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A USER'S GUIDE TO DISCRETE MORSE THEORY

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Abstract. A number of questions from a variety of areas of mathematics lead one to the problem of analyzing the topology of a simplicial complex. However, there are few general techniques available to aid us in this study. On the other hand, some very general theories have been developed for the study of smooth manifolds. One of the most powerful, and useful, of these theories is Morse Theory. In this paper we present a combinatorial adaptation of Morse Theory, which we call discrete Morse Theory, that may be applied to any simplicial complex (or more general cell complex). The goal of this paper is to present an overview of the subject of discrete Morse Theory that is sufficient both to understand the major applications of the theory to combinatorics, and to apply the the theory to new problems. We will not be presenting theorems in their most recent or most general form, and simple examples will often take the place of proofs. We hope to convey the fact that the theory is really very simple, and there is not much that one needs to know before one can become a "user".

0. Introduction

A number of questions from a variety of areas of mathematics lead one to the problem of analyzing the topology of a simplicial complex. We will see some examples in these notes. However, there are few general techniques available to aid us in this study. On the other hand, some very general theories have been developed for the study of smooth manifolds. One of the most powerful, and useful, of these theories is Morse Theory.

There is a very close relationship between the topology of a smooth manifold M and the critical points of a smooth function f on M. For example, if f is compact, then M must achieve a maximum and a minimum. Morse Theory is a far-reaching extension of this fact. Milnor's beautiful book [30] is the standard reference on this subject. In this paper we present an adaptation of Morse Theory that may be applied to any simplicial complex (or more general cell complex). There have been other adaptations of Morse Theory that can be applied to combinatorial spaces. For example, a Morse theory of piecewise linear functions appears in [26] and the very powerful "Stratified Morse Theory" was developed by Goresky and MacPherson [19], [20]. These theories, especially the latter, have each been successfully applied to prove some very striking results.

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We take a slightly different approach than that taken in these references. Rather than choosing a suitable class of continuous functions on our spaces to play the role of Morse functions, we will be assigning a single number to each cell of our complex, and all associated processes will be discrete. Hence, we have chosen the name *discrete Morse Theory* for the ideas we will describe.

Of course, these different approaches to combinatorial Morse Theory are not distinct. One can sometimes translate results from one of these theories to another by "smoothing out" a discrete Morse function, or by carefully replacing a continuous function by a discrete set of its values. However, that is not the path we will follow in this paper. Instead, we show that even without introducing any continuity, one can recreate, in the category of combinatorial spaces, a complete theory that captures many of the intricacies of the smooth theory, and can be used as an effective tool for a wide variety of combinatorial and topological problems.

The goal of this paper is to present an overview of the subject of discrete Morse Theory that is sufficient both to understand the major applications of the theory to combinatorics, and to apply the theory to new problems. We will not be presenting theorems in their most recent or most general form, and simple examples will often take the place of proofs. We hope to convey the fact that the theory is really very simple, and there is not much that one needs to know before one can become a "user". Those interested in a more complete presentation of the theory can consult the reference [10]. Earlier surveys of this work have appeared in [9] and [13].

1. CW Complexes

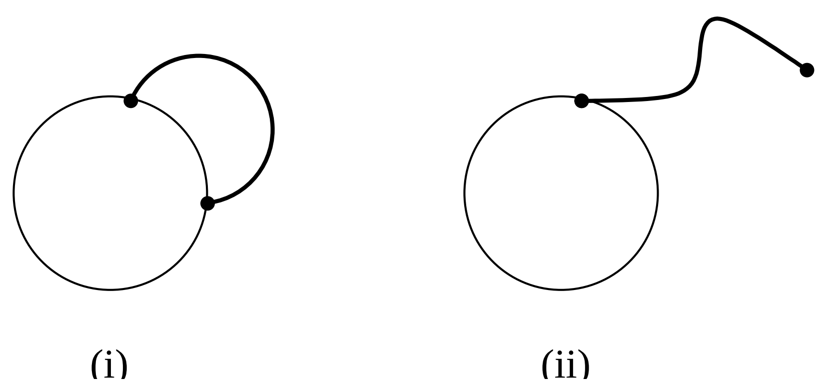
The main theorems of discrete (and smooth) Morse Theory are best stated in the language of CW complexes, so we begin with an overview of the basics of such complexes. J.H.C. Whitehead introduced CW complexes in his foundational work on homotopy theory, and all of the results in this section are due to him. The reader should consult [28] for a very complete introduction to this topic. In this paper we will consider only finite CW complexes, so many of the subtleties of the subject will not appear.

The building blocks of CW complexes are cells. Let B^d denote the closed unit ball in d-dimensional Euclidean space. That is, $B^d = \{x \in \mathbb{E}^d : |x| \leq 1\}$. The boundary of B^d is the unit $(d-1)$-sphere $S^{(d-1)} = \{x \in \mathbb{E}^d : |x| = 1\}$. A *$d$-cell* is a topological space which is homeomorphic to B^d. If σ is d-cell, then we denote by $\dot{\sigma}$ the subset of σ corresponding to $S^{(d-1)} \subset B^d$ under any homeomorphism between B^d and σ. A *cell* is a topological space which is a d-cell for some d.

The basic operation of CW complexes is the notion of *attaching a cell.* Let X be a topological space, σ a d-cell and $f : \dot{\sigma} \to X$ a continuous map. We let $X \cup_f \sigma$ denote the disjoint union of X and σ quotiented out by the equivalence relation that each point $s \in \dot{\sigma}$ is identified with $f(s) \in X$. We refer to this operation by saying that

$X \cup_f \sigma$ is the result of attaching the cell σ to X. The map f is called *the attaching map.*

We emphasize that the attaching map must be defined on all of $\dot{\sigma}$. That is, the entire boundary of σ must be "glued" to X. For example, if X is a circle, then Figure 1.1(i) shows one possible result of attaching a 1-cell to X. Attaching a 1-cell to X cannot lead to the space illustrated in Figure 1.1(ii) since the entire boundary of the 1-cell has not been "glued" to X.



(i). A 1-cell attached to a circle. (ii). This is not *a 1-cell attached to a circle*

Figure 1.1.

We are now ready for our main definition. A *finite CW complex* is any topological space X such that there exists a finite nested sequence

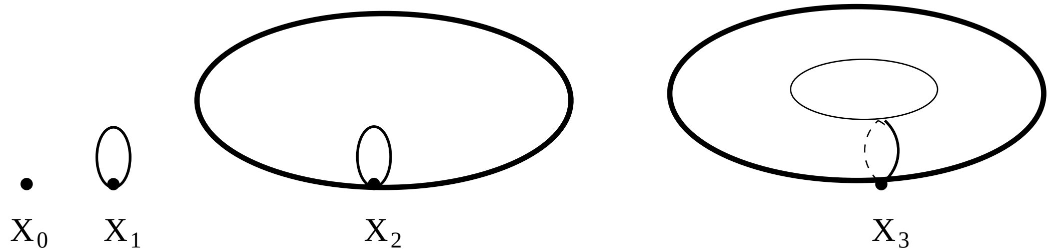
$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_n = X \tag{1.1}$$

such that for each $i = 0, 1, 2, \ldots, n$, X_i is the result of attaching a cell to $X_{(i-1)}$.

Note that this definition requires that X_0 be a 0-cell. If X is a CW complex, we refer to any sequence of spaces as in (1.1) as a *CW decomposition of* X. Suppose that in the CW decomposition (1.1), of the $n+1$ cells that are attached, exactly c_d are d-cells. Then we say that the CW complex X has a CW decomposition consisting of c_d d-cells for every d.

We note that a (closed) d-simplex is a d-cell. Thus a finite simplicial complex is a CW complex, and has a CW decomposition in which the cells are precisely the closed simplices.

In Figure 1.2 we demonstrate a CW decomposition of a 2-dimensional torus which, beginning with the 0-cell, requires attaching two 1-cells and then one 2-cell. Here we can see one of the most compelling reasons for considering CW complexes rather than just simplicial complexes. Every simplicial decomposition of the 2-torus has at least 7 vertices, 21 edges and 14 triangles.



A CW decomposition of a 2-torus

Figure 1.2.

It may seem that quite a bit has been lost in the transition from simplicial complexes to general CW complexes. After all, a simplicial complex is completely described by a finite amount of combinatorial data. On the other hand, the construction of a CW decomposition requires the choice of a number of continuous maps. However, if one is only concerned with the homotopy type of the resulting CW complex, then things begin to look a bit more manageable. Namely, the homotopy type of $X \cup_f \sigma$ depends only on the homotopy type of X and the homotopy class of f.

Theorem 1.3. *Let $h : X \to X'$ denote a homotopy equivalence, σ a cell, and $f_1 : \dot{\sigma} \to X$, $f_2 : \dot{\sigma} \to X'$ two continuous maps. If $h \circ f_1$ is homotopic to f_2, then $X \cup_{f_1} \sigma$ and $X' \cup_{f_2} \sigma$ are homotopy equivalent.*

An important special case is when h is the identity map. We state this case separately for future reference.

Corollary 1.4. *Let X be a topological space, σ a cell, and $f_1, f_2 : \dot{\sigma} \to X$ two continuous maps. If f_1 and f_2 are homotopic, then $X \cup_{f_1} \sigma$ and $X \cup_{f_2} \sigma$ are homotopy equivalent.*

(See Theorem 2.3 on page 120 of [28].) Therefore, the homotopy type of a CW complex is determined by the homotopy classes of the attaching maps. Since homotopy classes are discrete objects, we have now recaptured a bit of the combinatorial atmosphere that we seemingly lost when generalizing from simplicial complexes to CW complexes.

Let us now present some examples.

1) Suppose X is a topological space which has a CW decomposition consisting of exactly one 0-cell and one d-cell. Then X has a CW decomposition $\emptyset \subset X_0 \subset X_1 = X$. The space X_0 must be the 0-cell, and $X = X_1$ is the result of attaching the d-cell to X_0. Since X_0 consists of a single point, the only possible attaching map is the constant map. Thus X is constructed from taking a closed d-ball and identifying all of the points on its boundary. One can easily see that this implies that the resulting space is a d-sphere.

