int_{\Gamma_{ij}} \gamma_{ij}\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1}$$

$$\Gamma(\mathrm{s}\text{-}L^1(\mathbb{T}^d)) \Big\uparrow \; ?$$

$$E_h(u) := \frac{1}{2\sqrt{h}}\sum_{i,j=1}^{3}\int_{\mathbb{T}^d} \gamma_{ij}\, u_i K_h * u_j \,\mathrm{d}x$$

Triangular inequalities

$\gamma_{ij} \leq \gamma_{ik} + \gamma_{kj}$ for every $i, j, k \in \{1, 2, 3\}$

The approximate energy

$$E(u) \coloneqq \frac{1}{2}\sum_{i,j=1}^{3}\int_{\Gamma_{ij}} \gamma_{ij}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1}$$

$$\Gamma(\mathrm{s}\!-\!L^1(\mathbb{T}^d)) \Big\uparrow \ ?$$

$$E_h(u) \coloneqq \frac{1}{2\sqrt{h}}\sum_{i,j=1}^{3}\int_{\mathbb{T}^d} \gamma_{ij}(x)\,u_i K_h * u_j\,\mathrm{d}x$$

Triangular inequalities

$\gamma_{ij}(x) \leq \gamma_{ik}(x) + \gamma_{kj}(x)$ for every $i, j, k \in \{1, 2, 3\}$, $x \in \mathbb{T}^d$

$$E(u) := \int_{\Gamma} \psi(x,\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\,\mathrm{d}\mathcal{H}^{d-1}$$

$$\Gamma(\mathrm{s}\text{–}L^1(\mathbb{T}^d)) \Big\uparrow \ ?$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^{3} \int_{\mathbb{T}^d} \gamma_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$

Triangular inequalities

$\gamma_{ij}(x) \leq \gamma_{ik}(x) + \gamma_{kj}(x)$ for every $i, j, k \in \{1,2,3\}$, $x \in \mathbb{T}^d$

$$E(u) := \int_{\Gamma} \gamma_{PV}(x)\, \gamma(\nu)\, d\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\, d\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\, d\mathcal{H}^{d-1}$$

$\Gamma(\text{s}-L^1(\mathbb{T}^d))$ $\uparrow$?

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^{3} \int_{\mathbb{T}^d} \gamma_{ij}(x)\, u_i K_h * u_j \, dx$$

Triangular inequalities

$\gamma_{ij}(x) \leq \gamma_{ik}(x) + \gamma_{kj}(x)$ for every $i, j, k \in \{1, 2, 3\}$, $x \in \mathbb{T}^d$

The approximate energy: Obstacle

$$E(u) := \int_{\Gamma} \gamma_{PV}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1}$$

$$\Gamma(\mathrm{s}\text{–}L^1(\mathbb{T}^d)) \Big\uparrow \ ?$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^{3} \int_{\mathbb{T}^d} \gamma_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$

Triangular inequalities

$\gamma_{ij}(x) \le \gamma_{ik}(x) + \gamma_{kj}(x)$ for every $i, j, k \in \{1, 2, 3\}$, $x \in \mathbb{T}^d$

The approximate energy: Obstacle

$$E(u) := \int_{\Gamma} \gamma_{PV}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\,\cancel{\gamma(\nu)}\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\,\cancel{\gamma(\nu)}\,\mathrm{d}\mathcal{H}^{d-1}$$

$\Gamma(\mathrm{s}\text{-}L^1(\mathbb{T}^d))$ $\uparrow$ [C.-Švadlenka, '25]

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^{3} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$

Triangular inequalities

$\widetilde{\gamma}_{ij}(x) \leq \widetilde{\gamma}_{ik}(x) + \widetilde{\gamma}_{kj}(x)$ for every $i, j, k \in \{1, 2, 3\}$, $x \in \mathbb{T}^d$

