Viscoelasticity and accretive phase-change at finite strains

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SFB PDE Taming Complexity in Partial Differential Systems

Vienna School of Mathematics

Outline

- Model
- Assumptions
- Existence result

Model

Model

Evolution of a viscoelastic solid + phase change (accretive growth)

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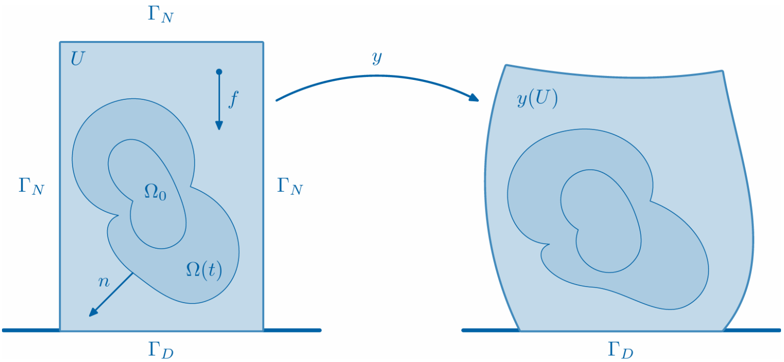
- Swelling in polymer gels
- Solidification processes
- Early tumor development



Model equations

Evolution of a viscoelastic solid

$$\begin{cases} -\operatorname{div}\left(\partial_{\nabla y} W_\varepsilon(\theta(x){-}t, \nabla y) + \partial_{\nabla \dot{y}} R_\varepsilon(\theta(x){-}t, \nabla y, \nabla \dot{y}) - \operatorname{div} \mathrm{D}H(\nabla^2 y)\right) \\ \qquad = f(\theta(x){-}t, x) \quad \text{in } [0,T] \times U, \\ \text{Boundary and initial conditions} \end{cases}$$

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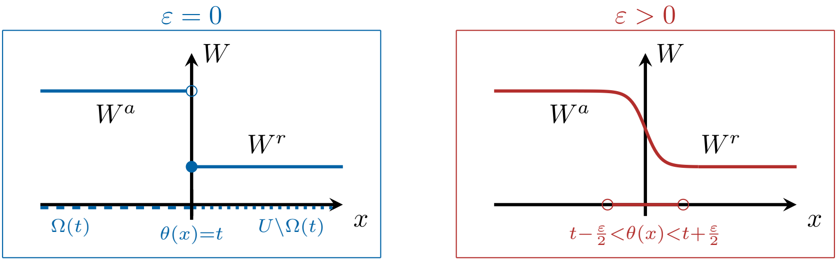

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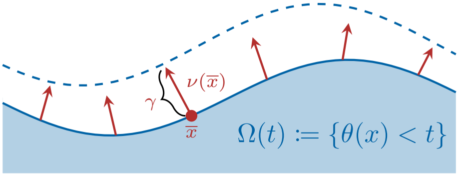

Notion of solution

Definition (Weak/viscosity solution)

$$(y,\theta) \in \big(L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))\big) \times C^{0,1}(\overline{U})$$

is a *weak/viscosity solution* to the initial-boundary-value problem if $\det \nabla y(t,\cdot) > 0$ for all $t \in (0,T)$, $y(0,\cdot) = y_0$, and

$$\int_0^T\int_U \big(\partial_F W_\varepsilon(\theta-t,\nabla y):\nabla z + \partial_{\dot F} R_\varepsilon(\theta-t,\nabla y,\nabla\dot y):\nabla z + \mathrm{D}H\big(\nabla^2 y\big)\vdots\nabla^2 z\big)\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T\int_U f(\theta-t)\cdot z\,\mathrm{d}x\mathrm{d}t \quad \forall z \in C^\infty(\overline{Q};\mathbb{R}^d) \text{ with } z = 0 \text{ on } \Sigma_D,$$

and θ is a viscosity solution to

$$\begin{cases} \gamma\big(y(\theta(x),x),\nabla y(\theta(x),x)\big)|\nabla(-\theta)(x)| = 1 \text{ in } U \setminus \overline{\Omega_0}, \\ \theta = 0 \text{ in } \Omega_0. \end{cases}$$