2) Suppose X is a topological space which has a CW decomposition consisting of exactly one 0-cell and n d-cells. Then X has a CW decomposition as in (1.1) such that

X_0 is the 0-cell, and for each $i = 1, 2, \ldots, n$ the space X_i is the result of attaching a d-cell to $X_{(i-1)}$. From the previous example, we know that X_1 is a d-sphere. The space X_2 is constructed by attaching a d-cell to X_1. The attaching map is a continuous map from a $(d-1)$-sphere to X_1. Every map of the $(d-1)$-sphere into X_1 is homotopic to a constant map (since $\pi_{(d-1)}(X_1) \cong \pi_{(d-1)}(S^d) \cong 0$). If the attaching map is actually a constant map, then it is easy to see that the space X_2 is the wedge of two d-spheres, denoted by $S^d \wedge S^d$. (The wedge of a collection of topological spaces is the space resulting from choosing a point in each space, taking the disjoint union of the spaces, and identifying all of the chosen points.) Since the attaching map must be homotopic to a constant map, Corollary 1.4 implies that X_2 is homotopy equivalent to a wedge of two d-spheres.

When constructing X_3 by attaching a d-cell to X_2, the relevant information is a map from S^{d-1} to X_2, and the homotopy type of the resulting space is determined by the homotopy class of this map. All such maps are homotopic to a constant map (since $\pi_{d-1}(X_2) \cong \pi_{d-1}(S^d \wedge S^d) \cong 0$). Since X_2 is homotopy equivalent to a wedge of two d-spheres, and the attaching map is homotopic to a constant map, it follows from Theorem 1.3 that X_3 is homotopy equivalent to the space that would result from attaching a d-cell to $S^d \wedge S^d$ via a constant map, i.e., X_3 is homotopy equivalent to a wedge of three d-spheres.

Continuing in this fashion, we can see that X must be homotopy equivalent to a wedge of n d-spheres.

The reader should not get the impression that the homotopy type of a CW complex is determined by the number of cells of each dimension. This is true only for very few spaces (and the reader might enjoy coming up with some other examples). The fact that wedges of spheres can, in fact, be identified by this numerical data partly explains why the main theorem of many papers in combinatorial topology is that a certain simplicial complex is homotopy equivalent to a wedge of spheres. Namely such complexes are the easiest to recognize. However, that does not explain why so many simplicial complexes that arise in combinatorics are homotopy equivalent to a wedge of spheres. I have often wondered if perhaps there is some deeper explanation for this.

3) Suppose that X is a CW complex which has a CW decomposition consisting of exactly one 0-cell, one 1-cell and one 2-cell. Let us consider a CW decomposition for X with these cells: $\emptyset \subset X_0 \subset X_1 \subset X_2 = X$. We know that X_0 is the 0-cell. Suppose that X_1 is the result of attaching the 1-cell to X_0. Then X_1 must be a circle, and X_2 arises from attaching a 2-cell to X_1. The attaching map is a map from the boundary of the 2-cell, i.e., a circle, to X_1 which is also a circle. Up to homotopy, such a map is determined by its winding number, which can be taken to be a nonnegative integer. If the winding number is 0, then without altering the homotopy type of X we may assume that the attaching map is a constant map, which yields that $X \sim S^1 \wedge S^2$ (where $\sim$ denotes homotopy equivalence). If the winding number is

1 then without altering the homotopy type of X we may assume that the attaching map is a homeomorphism, in which case X is a 2-dimensional disc. If the winding number is 2, then without altering the homotopy type of X we may assume that the attaching map is a standard degree 2 mapping (i.e., that wraps one circle around the other twice, with no backtracking). The reader should convince him/herself that the result in this case is that X is the 2-dimensional projective space $\mathbb{P}^2$. In fact, each winding number results in a homotopically distinct space. These spaces can be distinguished by their homology, since $H_1(X, \mathbb{Z})$ for the space X resulting from an attaching map with winding number n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

It seems that we are not quite done with this example, because we assumed that the 1-cell was attached before the 2-cell, and we must consider the alternative order, in which X_1 is the result of attaching a 2-cell to X_0. In this case, X_1 is a 2-sphere, and $X = X_2$ is the result of attaching a 1-cell to X_1. The attaching map is a map of S^0 into S^2. Since S^2 is connected (i.e., $\pi_0(S^2) = 0$) all such maps are homotopic to a constant map. Taking the attaching map to be a constant map yields that $X = S^1 \wedge S^2$. Thus adding the cells in this order merely resulted in fewer possibilities for the homotopy type of X. This is a general phenomenon. Generalizing the argument we just presented, using the fact that $\pi_i(S^d) = 0$ for $i < d$, yields the following statement.

Theorem 1.5. *Let*

$$\emptyset \subset X_0 \subset X_2 \subset \cdots \subset X_n = X \tag{1.2}$$

be a CW decomposition of a finite CW complex X. Then X is homotopy equivalent to a finite CW decomposition with precisely the same number of cells of each dimension as in (1.2), and with the cells attached so that their dimensions form a nondecreasing sequence.

I first learned of simplicial complexes in an algebraic topology course. They were introduced as a category of topological spaces for which it was rather easy to define homology and cohomology, i.e., in terms the simplicial chain- and cochain-complexes. One might be concerned that in the transition from simplicial complexes to CW complexes we have lost this ability to easily compute the homology. In fact, much of this computability remains. Let X be a CW complex with a fixed CW decomposition. Suppose that in this decomposition X is constructed from exactly c_d cells of dimension d for each $d = 0, 1, 2, \ldots, n = \dim(K)$, and let $C_d(X, \mathbb{Z})$ denote the space $\mathbb{Z}^{c_d}$ (more precisely, $C_d(X, \mathbb{Z})$ denotes the free abelian group generated by the d-cells of X, each endowed with an orientation). The following is one of the fundamental results in the theory of CW complexes.

Theorem 1.6. *There are boundary maps $\partial_d : C_d(X, \mathbb{Z}) \to C_{d-1}(X, \mathbb{Z})$, for each d, so that*

$$\partial_{d-1} \circ \partial_d = 0$$

and such that the resulting differential complex

$$0 \longrightarrow C_n(X,\mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(X,\mathbb{Z}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X,\mathbb{Z}) \longrightarrow 0$$

calculates the homology of X. That is, if we define

$$H_d(C,\partial) = \frac{\mathrm{Ker}(\partial_d)}{\mathrm{Im}(\partial_{d+1})}$$

then for each d

$$H_d(C,\partial) \cong H_d(X,\mathbb{Z})$$

where $H_d(X,\mathbb{Z})$ denotes the singular homology of X.

The actual definition of the boundary map ∂ is slightly nontrivial and we will not go into it here (see Ch. V Sec. 2 of [28] for the details). At first it may seem that without knowing this boundary map, there is little to be gained from Theorem 1.6. In fact, much can be learned from just knowing of the existence of such a boundary map. For example, let us choose a coefficient field $\mathbb{F}$, and tensor everything with $\mathbb{F}$ to get a differential complex

$$0 \longrightarrow C_n(X,\mathbb{F}) \xrightarrow{\partial_n} C_{n-1}(X,\mathbb{F}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X,\mathbb{F}) \longrightarrow 0$$

which calculates $H_*(X,\mathbb{F})$, where now $C_d(X,\mathbb{F}) \cong \mathbb{F}^{c_d}$. From basic linear algebra. we can deduce the following inequalities.

Theorem 1.7. *Let X be a CW complex with a fixed CW decomposition with c_d cells of dimension d for each d. Fix a coefficient field $\mathbb{F}$ and let b_* denote the Betti numbers of X with respect to $\mathbb{F}$, i.e., $b_d = \dim(H_d(X,\mathbb{F}))$.*
(i) (The Weak Morse Inequalities) For each d

$$c_d \geq b_d.$$

(ii) Let $\chi(X)$ denote the Euler characteristic of X, i.e.,

$$\chi(X) = b_0 - b_1 + b_2 - \dots.$$

then we also have

$$\chi(X) = c_0 - c_1 + c_2 - \dots.$$

As the name "Weak Morse Inequalities" implies, this theorem can be strengthened. The following inequalities, known as the "Strong Morse Inequalities" also follow from standard linear algebra.

Theorem 1.8 (The Strong Morse Inequalities)**.** *With all notation as in Theorem 1.7, for each $d = 0,1,2,\dots$*

$$c_d - c_{d-1} + c_{d-2} - \cdots + (-1)^d c_0 \geq b_d - b_{d-1} + b_{d-2} - \cdots + (-1)^d b_0.$$

Comparing Strong Morse Inequalities for consecutive values of d, and using the fact that $b_i = 0$ for i larger than the dimension of K, yields Theorem 1.7.

We mentioned earlier that a great benefit of passing from simplicial complexes to the more general CW complexes is that one often can use many fewer cells. Let us take another look at this phenomenon in light of the Morse inequalities. Consider the case where X is a two-dimensional torus, so that with respect to any coefficient field $b_0 = 1, b_1 = 2, b_2 = 1$. From the weak Morse inequalities, we have that for any CW decomposition,

$$c_0 \geq b_0 = 1$$
$$c_1 \geq b_1 = 2$$
$$c_2 \geq b_2 = 1.$$

A simplicial decomposition is a special case of a CW decomposition, so these inequalities are satisfied when c_d denotes the number of d-simplices in a fixed simplicial decomposition. However, every simplicial decomposition has at least 7 0-simplices, 21 1-simplices and 14 2-simplices, so these inequalities are far from equality. It is generally the case that for a simplicial decomposition these inequalities are very far from optimal, and hence are generally of little interest. On the other hand, earlier we demonstrated a CW decomposition of the two-torus with exactly one 0-cell, two 1-cells and one 2-cell. The inequalities tell us, in particular, that one cannot build a two-torus using fewer cells.

2. The Basics of Discrete Morse Theory

The discussion in the previous section leads us to an important question. Suppose one is given a finite simplicial complex X. Typically, we can expect that X has a CW decomposition with many fewer cells than in the original simplicial decomposition. How can one go about finding such an "efficient" CW decomposition for X? In this section we present a technique, discrete Morse Theory, which can be useful in such an investigation. (We note that the ideas we will describe can be applied with no modification at all to any finite regular CW complex, and with only minor modifications to a general finite CW complex. However, for simplicity, in this paper we will restrict attention to simplicial complexes.)

We begin by recalling that a finite simplicial complex is a finite set of vertices V, along with a set of subsets K of V. The set K satisfies two main properties:
1) $V \subseteq K$
2) If $\alpha \in K$ and $\beta \subseteq \alpha$ then $\beta \in K$.
By a slight abuse of notation, we will refer to the simplicial complex simply as K. The elements of K are called simplices. If $\alpha \in K$, and α contains $p+1$ vertices, then we say that the dimension of α is p, and we will sometimes denote this by $\alpha^{(p)}$. For simplices α and β we will use the notation $\alpha < \beta$ or $\beta > \alpha$ to indicate that α is a proper subset of β (thinking of α and β as subsets of V), and say that α *is a*

face of β. We emphasize that at this point we will not be placing any restrictions on the finite simplicial complexes under investigation. In particular, the complexes need not be manifolds (even though many of our examples will be). In Section 9 we will briefly indicate how some of our conclusions can be strengthened in the case that the complexes are assumed to have additional structure.