Surface tensions: Idea

$$E(u) := \int_\Gamma \gamma_{PV}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \frac{\gamma_{SP}(x)}{\gamma(\nu)}\gamma(\nu)\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \frac{\gamma_{SV}(x)}{\gamma(\nu)}\gamma(\nu)\mathrm{d}\mathcal{H}^{d-1}$$

$\Gamma(\text{s}-L^1(\mathbb{T}^d))$ $\uparrow$ [C.-Švadlenka, '25]

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j=1}^{3} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$

Idea

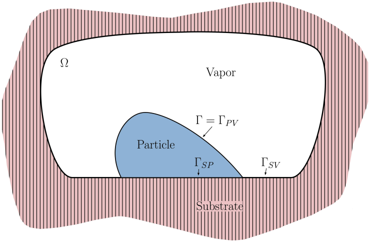
$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

Surface tensions: Assumptions

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E(u) = \int_\Gamma \gamma(\nu)\, \gamma_{PV}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\, \mathrm{d}\mathcal{H}^{d-1}$$



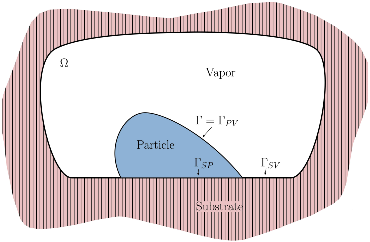
- $\gamma = \gamma(\nu) \in C^2(\mathbb{R}^d \setminus \{0\})$, γ^2 is convex, 1-homogeneous, $c_\gamma |\nu| \le \gamma(\nu) \le C_\gamma |\nu|$ [Bellettini-Paolini, '96]

Surface tensions: Assumptions

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E(u) = \int_\Gamma \gamma(\nu)\, \gamma_{PV}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\, \mathrm{d}\mathcal{H}^{d-1}$$



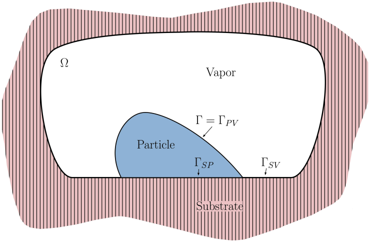
- $\gamma = \gamma(\nu) \in C^2(\mathbb{R}^d \setminus \{0\})$, γ^2 is convex, 1-homogeneous, $c_\gamma|\nu| \leq \gamma(\nu) \leq C_\gamma|\nu|$ [Bellettini-Paolini, '96]
- $\gamma_{PV} \in C^1(\overline{\Omega})$, $\gamma_{SP}, \gamma_{SV} \in C^1(\partial S)$ with $c \leq \gamma_{PV}, \gamma_{SP}, \gamma_{SV} \leq C$

Surface tensions: Assumptions

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E(u) = \int_\Gamma \gamma(\nu)\, \gamma_{PV}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\, \mathrm{d}\mathcal{H}^{d-1}$$



- $\gamma = \gamma(\nu) \in C^2(\mathbb{R}^d \backslash \{0\})$, γ^2 is convex, 1-homogeneous, $c_\gamma|\nu| \le \gamma(\nu) \le C_\gamma|\nu|$ [Bellettini-Paolini, '96]
- $\gamma_{PV} \in C^1(\overline{\Omega})$, $\gamma_{SP}, \gamma_{SV} \in C^1(\partial S)$ with $c \le \gamma_{PV}, \gamma_{SP}, \gamma_{SV} \le C$

$$\gamma\, \gamma_{PV} < \gamma_{SP} + \gamma_{SV} \quad \gamma_{SP} < \gamma\, \gamma_{PV} + \gamma_{SV} \quad \gamma_{SV} < \gamma\, \gamma_{PV} + \gamma_{SP}$$

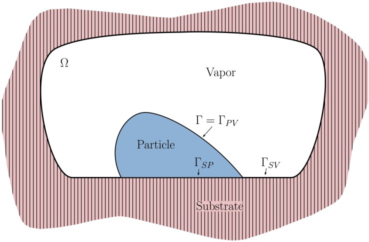
$$x \in \partial S,\ \nu \in \mathbb{S}^{d-1}$$