Assumptions

Assumptions: viscoelastic evolution

$$\int_0^T\int_U\Big(\partial_F W_\varepsilon(\theta-t,\nabla y)+\partial_{\dot F}R_\varepsilon(\theta-t,\nabla y,\nabla\dot y)\Big):\nabla z+\mathrm{D}H\big(\nabla^2 y\big)\vdots\nabla^2 z-f(\theta-t)\cdot z=0$$

$$R_\varepsilon(\sigma,F,\dot F):=(1-h_\varepsilon(\sigma))R^a(F,\dot F)+h_\varepsilon(\sigma)R^r(F,\dot F)$$

$$R^*(F,\dot F):=\frac{1}{2}\dot C:\mathbb{D}^*(C):\dot C,\quad C:=F^\top F,\ \dot C:=\dot F^\top F+F^\top\dot F$$

$$H(G)=|G|^p,\quad p>d$$

$$f\in W^{1,\infty}(\mathbb{R};L^2(U;\mathbb{R}^d))$$

Assumptions: viscoelastic evolution

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$$W_\varepsilon(\sigma,F)\coloneqq(1-h_\varepsilon(\sigma))V^a(F)+h_\varepsilon(\sigma)V^r(F)+V^J(F)$$

$$V^a,\,V^r,\,V^J\in C^1(\mathrm{GL}_+(d);[0,\infty))$$

$$V^a(F),\,V^r(F)\geq c_W|F|^p-\frac{1}{c_W}$$

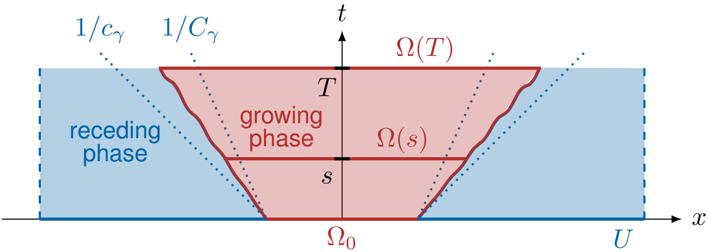
$$V^a(F)-V^r(F)\leq\frac{1}{c_W}(1+|F|^p)$$

$$\exists q>\frac{pd}{p-d}\;:\quad V^J(F)\geq\frac{c_W}{(\det F)^q}$$

Assumptions: accretive growth

$$\begin{cases} \gamma\big(y(\theta(x),x),\nabla y(\theta(x),x)\big)|\nabla(-\theta)(x)| = 1 \ \text{ in } \ U\setminus\overline{\Omega_0} \\ \theta = 0 \ \text{ in } \ \Omega_0 \subset\subset U \end{cases}$$

$$\gamma \in C^{0,1}(\mathbb{R}^d \times \mathrm{GL}_+(d)) \ \text{ with } \ 0 < c_\gamma \leq \gamma(\cdot) \leq C_\gamma$$



Existence result

Main result

Theorem (Existence and Energy equality)

For all given $\varepsilon \geq 0$ there exists a weak/viscosity solution $(y_\varepsilon, \theta_\varepsilon)$ and $(y_\varepsilon, \theta_\varepsilon) \to (y_0, \theta_0)$ as $\varepsilon \to 0$.

Main result

Theorem (Existence and Energy equality)

For all given $\varepsilon \geq 0$ there exists a weak/viscosity solution $(y_\varepsilon, \theta_\varepsilon)$ and $(y_\varepsilon, \theta_\varepsilon) \to (y_0, \theta_0)$ as $\varepsilon \to 0$.

Moreover, in the diffused-interface case $\varepsilon > 0$, (y, θ) fulfills for all $t \in [0, T]$

$$\int_U \big(W_\varepsilon(\theta - t, \nabla y) + H(\nabla^2 y) - f(\theta - t)\cdot y\big) - \big(W_\varepsilon(\theta, \nabla y_0) + H(\nabla^2 y_0) - f(\theta)\cdot y_0\big)\, \mathrm{d}x$$

$$= -\int_0^t \int_U 2R_\varepsilon(\theta - s, \nabla y, \nabla \dot{y}) + \dot{f}(\theta - s)\cdot y \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_U \partial_\sigma W_\varepsilon(\theta - s, \nabla y)\,\mathrm{d}x\,\mathrm{d}s.$$