A discrete Morse function on K is a function which, roughly speaking, assigns higher numbers to higher dimensional simplices, with at most one exception, locally, at each simplex. More precisely,

Definition 2.1. *A function*

$$f : K \longrightarrow \mathbb{R}$$

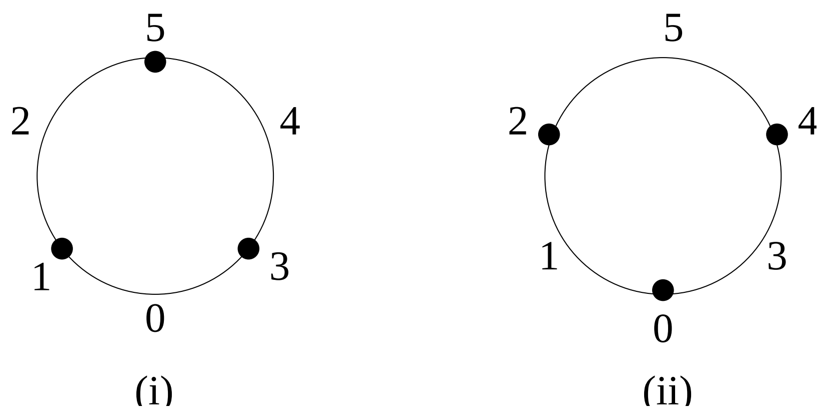
is a discrete Morse function if for every $\alpha^{(p)} \in K$

$$(1) \quad \#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} \leq 1,$$

and

$$(2) \quad \#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} \leq 1.$$

A simple example will serve to illustrate the definition. Consider the two complexes shown in Figure 2.2. Here we indicate functions by writing next to each simplex the value of the function on that simplex. The function (i) is not a discrete Morse function as the edge $f^{-1}(0)$ violates rule (2), since it has 2 lower dimensional "neighbors" on which f takes on higher values, and the vertex $f^{-1}(5)$ violates rule (1), since it has 2 higher dimensional "neighbors" on which f takes on lower values. The function (ii) is a Morse function. Note that a discrete Morse function is not a continuous function on K. Rather, it is an assignment of a single number to each simplex.



(i). This is not *a discrete Morse function. (ii). This is a discrete Morse function.*

Figure 2.2.

The other main ingredient in Morse Theory is the notion of a critical point.

Definition 2.3. *A simplex $\alpha^{(p)}$ is critical if*

$$(1) \quad \#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} = 0,$$

and

$$(2) \quad \#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} = 0.$$

For example, Figure 2.2(ii), the vertex $f^{-1}(0)$ and the edge $f^{-1}(5)$ are critical, and there are no other critical simplices.

We mention for later use that it follows from the axioms that a simplex cannot simultaneously fail both conditions in the test for criticality.

Lemma 2.4. *If K is a simplicial complex with a Morse function f, then for any simplex α, either*

$$(1) \quad \#\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} = 0,$$

or

$$(2) \quad \#\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} = 0.$$

(See Lemma 2.5 of [10].) This lemma will play a crucial role in Section 3.

We can now state the main theorem of discrete Morse Theory.

Theorem 2.5. *Suppose K is a simplicial complex with a discrete Morse function. Then K is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical simplex of dimension p.*

Rather than present a proof of this theorem, we will content ourselves here with a brief discussion of the main ideas. A discrete Morse function gives us a way to build the simplicial complex by attaching the simplices in the order prescribed by the function, i.e., adding first the simplices which are assigned the smallest values. More precisely, for any simplicial complex K with a discrete Morse function f, and any real number c, define the level subcomplex $K(c)$ by

$$K(c) = \cup_{f(\alpha) \leq c} \cup_{\beta \leq \alpha} \beta.$$

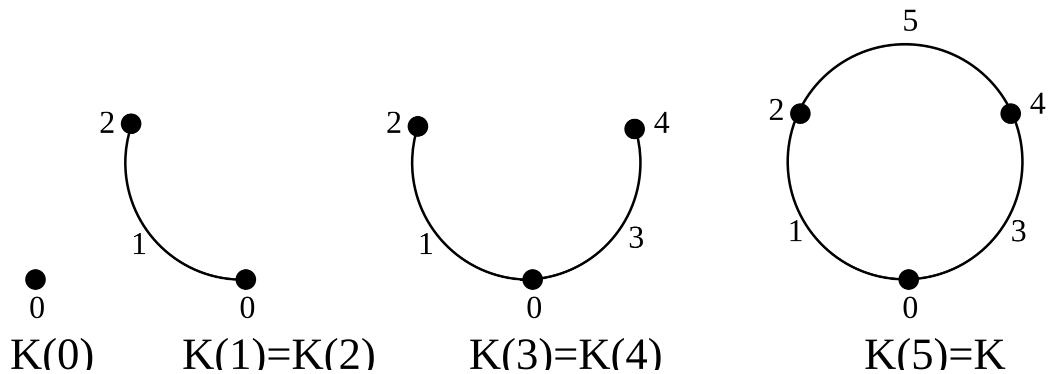
That is, $K(c)$ is the subcomplex consisting of all simplices α of K such that $f(\alpha) \leq c$ along with all of their faces.

Theorem 2.5 follows from two basic lemmas.

Lemma 2.6. *If there are no critical simplices α with $f(\alpha) \in (a, b]$, then $K(b)$ is homotopy equivalent to $K(a)$. (In fact, $K(b)$ collapses to $K(a)$ — this will be explained later.)*

Lemma 2.7. *If there is a single critical simplex α with $f(\alpha) \in (a, b]$ then there is a map $F : S^{(d-1)} \to K(a)$, where d is the dimension of α, such that $K(b)$ is homotopy equivalent to $K(a) \cup_F B^d$.*

In Figure 2.8 we illustrate all of the level subcomplexes in the case that K is the circle triangulated with 3 edges and 3 vertices, and f is the Morse function shown in Figure 2.2 (ii). Here we can see why these lemmas are true.

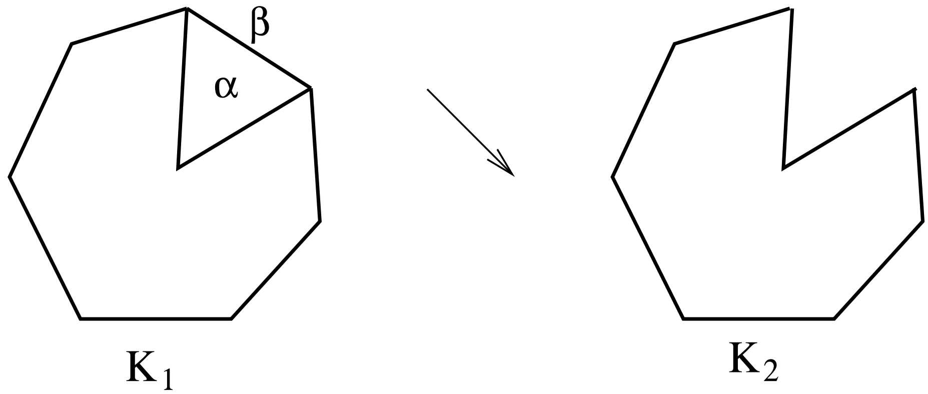


The level subcomplexes of the discrete Morse function shown in Figure 2.2(ii)

Figure 2.8.

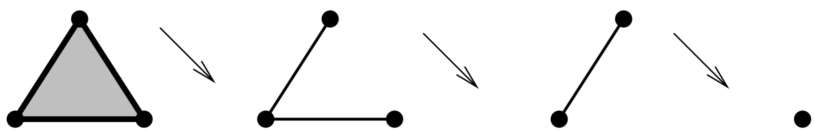
Let us begin with Lemma 2.6. Consider the transition from $K(0)$ to $K(1)$. We have not added any critical simplices, and, just as the lemma predicts, $K(0)$ and $K(1)$ are homotopy equivalent. Let us try to understand why the homotopy type did not change. To construct $K(1)$ from $K(0)$, we first have to add the edge $f^{-1}(1)$. This edge is not critical because it has a codimension-one face which is assigned a higher value, namely the vertex $f^{-1}(2)$. In order to have $K(1)$ be a subcomplex, we must also add this vertex. Thus we see that the edge $f^{-1}(1)$ in $K(1)$ has a *free face*, i.e., a face which is not the face of any other simplex in $K(1)$. We can deformation retract $K(1)$ to $K(0)$ by "pushing in" the edge $f^{-1}(1)$ starting at the vertex $f^{-1}(2)$.

This is a very general phenomenon. That is, it follows from the axioms for a discrete Morse function that for any simplicial complex with any discrete Morse function, when passing from one level subcomplex to the next the noncritical simplices are added in pairs, each of which consists of a simplex and a free face. Suppose that $K_2 \subset K_1$ are simplicial complexes, and K_1 has exactly two simplices α and β that are not in K_2, where β is a free face of α. Then it is easy to see that K_2 is a deformation retract of K_1, and hence K_1 and K_2 are homotopy equivalent (see Figure 2.9). This special sort of combinatorial deformation retract is called a *simplicial collapse*. If one can transform a simplicial complex K_1 into a subcomplex K_2 by simplicial collapses, then we say that K_1 *collapses to* K_2, and we indicate this by $K_1 \searrow K_2$. Figure 2.10 shows a 2-dimensional simplex collapsing to one of its vertices.



A simplicial collapse.

Figure 2.9.



A 2-simplex collapsing to a vertex.

Figure 2.10.

The process of simplicial collapse was studied by J.H.C. Whitehead, and he defined *simple homotopy equivalence* to be the equivalence relation generated by simplicial collapse. This indicates that discrete Morse Theory may be particularly useful when working in the category of simple homotopy equivalence.

Now let us turn to Lemma 2.7 and investigate what happens when one adds a critical simplex, for example when making the transition from $K(4)$ to $K(5)$. In this case we are adding a critical edge. We can see clearly from the illustration that we pass from $K(4)$ to $K(5)$ by attaching a 1-cell, just as predicted by Lemma 2.7. To see why this works in general, consider a critical d-simplex α. It follows from the definition of a critical simplex that each face of α is assigned a smaller value than α, which implies in turn that each face of α appears in a previous level subcomplex. Thus the entire boundary of α appears in an earlier level subcomplex, so that when it comes time to add α, we must "glue it in" along its entire boundary. This is precisely the process of attaching a d-cell.