Surface tensions: Assumptions

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E(u) = \int_{\Gamma} \gamma(\nu)\, \gamma_{PV}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\, \mathrm{d}\mathcal{H}^{d-1}$$



- $\gamma = \gamma(\nu) \in C^2(\mathbb{R}^d \setminus \{0\})$, γ^2 is convex, 1-homogeneous, $c_\gamma|\nu| \le \gamma(\nu) \le C_\gamma|\nu|$ [Bellettini-Paolini, '96]
- $\gamma_{PV} \in C^1(\overline{\Omega})$, $\gamma_{SP}, \gamma_{SV} \in C^1(\partial S)$ with $c \le \gamma_{PV}, \gamma_{SP}, \gamma_{SV} \le C$

$$C_\gamma\, \gamma_{PV} < \gamma_{SP} + \gamma_{SV} \quad \gamma_{SP} < c_\gamma\, \gamma_{PV} + \gamma_{SV} \quad \gamma_{SV} < c_\gamma\, \gamma_{PV} + \gamma_{SP}$$

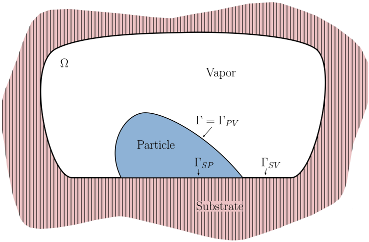
$$x \in \partial S$$

Surface tensions: Construction

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j \in \{P,V,S\}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$



- $\gamma_{PV} \in C^1(\overline{\Omega})$, $\gamma_{SP}, \gamma_{SV} \in C^1(\partial S)$ with $c \leq \gamma_{PV}, \gamma_{SP}, \gamma_{SV} \leq C$

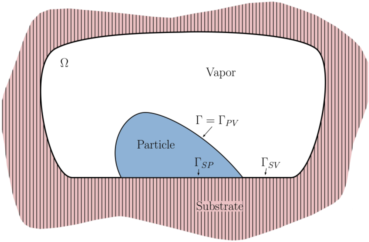
$$C_\gamma\, \gamma_{PV} < \gamma_{SP} + \gamma_{SV} \qquad \gamma_{SP} < c_\gamma\, \gamma_{PV} + \gamma_{SV} \qquad \gamma_{SV} < c_\gamma\, \gamma_{PV} + \gamma_{SP}$$

$$x \in \partial S$$

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\nu_S(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\nu_S(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j \in \{P,V,S\}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \, \mathrm{d}x$$



- $\widetilde{\gamma}_{PV} \in C^1(\mathbb{T}^d)$, $\gamma_{SP}, \gamma_{SV} \in C^1(\partial S)$ with $c \leq \widetilde{\gamma}_{PV}, \gamma_{SP}, \gamma_{SV} \leq C$

$$\begin{cases} \Delta \widetilde{\gamma}_{PV} = 0 & \text{in } \mathbb{T}^d \setminus \Omega \\ \widetilde{\gamma}_{PV} = \gamma_{PV} & \text{on } \partial\Omega = \partial S \end{cases}$$

$$C_\gamma \widetilde{\gamma}_{PV} < \gamma_{SP} + \gamma_{SV} \quad \gamma_{SP} < c_\gamma \widetilde{\gamma}_{PV} + \gamma_{SV} \quad \gamma_{SV} < c_\gamma \widetilde{\gamma}_{PV} + \gamma_{SP}$$

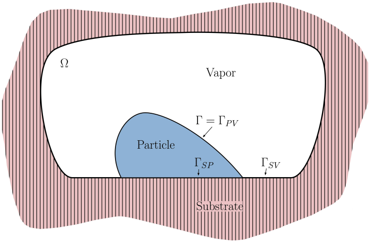
$$x \in \partial S$$

Surface tensions: Construction

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\gamma_{SP}(x)}{\gamma(\widehat{\nu}(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\gamma_{SV}(x)}{\gamma(\widehat{\nu}(x))}, \quad x \in \partial S = \Gamma_{SP} \cup \Gamma_{SV}$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j \in \{P,V,S\}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$