In the sharp-interface case $\varepsilon = 0$, for all $t \in [0, T]$,

$$\int_U \big(W_0(\theta - t, \nabla y) + H(\nabla^2 y) - f(\theta - t)\cdot y\big) - \big(W_0(\theta, \nabla y_0) + H(\nabla^2 y_0) - f(\theta)\cdot y_0\big)\, \mathrm{d}x$$

$$= -\int_0^t \int_U 2R_0(\theta - s, \nabla y, \nabla \dot{y}) + \dot{f}(\theta - s)\cdot y \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\{\theta = s\}} \frac{V^a(\nabla y) - V^r(\nabla y)}{|\nabla \theta|}\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s.$$

Energy inequality estimate

$$\int_U \left(W_0(\theta-t,\nabla y)+H(\nabla^2 y)-f(\theta-t)\cdot y\right)-\left(W_0(\theta,\nabla y_0)+H(\nabla^2 y_0)-f(\theta)\cdot y_0\right)\,\mathrm{d}x$$

$$=-\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)+\dot f(\theta-s)\cdot y\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

$$\|y(t)\|^p_{W^{2,p}(U)}\lesssim\int_U\left(W_0(\theta-t,\nabla y)+H(\nabla^2 y)\right)\mathrm{d}x+\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)\,\mathrm{d}x\,\mathrm{d}s$$

Energy inequality estimate

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$$=-\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)+\dot f(\theta-s)\cdot y\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

$$\|y(t)\|^p_{W^{2,p}(U)}\lesssim\int_U\big(W_0(\theta-t,\nabla y)+H(\nabla^2 y)\big)\,\mathrm{d}x+\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)\,\mathrm{d}x\,\mathrm{d}s$$

$$\lesssim E_0+\|f\|_{W^{1,\infty}(0,T;L^2(U))}+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

Energy inequality estimate

$$\int_U \big(W_0(\theta-t,\nabla y)+H(\nabla^2 y)-f(\theta-t)\cdot y\big)-\big(W_0(\theta,\nabla y_0)+H(\nabla^2 y_0)-f(\theta)\cdot y_0\big)\,\mathrm{d}x$$

$$=-\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)+\dot f(\theta-s)\cdot y\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

$$\|y(t)\|^p_{W^{2,p}(U)}\lesssim\int_U\big(W_0(\theta-t,\nabla y)+H(\nabla^2 y)\big)\,\mathrm{d}x+\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)\,\mathrm{d}x\,\mathrm{d}s$$

$$\lesssim E_0+\|f\|_{W^{1,\infty}(0,T;L^2(U))}+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

$$\lesssim\int_0^t\|\nabla y(s)\|^p_{L^\infty(U)}\int_{\{\theta=s\}}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s\lesssim\int_0^t\|y(s)\|^p_{W^{2,p}(U)}\int_{\{\theta=s\}}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

control from above

Energy inequality estimate

$$\int_U \big(W_0(\theta-t,\nabla y)+H(\nabla^2 y)-f(\theta-t)\cdot y\big)-\big(W_0(\theta,\nabla y_0)+H(\nabla^2 y_0)-f(\theta)\cdot y_0\big)\;\mathrm{d}x$$

$$=-\int_0^t\int_U 2R_0\,(\theta-s,\nabla y,\nabla\dot y)+\dot f(\theta-s)\cdot y\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

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$$\lesssim E_0+\|f\|_{W^{1,\infty}(0,T;L^2(U))}+\int_0^t\int_{\{\theta=s\}}\frac{V^a(\nabla y)-V^r(\nabla y)}{|\nabla\theta|}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

$$\lesssim\int_0^t\|\nabla y(s)\|^p_{L^\infty(U)}\int_{\{\theta=s\}}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s\lesssim\int_0^t\|y(s)\|^p_{W^{2,p}(U)}\int_{\{\theta=s\}}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s$$

$$\overset{\text{Grönwall}}{\Longrightarrow}\quad\|y(t)\|^p_{W^{2,p}(U)}\lesssim\exp\left(\int_0^t\int_{\{\theta=s\}}\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s\right)\lesssim\exp\big(|\Omega(t)|\big)$$

What comes next?

- Backstrain
- Temperature
- One phase problem on a growing domain
- Different growth models

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[3] G. Zurlo, L. Truskinovsky. Inelastic surface growth. *Mech. Res. Comm.* 93 (2018), 174–179.

Thank you for your attention!