This completes our discussion of the proof.

Perhaps this is a good time to point out that one can define a discrete Morse function on any simplicial complex. Namely, one can simply let $f(\alpha) = \dim(\alpha)$ for each simplex α. In this case, every simplex is critical, and Theorem 2.5 is a rather uninteresting tautology. However, as we will see in examples, one can often construct discrete Morse functions with many fewer critical simplices.

Let K be a simplicial complex with a discrete Morse function. Let m_p denote the number of critical simplices of dimension p. Let $\mathbb{F}$ be any field, and $b_p = \dim H_p(K, \mathbb{F})$ the p^{th} Betti number with respect to $\mathbb{F}$. Combining Theorems 2.5, 1.7 and 1.8, and the fact that homotopy equivalent spaces have isomorphic homology, we have the following inequalities.

Theorem 2.11. I. The Weak Morse Inequalities.
(i)For each $p = 0, 1, 2, \ldots, n$ (where n is the dimension of K)

$$m_p \geq b_p.$$

(ii) $m_0 - m_1 + m_2 - \cdots + (-1)^n m_n = b_0 - b_1 + b_2 - \cdots + (-1)^n b_n \ \ [= \chi(K)]$.

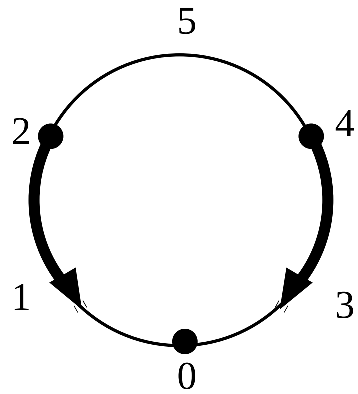
II. The Strong Morse Inequalities.
For each $p = 0, 1, 2, \ldots, n, n+1$,

$$m_p - m_{p-1} + \cdots + (-1)^p m_0 \geq b_p - b_{p-1} + \cdots + (-1)^p b_0.$$

3. Gradient Vector Fields

Any ambitious reader who has already started trying some examples will have noticed that the theory as presented in the previous section can be a bit unwieldy. After all, how is one to go about assigning numbers to each of the simplices of a simplicial complex so that they satisfy the axioms of a discrete Morse function? Fortunately, in practice one need not actually find a discrete Morse function. Finding the gradient vector field of the Morse function is sufficient. This requires a bit of explanation.

Let us now return to the example in Figure 2.2(ii). Noncritical simplices occur in pairs. For example, the edge $f^{-1}(1)$ is not critical because it has a "lower dimensional neighbor" which is assigned a higher value, i.e., the vertex $f^{-1}(2)$. Similarly, the vertex $f^{-1}(2)$ is not critical because it has a "higher dimensional neighbor" which is assigned a lower value, i.e., the edge $f^{-1}(1)$. We indicate this pairing by drawing an arrow from the vertex $f^{-1}(2)$, pointing into the edge $f^{-1}(1)$. Similarly, we draw an arrow from the vertex $f^{-1}(4)$ pointing into the edge $f^{-1}(3)$. (See Figure 3.1.) One can think of these arrows as pictorially indicating the simplicial collapse that is referred to in the proof of Lemma 2.6.

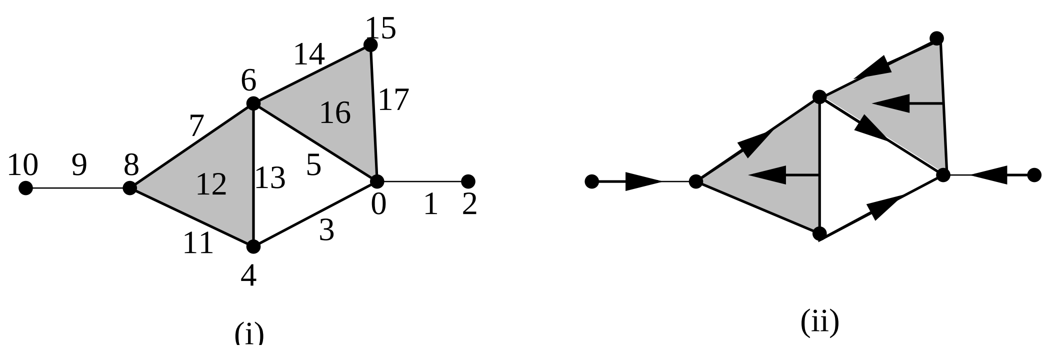


The gradient vector field of the Morse function shown in Figure 2.2 (ii).

Figure 3.1.

We can apply this process to any simplicial complex with a discrete Morse function. The arrows are drawn as follows. Suppose $\alpha^{(p)}$ is a non-critical simplex with $\beta^{(p+1)} > \alpha$ satisfying $f(\beta) \leq f(\alpha)$. We then draw an arrow from α to β. Figure 3.2 illustrates a more complicated example. Note that the discrete Morse function drawn in this figure has one critical vertex, $f^{-1}(0)$, and one critical edge, $f^{-1}(11)$. Theorem 2.5 implies this simplicial complex is homotopy equivalent to a CW complex with exactly one 0-cell and one 1-cell, i.e., a circle.

It follows from Lemma 2.4 that that every simplex α satisfies exactly one of the following:
(i) α is the tail of exactly one arrow.
(ii) α is the head of exactly one arrow.
(iii) α is neither the head nor the tail of an arrow.
Note that a simplex is critical if and only if it is neither the tail nor the head of any arrow. These arrows can be viewed as the discrete analogue of the gradient vector field of the Morse function. (To be precise, when we say "gradient vector field" we are really referring to the negative of the gradient vector field.)



Another example of a gradient vector field

Figure 3.2.

As we will see in examples later, these arrows are much easier to work with than the original discrete Morse function. In fact, this gradient vector field contains all of the information that we will need to know about the function for most applications. The upshot is that if one is given a simplicial complex and one wishes to apply the theory of the previous section, one need not find a discrete Morse function. One "only" needs to find a gradient vector field.

This leads us to the following question. Suppose we attach arrows to the simplices so that each simplex satisfies exactly one of properties (i),(ii),(iii) above. Then how do we know if that set of arrows is the gradient vector field of a discrete Morse function? This is the question we will answer in the remainder of this section.

Let K be a simplicial complex with a discrete Morse function f. Then rather than thinking about the discrete gradient vector field V of f as a collection of arrows, we may equivalently describe V as a collection of pairs $\{\alpha^{(p)} < \beta^{(p+1)}\}$ of simplices of K, where $\{\alpha^{(p)} < \beta^{(p+1)}\}$ is in V if and only if $f(\beta) \leq f(\alpha)$. In other words, $\{\alpha^{(p)} < \beta^{(p+1)}\}$ is in V if and only if we have drawn an arrow that has α as its tail, and β as its head. The properties of a discrete Morse function imply that each simplex is in at most one pair of V. This leads us to the following definition.

Definition 3.3. *A discrete vector field V on K is a collection of pairs $\{\alpha^{(p)} < \beta^{(p+1)}\}$ of simplices of K such that each simplex is in at most one pair of V.*

Such pairings were studied in [41] and [8] as a tool for investigating the possible f-vectors for a simplicial complex. Here we take a different point of view. If one has a smooth vector field on a smooth manifold, it is quite natural to study the dynamical system induced by flowing along the vector field. One can begin the same sort of study for any discrete vector field. In [12] we present a study of the dynamics associated to a discrete vector field. Here, we present just enough to continue our discussion of discrete Morse Theory.

Given a discrete vector field V on a simplicial complex K, a $V-$path is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)} \tag{3.1}$$

such that for each $i = 0, \ldots r$, $\{\alpha < \beta\} \in V$ and $\beta_i > \alpha_{i+1} \neq \alpha_i$. We say such a path is a *non-trivial closed path* if $r \geq 0$ and $\alpha_0 = \alpha_{r+1}$. If V is the gradient vector field of a discrete Morse function f, then we sometimes refer to a V-path as a *gradient path of f*.

One idea behind this definition is the following result.

Theorem 3.4. *Suppose V is the gradient vector field of a discrete Morse function f. Then a sequence of simplices as in (3.1) is a V-path if and only if $\alpha_i < \beta_i > \alpha_{i+1}$ for each $i = 0, 1, \ldots, r$, and*

$$f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \cdots \geq f(\beta_r) > f(\alpha_{r+1}).$$

That is, the gradient paths of f are precisely those "continuous" sequences of simplices along which f is decreasing. In particular, this theorem implies that if V is a gradient vector field, then there are no nontrivial closed V-paths. In fact, the main result of this section is that the converse is true.

Theorem 3.5. *A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed V-paths.*

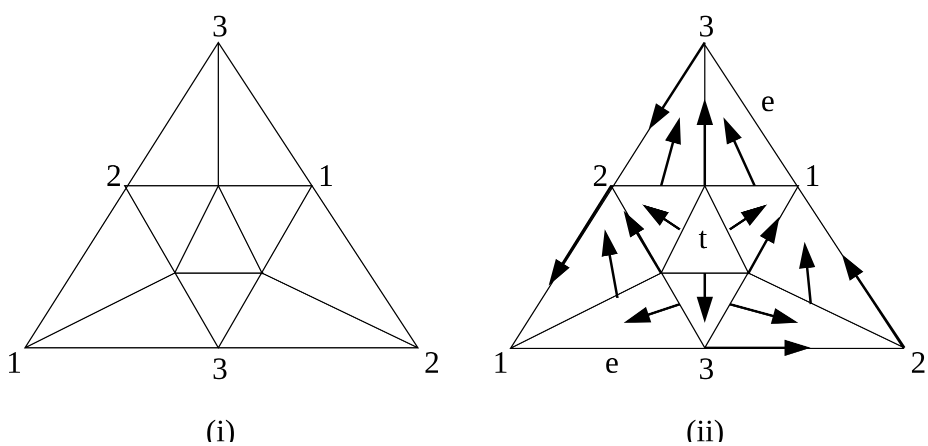
We will not prove this theorem here. However, many readers may notice the similarity with the following standard theorem from the subject of directed graphs.