- $\widetilde{\gamma}_{PV} \in C^1(\mathbb{T}^d)$, $\gamma_{SP}, \gamma_{SV} \in C^1(\partial S)$ with $c \leq \widetilde{\gamma}_{PV}, \gamma_{SP}, \gamma_{SV} \leq C$

$$\begin{cases} \Delta \widehat{\nu} = 0 & \text{in } B_\delta(\partial S) \\ \widehat{\nu} = \nu_S & \text{on } \partial\Omega = \partial S \end{cases}$$

$$C_\gamma\, \widetilde{\gamma}_{PV} < \gamma_{SP} + \gamma_{SV} \quad \gamma_{SP} < c_\gamma\, \widetilde{\gamma}_{PV} + \gamma_{SV} \quad \gamma_{SV} < c_\gamma\, \widetilde{\gamma}_{PV} + \gamma_{SP}$$

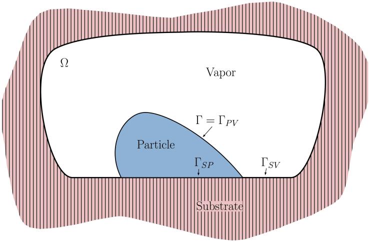
$$x \in \partial S$$

Surface tensions: Construction

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\widehat{\gamma}_{SP}(x)}{\gamma(\widehat{\nu}(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\widehat{\gamma}_{SV}(x)}{\gamma(\widehat{\nu}(x))}, \quad x \in \mathbb{T}^d$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j\in\{P,V,S\}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \,\mathrm{d}x$$



- $\widetilde{\gamma}_{PV} \in C^1(\mathbb{T}^d)$, $\widetilde{\gamma}_{SP}, \widetilde{\gamma}_{SV} \in C^1(\mathbb{T}^d)$ with $c \leq \widetilde{\gamma}_{PV}, \widetilde{\gamma}_{SP}, \widetilde{\gamma}_{SV} \leq C$

$$\begin{cases} \Delta\widehat{\gamma}_{SP} = 0 & \text{in } B_\delta(\partial S) \\ \widehat{\gamma}_{SP} = \gamma_{SP} & \text{on } \partial\Omega = \partial S \end{cases}$$

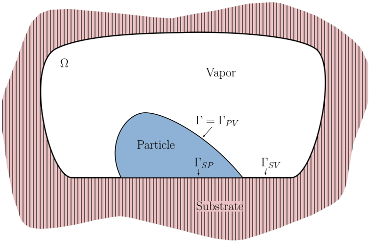
$$C_\gamma \widetilde{\gamma}_{PV} < \widetilde{\gamma}_{SP} + \widetilde{\gamma}_{SV} \quad \widetilde{\gamma}_{SP} < c_\gamma \widetilde{\gamma}_{PV} + \widetilde{\gamma}_{SV} \quad \widetilde{\gamma}_{SV} < c_\gamma \widetilde{\gamma}_{PV} + \widetilde{\gamma}_{SP}$$

$$x \in \mathbb{T}^d$$

Idea

$$\widetilde{\gamma}_{SP}(x) := \frac{\widehat{\gamma}_{SP}(x)}{\gamma(\widehat{\nu}(x))}, \quad \widetilde{\gamma}_{SV}(x) := \frac{\widehat{\gamma}_{SV}(x)}{\gamma(\widehat{\nu}(x))}, \quad x \in \mathbb{T}^d$$

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j \in \{P,V,S\}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}(x)\, u_i K_h * u_j \, \mathrm{d}x$$