Theorem 3.6. *Let G be a directed graph. Then there is a real-valued function of the vertices that is strictly decreasing along each directed path if and only if there are no directed loops.*

We will show in Section 6 that, in fact, Theorem 3.6 implies Theorem 3.5. The power of Theorem 3.5 is indicated in the next two sections in which we construct some discrete vector fields and use Theorem 3.5 to verify that they are gradient vector fields.

4. Our First Example: The Real Projective Plane

Figure 4.1 (i) shows a triangulation of the real projective plane $\mathbb{P}^2$. Note that the vertices along the boundary with the same labels are to be identified, as are the edges whose endpoints have the same labels. In Figure 5(ii) we illustrate a discrete vector field V on this simplicial complex. One can easily see that there are no closed V-paths (since all V-paths go to the boundary of the figure and there are no closed V-paths on the boundary), and hence is a gradient vector field. The only simplices which are neither the head nor the tail of an arrow are the vertex labelled 1, the edge e, and the triangle t. Thus, by Theorem 2.5, the projective plane is homotopy equivalent to a CW complex with exactly one 0-cell, one 1-cell and one 2-cell. (Of course, we already knew this from our discussion of Example 3 in Section 2.)



(i) A triangulation of the real projective plane. (ii) A discrete gradient vector field on $\mathbb{P}^2$.

Figure 4.1.

This example gives rise to two potential concerns. The first is that from the main theorem we learn only a statement about "homotopy equivalence". This is sufficient if one is only interested in calculating homology or homotopy groups. However, one might be interested in determining the (PL-)homeomorphism type of the complex. This is possible, in some cases, using deep results of J.H.C. Whitehead. We revisit this topic in Section 8.

The second potential point of concern is that as we saw in Section 2 there are an infinite number of different homotopy types of CW complexes which can be built from exactly one 0-cell, one 1-cell and one 2-cell. One might wonder if Morse Theory can give us any additional information as to how the cells are attached. In fact, one can deduce much of this information if one has enough information about the gradient paths of the Morse function. This point is discussed further in Section 7, where we will return to this example of the triangulated projective plane.

5. Our Second Example: The Complex of Not Connected Graphs

A number of fascinating simplicial complexes arise from the study of monotone graph properties. Let K_n denote the complete graph on n vertices, and suppose we have labelled the vertices 1,2,. . . ,n. Let $\mathcal{G}_n$ denote the spanning subgraphs of K_n, that is, the subgraphs of K_n that contain all n vertices. A subset $\mathcal{P} \subset \mathcal{G}_n$ is called a *graph property* of graphs with n vertices if inclusion in $\mathcal{P}$ only depends on the isomorphism type of the graph. That is, $\mathcal{P}$ is a graph property if for all pairs of graphs $G_1, G_2 \in \mathcal{G}_n$, if G_1 and G_2 are isomorphic (ignoring the labellings on the vertices) then $G_1 \in \mathcal{P}$ if and only if $G_2 \in \mathcal{P}$. A graph property $\mathcal{P}$ of graphs with n vertices is said to be *monotone decreasing* if for any graphs $G_1 \subset G_2 \in \mathcal{G}_n$, if $G_2 \in \mathcal{P}$ then $G_1 \in \mathcal{P}$.

Monotone decreasing properties abound in the study of graph theory. Here are some typical examples: graphs having no more than k edges (for any fixed k), graphs such that the degree of every vertex is less that δ (for any fixed δ), graphs which are not connected, graphs which are not i-connected (for any fixed i), graphs which do not have a Hamiltonian cycle, graphs which do not contain a minor isomorphic to H (for any fixed graph H), graphs which are r-colorable (for any fixed r), and bipartite graphs.

Any monotone decreasing graph property $\mathcal{P}$ gives rise to a simplicial complex $\mathcal{K}$ where the d-simplices of $\mathcal{K}$ are the graphs $G \in \mathcal{P}$ which have $d+1$ edges. In particular, if G is a d-simplex in $\mathcal{K}$, then the faces of G are all of the nontrivial spanning subgraphs of G (the monotonicity of $\mathcal{P}$ implies that each of these graphs is in $\mathcal{K}$). Said in another way, if $\mathcal{P}$ is nonempty, then the vertices of $\mathcal{K}$ are the edges of K_n, and a collection of vertices in $\mathcal{K}$ span a simplex if the spanning subgraph of K_n consisting of all edges which correspond to these vertices lies in $\mathcal{P}$.

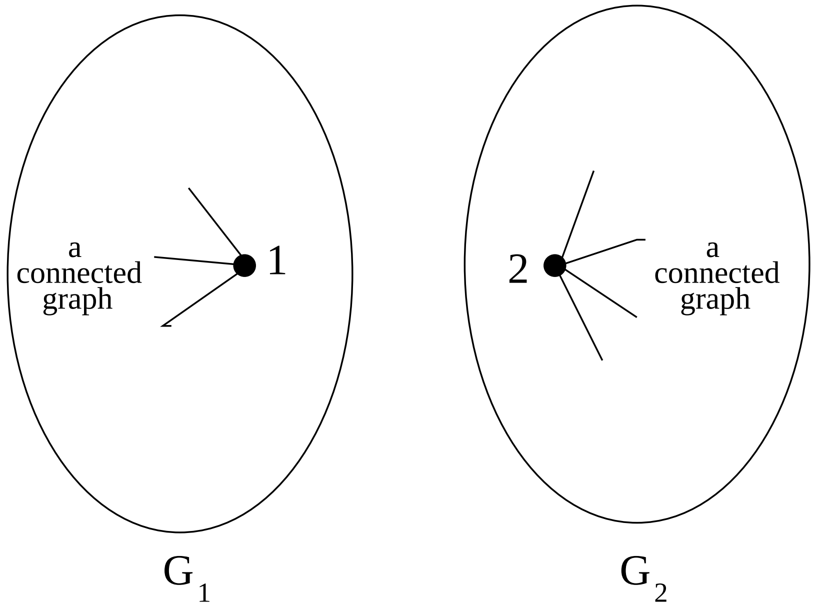
The simplicial complexes induced by many of the above-mentioned monotone decreasing graph properties have been studied using the techniques of this paper. See for example [6], [7], [21], [22], [27], [37]. These papers contain some beautiful mathematics in which the authors construct, "by hand", explicit discrete gradient vector fields, along the way illuminating some of the intricate finer structures of the graph properties.

Some monotone graph properties have recently been the focus of intense interest because of their relation to knot theory. Unfortunately this is probably not a good time for an in depth discussion of this fascinating topic. We will mention only that Vassiliev has shown how one can derive "finite type knot invariants" from the study of the space of "singular knots" (i.e., maps from S^1 to $\mathbb{R}^3$ which are not embeddings). The homology of the simplicial complexes of not connected and not 2-connected graphs show up in his spectral sequence calculation of the homology of this space. This is explained in [43], where Vassiliev derives the homotopy type of the complex of not connected graphs. In [42] and [1], the topology of the space of not 2-connected graphs is determined, with discrete Morse Theory playing a minor role in the latter reference. This topic is reexamined in [37], in which the entire investigation is framed in the language of discrete Morse Theory. Discrete Morse Theory is used to determine the topology of not 3-connected graphs in [21].

In this section, we will provide an introduction to this work by taking a look at the simpler case of the complex of not connected graphs. We will show how the ideas of this paper may be used to determine the topology of $\mathcal{N}_n$, the simplicial complex of not connected graphs on n vertices. Let me begin by pointing out that this complex can be well studied by more classical methods, and the answer has also been found by Vassiliev in [43]. The only novelty of this section is our use of discrete Morse Theory.

Our goal is to construct a discrete gradient vector field V on $\mathcal{N}_n$, the simplicial complex of all not-connected graphs with the vertex set $\{1, 2, 3, \ldots, n\}$. The construction will be in steps. Let V_{12} denote the discrete vector field consisting of all pairs $\{G, G + (1, 2)\}$, where G is any graph in $\mathcal{N}_n$ which does not contain the edge $(1, 2)$ and such that $G + (1, 2) \in \mathcal{N}_n$. Another way of describing V_{12} is that if G is any graph in $\mathcal{N}_n$ which contains the edge $(1, 2)$, then $G - (1, 2)$ and G are paired in V_{12}. Actually, there is one exception to this rule. Let G^* denote the graph consisting of only the single edge (1,2). Then $G^* - (1, 2)$ is the empty graph, which corresponds to the empty simplex in $\mathcal{N}_n$, and may not be paired in a discrete vector field. Thus, G^* is unpaired in V_{12}.

The graphs in $\mathcal{N}_n$ other than G^* which are unpaired in V_{12} are those that do not contain the edge $(1, 2)$ and have the property that $G + (1, 2) \notin \mathcal{N}_n$. That is, those disconnected graphs G with the property that $G + (1, 2)$ is connected. Such a graph must have exactly two connected components, one of which contains the vertex labelled 1, and one which contains the vertex labelled 2. We denote these connected components by G_1 and G_2, resp. See Figure 5.1.



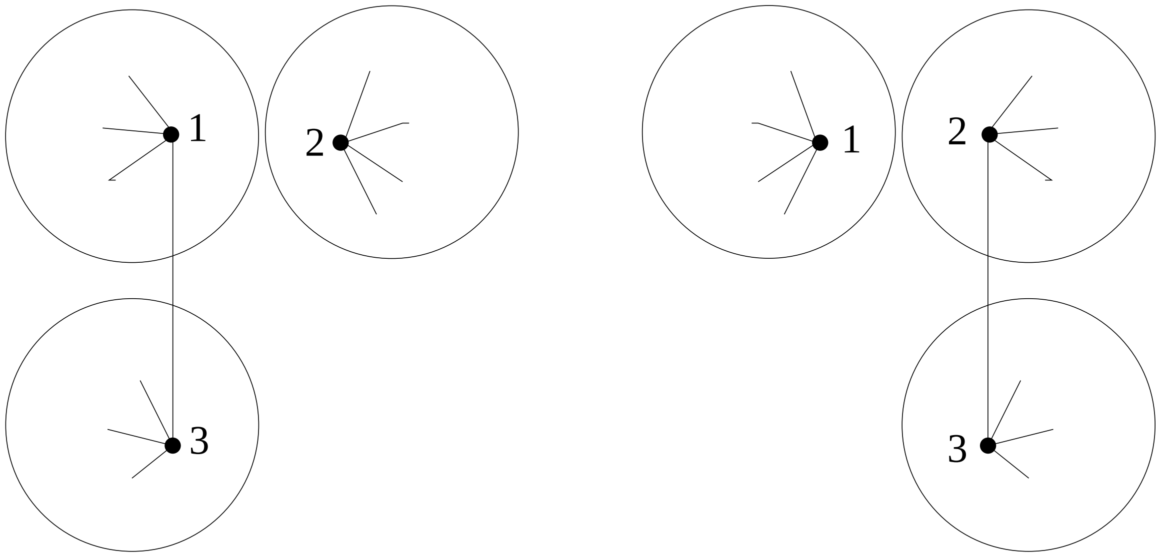
The graphs other than G^* which are unpaired in the vector field V_{12}.