- $\widetilde{\gamma}_{PV} \in C^1(\mathbb{T}^d)$, $\widetilde{\gamma}_{SP}, \widetilde{\gamma}_{SV} \in C^1(\mathbb{T}^d)$ with $c \leq \widetilde{\gamma}_{PV}, \widetilde{\gamma}_{SP}, \widetilde{\gamma}_{SV} \leq C$

$$C_\gamma\, \widetilde{\gamma}_{PV} \leq \widetilde{\gamma}_{SP} + \widetilde{\gamma}_{SV} \quad \widetilde{\gamma}_{SP} \leq c_\gamma\, \widetilde{\gamma}_{PV} + \widetilde{\gamma}_{SV} \quad \widetilde{\gamma}_{SV} \leq c_\gamma\, \widetilde{\gamma}_{PV} + \widetilde{\gamma}_{SP}$$

$$x \in \mathbb{T}^d$$

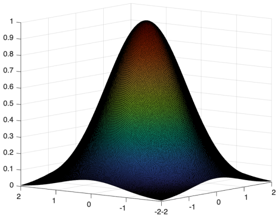
The Γ-convergence result

Kernel

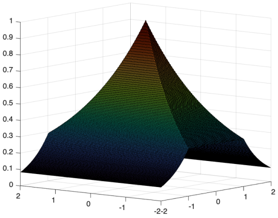
$$K(-x) = K(x), \quad K \in L^1(\mathbb{R}^d), \quad K(x) \geq 0, \quad \int_{\mathbb{R}^d} K(x)\,\mathrm{d}x = 1$$

$$\gamma(\nu) = \frac{1}{2}\int_0^{+\infty} r^d \int_{\mathbb{S}^{d-1}} |\xi \cdot \nu| K(r\xi)\,\mathrm{d}\xi\,\mathrm{d}r$$

$$|x|K(x) \leq c_K K(x/2), \qquad K \geq a \text{ in } B_b(0)$$



$\gamma(\nu)=|\nu|$



$\gamma(\nu)=\max\{2|\nu_x|,|\nu_y|\}$

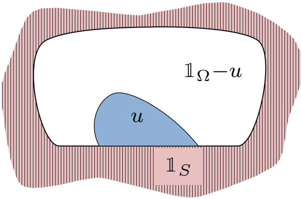
Γ-convergence result

Theorem (C.-Švadlenka, '25)

$$E_h(u) := \frac{1}{2\sqrt{h}} \sum_{i,j\in\{P,V,S\}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{ij}\, u_i K_h * u_j \,\mathrm{d}x$$

$$\Gamma(\mathrm{s}-L^1(\mathbb{T}^d)) \Big\downarrow \; h \to 0$$

$$E(u) := \int_\Gamma \gamma_{PV}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\,\mathrm{d}\mathcal{H}^{d-1}$$



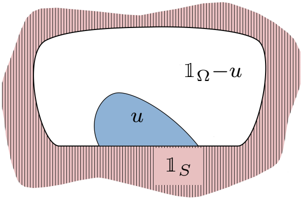
Γ-convergence result

Theorem (C.-Švadlenka, '25)

$$E_h(u) := \frac{1}{\sqrt{h}} \int_{\mathbb{T}^d} \widetilde{\gamma}_{PV}\, u K_h * (\mathbb{1}_\Omega - u) + \widetilde{\gamma}_{SP}\, u K_h * \mathbb{1}_S + \widetilde{\gamma}_{SV}\, (\mathbb{1}_\Omega - u) K_h * \mathbb{1}_S \,\mathrm{d}x$$

$$\Gamma(\mathrm{s}\text{–}L^1(\mathbb{T}^d)) \Big\downarrow \; h \to 0$$

$$E(u) := \int_\Gamma \gamma_{PV}(x)\,\gamma(\nu)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SP}} \gamma_{SP}(x)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Gamma_{SV}} \gamma_{SV}(x)\,\mathrm{d}\mathcal{H}^{d-1}$$