Figure 5.1.

Let G be a graph other than G^* which is unpaired in V_{12}, and consider vertex 3. This vertex must either be in G_1 or G_2. Suppose that vertex 3 is in G_1. If G does not contain the edge $(1,3)$ then $G+(1,3)$ is also unpaired in V_{12}, so we can pair G with $G+(1,3)$. If vertex 3 is in G_1, then the graph G is still unpaired if and only if G contains the edge (1,3) and $G-(1,3)$ is the union of three connected components, one containing vertex 1, one containing vertex 2, and one containing vertex 3.

Similarly, if vertex 3 is in G_2 and G does not contain the edge $(2,3)$, then pair G with $G+(2,3)$. Let V_3 denote the resulting discrete vector field.

The unpaired graphs in V_3 are G^* and those that either contain the edge (1,3) and have the property that $G-(1,3)$ is the union of three connected components, one containing vertex 1, one containing vertex 2, and one containing vertex 3, or contain the edge (2,3) and have the property that $G-(2,3)$ is the union of three connected components, one containing vertex 1, one containing vertex 2, and one containing vertex 3. We illustrate these graphs in Figure 5.2. The circles in this figure indicate connected graphs.



The graphs other than G^* which are unpaired in the vector field V_3.

Figure 5.2.

Now consider the location of the vertex labelled 4, and pair any graph G which is unpaired in V_3 with $G+(1,4)$, $G+(2,4)$, or $G+(3,4)$ if possible (at most one of these graphs is unpaired in V_3). Call the resulting discrete vector field V_4. We continue in this fashion, considering in turn the vertices labelled $5, 6, \dots, n$. Let V_i denote the discrete vector field that has been constructed after the consideration of vertex i, and $V = V_n$ the final discrete vector field. When we are done the only unpaired graphs in V will be G^* and those graphs that are the union of two connected trees, one containing the vertex 1 and one containing the vertex 2. In addition, both trees have the property that the vertex labels are increasing along every ray starting from the vertex 1 or the vertex 2. There are precisely $(n-1)!$ such graphs, and they each have $n-2$ edges, and hence correspond to an $(n-3)$-simplex in $\mathcal{N}_n$.

It remains to see that the discrete vector field V is a gradient vector field, i.e., that there are no closed V-paths. We first check that V_{12} is a gradient vector field. Let $\gamma = \alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}$ denote a V_{12}-path. Then α_0 must be the "tail of an arrow", i.e., the smaller graph of some pair in V_{12}, with β_0 being the head of the arrow, i.e., $\beta_0 = \alpha_0 + (1,2)$. The simplex α_1 is a codimension-one face of β_0 other than α_0. Thus, α_1 corresponds to a graph of the form $\alpha_0 + (1,2) - e$, where e is an edge of α_0 other than (1,2). Since α_1 contains the edge $(1,2)$ it is the "head of an arrow" in V_{12}, i.e., the larger graph of some pair in V_{12}, which implies that γ cannot be continued to a longer V_{12}-path. This certainly implies that there are no closed V_{12}-paths.

The same sort of argument will work for V. Recall that V is constructed in stages, by first considering the edge (1,2) and then the vertices 3, 4, 5, $\dots$ in order. Let

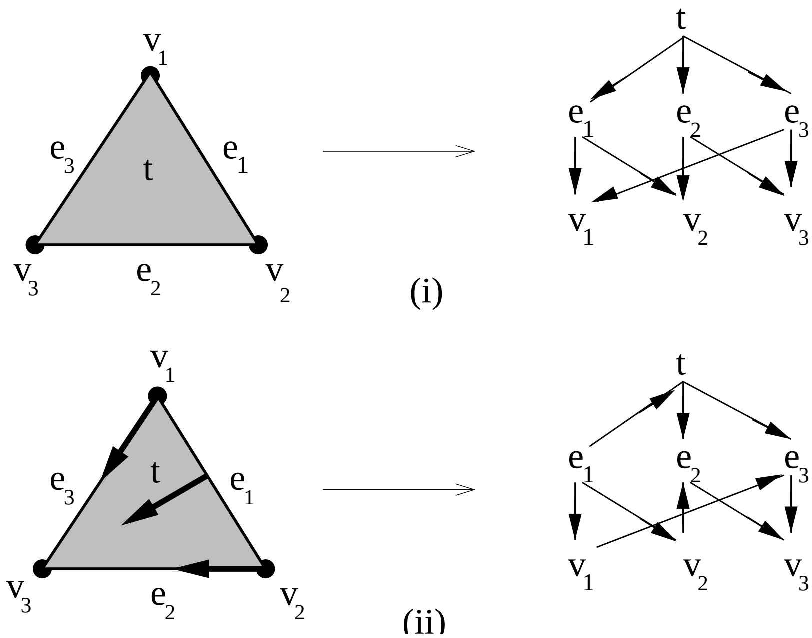
$\gamma = \alpha_0, \beta_0, \alpha_1$ denote a V-path. In particular, α_0 and β_0 must be paired in V. The reader can check that if α_0 and β_0 are first paired in V_i, $i \geq 3$, then either α_1 is the head of an arrow in V_i, in which case the V-path cannot be continued, or α_1 is paired in V_{i-1}. It follows by induction that there can be no closed V-paths.

In summary, V is a discrete gradient vector field on $\mathcal{N}_n$ with exactly one unpaired vertex, and $(n-1)!$ unpaired $(n-3)$-simplices. We can now apply Theorem 2.5 to conclude

Theorem 5.3 ([43]). *The complex $\mathcal{N}_n$ of not connected graphs on n-vertices is homotopy equivalent to the wedge of $(n-1)!$ spheres of dimension $(n-3)$.*

6. A Combinatorial Point of View

The notion of a gradient vector field has a very nice purely combinatorial description due to Chari [6], using which we can recast the Morse Theory in an appealing form. We begin with the Hasse diagram of K, that is, the partially ordered set of simplices of K ordered by the face relation. Consider the Hasse diagram as a directed graph. The vertices of the graph are in 1-1 correspondence with the simplices of K, and there is a directed edge from β to α if and only if α is a codimension-one face of β. (See Figure 6.1 (i).) Now let V be a combinatorial vector field. We modify the directed graph as follows. If $\{\alpha < \beta\} \in V$ then reverse the orientation of the edge between α and β, so that it now goes from α to β. (See Figure 6.1(ii).) A V-path can be thought of as a directed path in this modified graph. There are some directed paths in this modified Hasse diagram which are not V-paths as we have defined them. However, the following result is not hard to check.



From a discrete vector field to a directed Hasse diagram.

Figure 6.1.

Theorem 6.2. *There are no nontrivial closed V-paths if and only if there are no nontrivial closed directed paths in the corresponding directed Hasse diagram.*

Thus, in this combinatorial language, a discrete vector field is a partial matching of the Hasse diagram, and a discrete vector field is a gradient vector field if the partial matching is acyclic in the above sense. Note that using Theorem 6.2, we can see that Theorem 3.5 does follow from Theorem 3.6.

We can now restate some of our earlier theorems in this language. There is a very minor complication in that one usually includes the empty set as an element of the Hasse diagram (considered as a simplex of dimension -1) while we have not considered the empty set previously.

Theorem 6.3. *Let V be an acyclic partial matching of the Hasse diagram of K (of the sort described above — assume that the empty set is not paired with another simplex). Let u_p denote the number of unpaired p-simplices. Then K is homotopy equivalent to a CW-complex with exactly u_p cells of dimension p, for each $p \geq 0$.*

An important special case is when V is a complete matching, that is, every simplex (this time including the empty simplex) is paired with another simplex. In this case, Lemma 2.9 implies the following result.

Theorem 6.4. *Let V be a complete acyclic matching of the Hasse diagram of K, then K collapses onto a vertex, so that, in particular, K is contractible.*

This result was used in a very interesting fashion in [1].

7. The Morse Complex

In this section we will see how knowledge of the gradient paths of a discrete Morse function on a space K can allow one to strengthen the conclusions of the main theorems. In particular, rather than just knowing the number of cells in a CW decomposition for K, one can calculate the homology exactly.

Let K be a simplicial complex with a Morse function f. Let $C_p(X,\mathbb{Z})$ denote the space of p-simplicial chains, and $\mathcal{M}_p \subseteq C_p(X,\mathbb{Z})$ the span of the critical p-simplices. We refer to $\mathcal{M}_*$ as the space of Morse chains. If we let m_p denote the number of critical p-simplices, then we obviously have

$$\mathcal{M}_p \cong \mathbb{Z}^{m_p}.$$

Since homotopy equivalent spaces have isomorphic homology, the following theorem follows from Theorems 2.5 and 1.6.

Theorem 7.1. *There are boundary maps $\tilde{\partial}_d : \mathcal{M}_p \to \mathcal{M}_{d-1}$, for each d, so that*

$$\tilde{\partial}_{d-1} \circ \tilde{\partial}_d = 0$$

and such that the resulting differential complex

$$0 \longrightarrow \mathcal{M}_n \xrightarrow{\tilde{\partial}_n} \mathcal{M}_{n-1} \xrightarrow{\tilde{\partial}_{n-1}} \cdots \xrightarrow{\tilde{\partial}_1} \mathcal{M}_0 \longrightarrow 0 \tag{7.1}$$

calculates the homology of X. That is, if we define

$$H_d(\mathcal{M},\tilde{\partial}) = \frac{\operatorname{Ker}(\tilde{\partial}_d)}{\operatorname{Im}(\tilde{\partial}_{d+1})}$$

then for each d

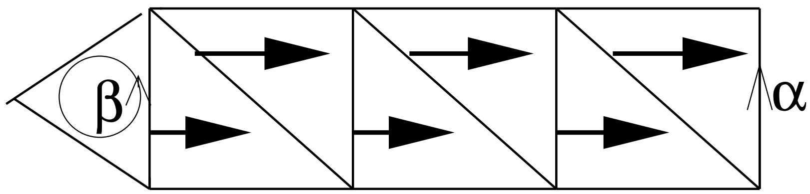
$$H_d(\mathcal{M},\tilde{\partial}) \cong H_d(X,\mathbb{Z}).$$

In fact, this statement is equivalent to the Strong Morse inequalities. The main goal of this section is to present an explicit formula for the boundary operator $\tilde{\partial}$. This requires a closer look at of the notion of a gradient path. Let α and $\tilde{\alpha}$ be p-simplices. Recall from Section 7 that a *gradient path* from $\tilde{\alpha}$ to α is a sequence of simplices

$$\tilde{\alpha} = \alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)} = \alpha$$

such that $\alpha_i < \beta_i > \alpha_{i+1}$ for each $i = 0,1,2,\ldots,r$ and $f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \cdots \geq f(\beta_r) > f(\alpha_{r+1})$. Equivalently, if V is the gradient vector field of f, we require that for each i, α_i and β_i be paired in V and $\beta_i > \alpha_{i+1} \neq \alpha_i$. In Figure 7.2 we show a single gradient path from the boundary of a critical 2-simplex β to a critical edge α, where the arrows indicate the gradient vector field V.