Proof: Limsup-inequality

$\forall u \in BV(\Omega;\{0,1\})\ \exists (u_h)_h \subset BV(\Omega;[0,1])$ with $u_h \to u$ in $L^1(\Omega)$

such that $E(u) \geq \limsup_{h\to 0} E_h(u_h)$

Proof: Limsup-inequality

$\forall u \in BV(\Omega;\{0,1\})\ \exists (u_h)_h \subset BV(\Omega;[0,1])$ with $u_h \to u$ in $L^1(\Omega)$

such that $E(u) \geq \limsup_{h\to 0} E_h(u_h)$

Lemma (Pointwise convergence)

If $u \in BV(\Omega;\{0,1\})$*, then*

$$\lim_{h\to 0} E_h(u) = E(u).$$

Proof: Limsup-inequality

$$\forall u \in BV(\Omega;\{0,1\})\ \exists (u_h)_h \subset BV(\Omega;[0,1]) \text{ with } u_h \to u \text{ in } L^1(\Omega)$$
$$\text{such that } E(u) \geq \limsup_{h\to 0} E_h(u_h)$$

Lemma (Pointwise convergence)

If $u \in BV(\Omega;\{0,1\})$, *then*

$$\lim_{h\to 0} E_h(u) = E(u).$$

$\limsup$-inequality ✓

Proof: Liminf-inequality

$\forall (u_h)_h \subset BV(\Omega; [0,1])$ with $u_h \to u$ in $L^1(\Omega)$

such that $E(u) \leq \liminf_{h\to 0} E_h(u_h)$

Proof: Liminf-inequality

$$\forall (u_h)_h \subset BV(\Omega;[0,1]) \text{ with } u_h \to u \text{ in } L^1(\Omega)$$
$$\text{such that } E(u) \le \liminf_{h\to 0} E_h(u_h)$$

Lemma (Approximate monotonicity [cf. Laux-Otto, '16])

For every $u \in BV(\Omega;[0,1])$, *every* $h > 0$, *and* $N \in \mathbb{N}_+$, *we have*

$$E_{N^2 h}(u) \le \underbrace{\left(1 + cN\sqrt{h}\right)}_{\textit{inhomogeneity}} E_h(u).$$

Proof: Liminf-inequality

$$\forall (u_h)_h \subset BV(\Omega;[0,1]) \text{ with } u_h \to u \text{ in } L^1(\Omega)$$

$$\text{such that } E(u) \le \liminf_{h\to 0} E_h(u_h)$$

Lemma (Approximate monotonicity [cf. Laux-Otto, '16])

For every $u \in BV(\Omega;[0,1])$, every $h > 0$, and $N \in \mathbb{N}_+$, we have

$$E_{N^2 h}(u) \le \underbrace{\left(1 + cN\sqrt{h}\right)}_{\textit{inhomogeneity}} E_h(u).$$

Fix h_0 and let $h \to 0$ with $N_h^2 h = h_0$ for some $N_h \in \mathbb{N}$

$$E_{N_h^2 h}(u_h) \le \left(1 + cN_h\sqrt{h}\right) E_h(u_h)$$

Proof: Liminf-inequality

$\forall (u_h)_h \subset BV(\Omega;[0,1])$ with $u_h \to u$ in $L^1(\Omega)$

such that $E(u) \leq \liminf_{h\to 0} E_h(u_h)$

Lemma (Approximate monotonicity [cf. Laux-Otto, '16])

For every $u \in BV(\Omega;[0,1])$, every $h > 0$, and $N \in \mathbb{N}_+$, we have

$$E_{N^2 h}(u) \leq \underbrace{\left(1 + cN\sqrt{h}\right)}_{\text{inhomogeneity}} E_h(u).$$

Fix h_0 and let $h \to 0$ with $N_h^2 h = h_0$ for some $N_h \in \mathbb{N}$

$$E_{h_0}(u_h) \leq \left(1 + c\sqrt{h_0}\right) E_h(u_h)$$

Proof: Liminf-inequality

$\forall (u_h)_h \subset BV(\Omega;[0,1])$ with $u_h \to u$ in $L^1(\Omega)$

such that $E(u) \leq \liminf_{h\to 0} E_h(u_h)$

Lemma (Approximate monotonicity [cf. Laux-Otto, '16])

For every $u \in BV(\Omega;[0,1])$, every $h > 0$, and $N \in \mathbb{N}_+$, we have

$$E_{N^2 h}(u) \leq \underbrace{\left(1 + cN\sqrt{h}\right)}_{\text{inhomogeneity}} E_h(u).$$

Fix h_0 and let $h \to 0$ with $N_h^2 h = h_0$ for some $N_h \in \mathbb{N}$

$$E_{h_0}(u) \leq \left(1 + c\sqrt{h_0}\right) \liminf_{h\to 0} E_h(u_h)$$

Proof: Liminf-inequality

$\forall (u_h)_h \subset BV(\Omega;[0,1])$ with $u_h \to u$ in $L^1(\Omega)$

such that $E(u) \leq \liminf_{h\to 0} E_h(u_h)$

Lemma (Approximate monotonicity [cf. Laux-Otto, '16])

For every $u \in BV(\Omega;[0,1])$, every $h > 0$, and $N \in \mathbb{N}_+$, we have

$$E_{N^2 h}(u) \leq \underbrace{\left(1 + cN\sqrt{h}\right)}_{\text{inhomogeneity}} E_h(u).$$

Fix h_0 and let $h \to 0$ with $N_h^2 h = h_0$ for some $N_h \in \mathbb{N}$

$$E_{h_0}(u) \leq \left(1 + c\sqrt{h_0}\right) \liminf_{h\to 0} E_h(u_h)$$

$\liminf$-inequality ✓

Conclusions

How is our result different from the one in [Esedoḡlu-Otto, ‘15]?

- The surface densities are inhomogeneous
- We can consider different anisotropies (obstacle)

Conclusions

How is our result different from the one in [Esedoḡlu-Otto, ‘15]?

- The surface densities are inhomogeneous
- We can consider different anisotropies (obstacle)

What’s next? Convergence of the MBO scheme (soon?)

- anisotropic particle γ (see [Laux-Otto, ‘16])
- inhomogeneous obstacle γ_{SP}, γ_{SV} (see [Laux-Otto, ‘16])
- volume preservation constraint (see [Laux-Swartz, ‘17])

Conclusions

How is our result different from the one in [Esedoḡlu-Otto, ‘15]?

- The surface densities are inhomogeneous
- We can consider different anisotropies (obstacle)

What’s next? Convergence of the MBO scheme (less soon?)

- anisotropic particle γ (see [Laux-Otto, ‘16])
- inhomogeneous obstacle γ_{SP}, γ_{SV} (see [Laux-Otto, ‘16])
- volume preservation constraint (see [Laux-Swartz, ‘17])
- and inhomogeneous particle γ_{PV}?

Conclusions

How is our result different from the one in [Esedoḡlu-Otto, ‘15]?

- The surface densities are inhomogeneous
- We can consider different anisotropies (obstacle)

What’s next? Convergence of the MBO scheme (less soon?)

- anisotropic particle γ (see [Laux-Otto, ‘16])
- inhomogeneous obstacle γ_{SP}, γ_{SV} (see [Laux-Otto, ‘16])
- volume preservation constraint (see [Laux-Swartz, ‘17])
- and inhomogeneous particle γ_{PV}?

Thank you for your attention!

Bibliography

[⋆] A. C., K. Švadlenka. Convergence of thresholding energies for anisotropic mean curvature flow on inhomogeneous obstacle. *Preprint: ArXiv:2503.20524*.

[1] G. Bellettini, M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.* 25 (1996), no. 3, pp. 537–566.

[2] S. Esedoḡlu, F. Otto. Threshold dynamics for networks with arbitrary surface tensions. *Comm. Pure Appl. Math.* 68 (2015), no. 5, pp. 808–864.

[3] T. Laux, F. Otto. Convergence of the thresholding scheme for multi-phase mean-curvature flow. *Calc. Var. Partial Differential Equations* 55 (2016), no. 5, Art. 129, 74 pp.