Given a gradient path as shown in Figure 7.2, an orientation on β induces an orientation on α. We will not state the precise definition here. The idea is that one "slides" the orientation from β along the gradient path to α. For example, for the gradient path shown in Figure 7.2, the indicated orientation on β induces the indicated orientation on α.



A gradient path from the boundary of β to α.

Figure 7.2.

We are now ready to state the desired formula.

Theorem 7.3. *Choose an orientation for each simplex. Then for any critical $(p+1)$-simplex β set*

$$\tilde{\partial}\beta = \sum_{\text{critical } \alpha^{(p)}} c_{\alpha,\beta}\alpha \tag{7.2}$$

where

$$c_{\alpha,\beta} = \sum_{\gamma \in \Gamma(\beta,\alpha)} m(\gamma)$$

where $\Gamma(\beta,\alpha)$ is the set of gradient paths which go from a maximal face of β to α. The multiplicity $m(\gamma)$ of any gradient path γ is equal to ± 1, depending on whether, given γ, the orientation on β induces the chosen orientation on α, or the opposite orientation. With this differential, the complex (7.1) computes the homology of K.

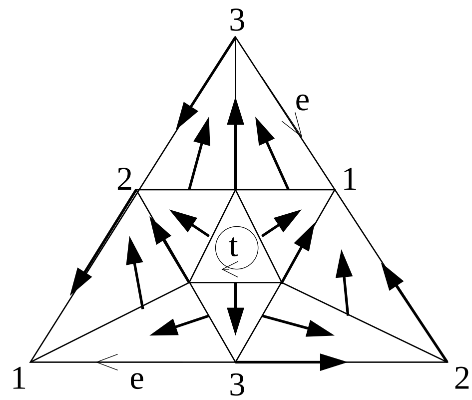
A proof of this theorem appears in Section 8 of [10]. We refer to the complex (7.1) with the differential (7.2) as the Morse complex (it goes by many different names in the literature). An extensive study of the Morse complex in the smooth category appears in [36].

We end this section with a demonstration of how the ideas of this section may be applied to the example of the real projective plane $\mathbb{P}^2$ as illustrated in Figure 4.1(ii). We saw in Section 2 how discrete Morse Theory can help us see that $\mathbb{P}^2$ has a CW decomposition with exactly one 0-cell, one 1-cell and one 2-cell. Here we will see how Morse Theory can distinguish between the spaces which have such a CW decomposition. In Figure 7.4 we redraw the gradient vector field, and indicate a chosen orientation on the critical edge e and the critical triangle t. Let us now calculate the boundary map in the Morse complex. To calculate $\tilde{\partial}(e)$, we must count all of the gradient paths from the boundary of e to v. There are precisely two such paths. Namely, following the unique gradient path beginning at each endpoint of e leads us to v. (The gradient path beginning at the head of e is the trivial path of 0 steps.) Since the orientation of e induces a $+$ orientation on the head of e, and a $-$ orientation on the tail of e, adding these two paths with their corresponding signs leads us to the formula that $\tilde{\partial}(e) = 0$. It can be seen from the illustration that there are precisely two gradient paths from the boundary of t to e, and, with the illustrated orientation for t, both induce the chosen orientation on e, so that $\tilde{\partial}(t) = 2e$. Therefore the homology of the real projective plane can be calculated from the following differential complex.

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

Thus we see that

$$H_0(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_2(\mathbb{P}^2, \mathbb{Z}) \cong 0.$$



A gradient vector field on the real projective plane.

Figure 7.4.

8. Sphere Theorems

As mentioned in our discussion at the end of Section 4, one can sometimes use discrete Morse Theory to make statements about more than just the homotopy type of the simplicial complex. One can sometimes classify the complex up to homeomorphism or combinatorial equivalence. This will be a very short section, as this topic seems a bit far from the main thrust of this paper. In addition, some terms will unfortunately have to be defined only cursorily or not at all. So far, we have not placed any restrictions on the simplicial complexes under consideration. The main idea of this section is that if our simplicial complex has some additional structure, then one may be able to strengthen the conclusion. This idea rests on some very deep work of J.H.C. Whitehead [44].

Recall that a simplicial complex K is a *combinatorial d-ball* if K and the standard d-simplex Σ_d have isomorphic subdivisions. A simplicial complex K is a *combinatorial $(d-1)$-sphere*, if K and $\dot{\Sigma}_d$ have isomorphic subdivisions (where $\dot{\Sigma}_d$ denotes the boundary of Σ_d with its induced simplicial structure). A simplicial complex K is a *combinatorial d-manifold with boundary* if the link of every vertex is either a combinatorial $(d-1)$-sphere or a combinatorial $(d-1)$-ball. The following is a special case of the main theorem of [44].

Theorem 8.1. *Let K be a combinatorial d-manifold with boundary which simplicially collapses to a vertex. Then K is a combinatorial d-ball.*

It is with this theorem (and its generalizations) that one can strengthen the conclusion of Theorem 2.5 beyond homotopy equivalence. We present just one example.

Theorem 8.2. *Let X be a combinatorial d-manifold with a discrete Morse function with exactly two critical simplices. Then X is a combinatorial d-sphere.*

The proof is quite simple (given Theorem 8.1). If X is a combinatorial d-manifold with a discrete Morse function f with exactly two critical simplices, then the critical simplices must be the minimum of f, which must occur at a vertex v, and the maximum of f, which must occur at a d-simplex α. Then $X - \alpha$ is a combinatorial d-manifold with boundary with a discrete Morse function with only a single critical simplex, namely the vertex v. It follows from Lemma 2.6 that $X - \alpha$ collapses to v. Whitehead's theorem now implies that $X - \alpha$ is a combinatorial d-ball, which implies that X is a combinatorial d-sphere.

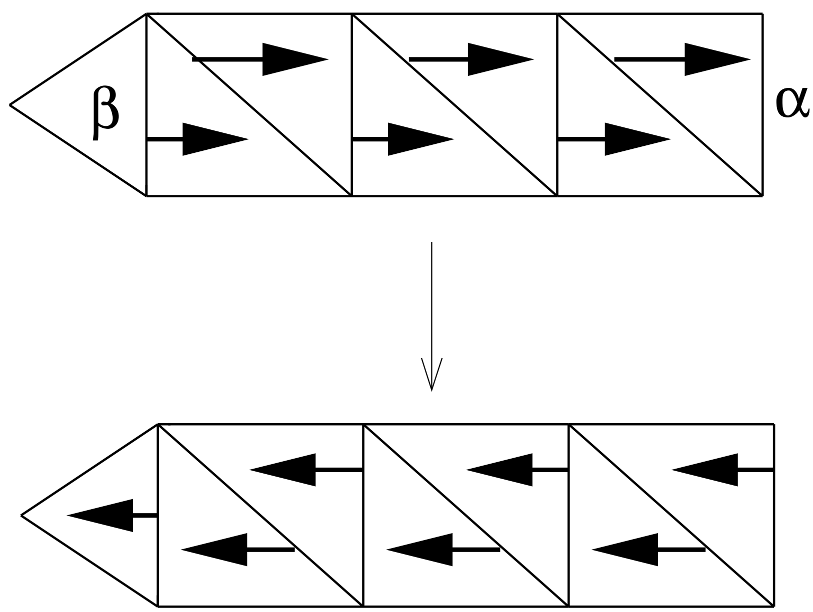
9. Cancelling Critical Points

One of the main problems in Morse Theory, whether in the combinatorial or smooth setting, is to find a Morse function for a given space with the fewest possible critical points (much of the book [38] is devoted to this topic). In general this is a very difficult problem, since, in particular, it contains the Poincaré conjecture — spheres can be recognized as those spaces which have a Morse function with precisely 2 critical points. In [31], Milnor presents Smale's proof [40] of the higher dimensional Poincaré conjecture (in fact, a proof is presented of the more general h-cobordism theorem) completely in the language of Morse Theory. Drastically oversimplifying matters, the proof of the higher Poincaré conjecture can be described as follows. Let M be a smooth manifold of dimension ≥ 5 which is homotopy equivalent to a sphere. Endow M with a (smooth) Morse function f. One then proceeds to show that the critical points of f can be cancelled out in pairs until one is left with a Morse function with exactly two critical points, which implies that M is a (topological) sphere.

A key step in this proof is the "cancellation theorem" which provides a sufficient condition for two critical points to be cancelled (see Theorem 5.4 in [31], which Milnor calls "The First Cancellation Theorem", or the original proof in [33]). In this section we will see that the analogous theorem holds for discrete Morse functions. Moreover, in the combinatorial setting the proof is much simpler. The main result is that if $\alpha^{(p)}$ and $\beta^{(p+1)}$ are 2 critical simplices, and if there is exactly 1 gradient path from the boundary of β to α, then α and β can be cancelled. More precisely,

Theorem 9.1. *Suppose f is a discrete Morse function on M such that $\beta^{(p+1)}$ and $\alpha^{(p)}$ are critical, and there is exactly one gradient path from the boundary β to α. Then there is another Morse function g on M with the same critical simplices except that α and β are no longer critical. Moreover, the gradient vector field associated to g is equal to the gradient vector field associated to f except along the unique gradient path from the boundary β to α.*

In the smooth case, the proof, either as presented originally by Morse in [33] or as presented in [31], is rather technical. In our discrete case the proof is simple. If, in the top drawing in Figure 9.2, the indicated gradient path is the only gradient path from the boundary of β to α, then we can reverse the gradient vector field along this path, replacing the figure by the vector field shown in the bottom drawing in Figure 9.2.



Cancelling critical points.

Figure 9.2.

The uniqueness of the gradient path implies that the resulting discrete vector field has no closed orbits, and hence, by Theorem 3.5, is the gradient vector field of some Morse function. Moreover, α and β are not critical for this new Morse function, while the criticality of all other simplices is unchanged. This completes the proof.

The proof in the smooth case proceeds along the same lines. However, in addition to turning around those vectors along the unique gradient path from β to α, one must also adjust all nearby vectors so that the resulting vector field is smooth. Moreover, one must check that the new vector field is the gradient of a function, so that, in particular, modifying the vectors did not result in the creation of a closed orbit. This is an example of the sort of complications which arise in the smooth setting, but which do not make an appearance in the discrete theory.

This theorem was recently put to very good use in [2], in which discrete Morse Theory is used to determine the homotopy type of some simplicial complexes arising in the study of partitions. It is fascinating, and quite pleasing, to see the same idea play a central role in two subjects, the Poincaré conjecture and the study of partitions, which seem to have so little to do with one another.

10. Morse Theory and Evasiveness

So far, we have indicated some applications of discrete Morse Theory to combinatorics and topology. We now present an application to computer science. The reader should see the reference [14] for a more complete treatment of the content of this section.

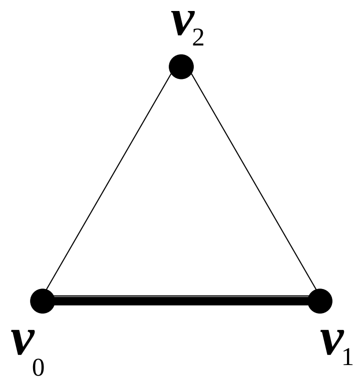
The problem we study is a topological version of a standard type of "search problem". The generalized version that we will present first appeared in [35]. Let S be an n-dimensional simplex, with vertices $v_0, v_1, \dots, v_n$, and K a subcomplex of S which is known to you. Let σ be a face of S which is not known to you. Your goal is to determine if σ is in K. In particular, you need not determine the face σ, just whether or not it is in K. You are permitted to ask questions of the form "Is v_i in σ?". You may use the answers to the questions you have already asked in determining which vertex to ask about next. Of course, you can determine if σ is in K by asking $n+1$ questions, since by asking about all $n+1$ vertices you can completely determine σ. You win this game if you answer the given question after asking fewer than $n+1$ questions.

Say that K is *nonevasive* if there is a winning strategy for this game, i.e there is a question algorithm that determines whether or not $\sigma \in K$ in fewer than $n+1$ questions, no matter what σ is. Say K is *evasive* otherwise.

Kahn, Saks and Sturtevant proved the following relationship between the evasiveness of K and its algebraic topology.

Theorem 10.1. *If $\tilde{H}_*(K) \neq 0$, where $\tilde{H}_*(K)$ denotes the reduced homology of K, then K is evasive.*

In fact, they proved something stronger, and we will come back to this point later. We illustrate the previous theorem with a simple example. Let S be the 2-simplex shown in Figure 10.2, spanned by the vertices v_0, v_1 and v_2, with K the subcomplex consisting of the edge $[v_0, v_1]$ together with the vertex v_2.



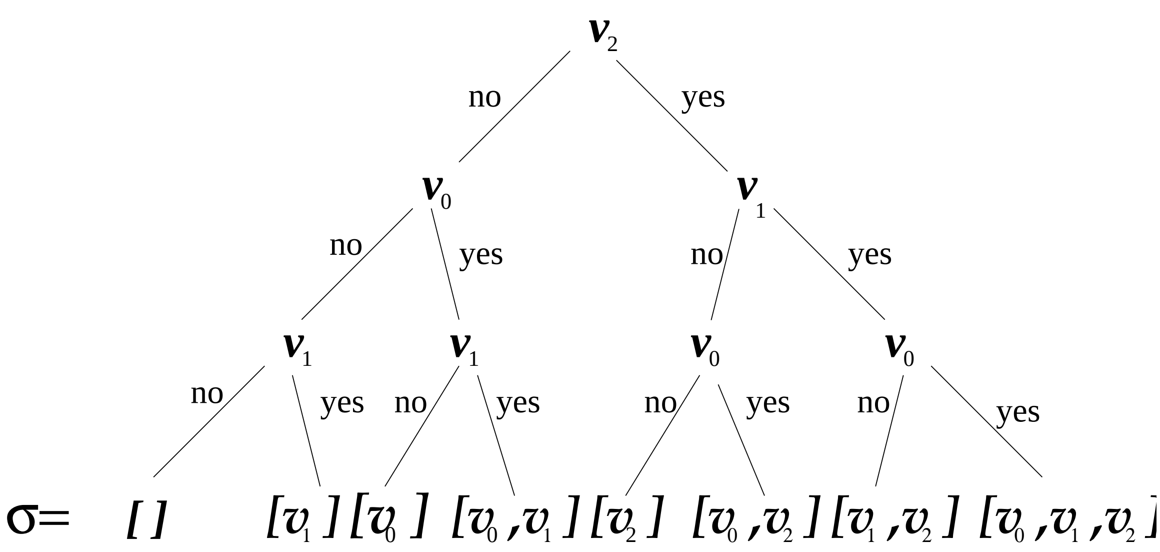
An example of an evasive subcomplex of the 2-dimensional simplex.

Figure 10.2.

A possible guessing algorithm is shown in Figure 10.3. Define an *evader* of a guessing algorithm to be a face σ of S with the property that when questions are asked in the order determined by the algorithm one must ask all three questions before it is known whether or not σ is in K. In particular, the evaders of the illustrated guessing algorithm are:

$$\sigma = [v_2], \quad [v_0, v_2]$$

Note that the subcomplex K has nonzero reduced homology, so the theorem of Kahn, Saks and Sturtevant guarantees that every guessing algorithm has some evaders.



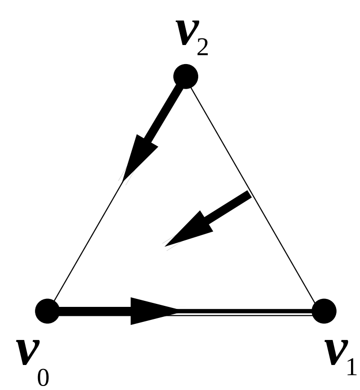
A guessing algorithm

Figure 10.3.

Morse Theory comes to the fore when one observes that a guessing algorithm induces a discrete vector field on S. For example, the guessing algorithm shown in Figure 10.3 induces the vector field

$$\begin{aligned} V = \quad & \{\, \{\emptyset < [v_1]\}, \{[v_0] < [v_0, v_1]\}, \\ & \{[v_2] < [v_0, v_2]\}, \{[v_1, v_2] < [v_0, v_1, v_2]\} \,\} \end{aligned}$$

That is, V consists of those pairs of faces of S which are not distinguished by the guessing algorithm until the last question. There is slight subtlety here in that a guessing algorithm pairs a vertex with the empty simplex $\emptyset$, while in our original definition, it was not permitted to pair a simplex with $\emptyset$. Thus, to get a true discrete vector field, we must remove this pair from V. (It is precisely this subtle point that results in the reduced homology of K being the relevant measure of topological complexity, rather than the nonreduced homology.) However, for simplicity, from now on we will simply ignore this technical point.



The vector field induced by the guessing algorithm shown in Figure 10.3.

Figure 10.4.

Theorem 10.5. *This induced vector field is always a gradient vector field.*

We will postpone the proof of this result until the end of this section.

Now restrict V to K (by taking only those pairs in V such that both simplices are in K). For example, in our example, this results in the vector field

$$V_K = \{\{[v_0] < [v_0, v_1]\}\}.$$

From the previous theorem, V has no closed orbits. Any discrete vector field consisting of a subset of the pairs of V has fewer paths, and hence also has no closed orbits. Therefore, V_K is a gradient vector field on K. Note that V pairs every face of S with another face, and hence there are no critical simplices (we are continuing to ignore for now the simplex which is paired with the emptyset). Thus, the critical simplices of V_K are precisely the simplices of K which are paired in V with a face of S which is not in K. These are precisely the simplices of K which are the evaders of the guessing algorithm.

The Morse inequalities of Theorem 2.11 (i) imply that the number of evaders in K is at least $\dim \widetilde{H}_*(K)$. Evaders occur in pairs, with each pair having one face of K and one face not in K. This yields the following quantitative refinement of Theorem 10.1.

Theorem 10.6. *For any guessing algorithm*

$$\#\ \textit{of evaders}\ \geq 2\dim \widetilde{H}_*(K)$$

Suppose that K is nonevasive. Then there is some guessing algorithm which has no evaders. From our above discussion we have seen that this implies that K has a gradient vector field with no critical simplices. Actually, this is not quite true. The gradient vector field must have a critical vertex — the vertex that is paired with the empty set — this is that minor technicality that we have been ignoring. Applying Lemma 2.6 yields the following strengthening of Theorem 10.1.

Theorem 10.7. *If K is nonevasive, then K simplicially collapses to a point.*

This theorem appears in [23]. The interested reader can consult [14] for some additional refinements of this theorem.

We end this section with a proof of Theorem 10.5. Fix a subcomplex K of an n-simplex S, and a guessing algorithm. Associate to each p-simplex α of S the sequence of integers

$$n(\alpha) = n_0(\alpha) < n_1(\alpha) < \cdots < n_p(\alpha)$$

where the $n_i(\alpha)$'s are the numbers of the questions answered "yes" if $\sigma = \alpha$.

If V is the vector field induced by the guessing algorithm and

$$\alpha_0^{(p)},\quad \beta_0^{(p+1)},\quad \alpha_1^{(p)}$$

is a V-path, then $\{\alpha_0, \beta_0\}$ is in V, which means that α_0 and β_0 are not distinguished until the $(n+1)^{st}$ question. Thus,

$$n(\beta_0) = n_0(\alpha_0) < n_1(\alpha_0) < \cdots < n_p(\alpha_0) < n+1.$$

We now observe that the vertices of a_1 are a subset of the vertices of b_0. Suppose the vertex of β_0 which is not in α_1 is the vertex tested in question $n_i(\beta_0)$. Then we must have $i \neq n+1$ (since $\alpha_0 \neq \alpha_1$). This demonstrates that

$$n(\alpha_1) = n_0(\alpha_0) < n_1(\alpha_0) < \cdots < n_{i-1}(\alpha_0) < n_i(\alpha_1) < \ldots$$

for some $i < n+1$, and such that $n_i(\alpha_1) > n_i(\alpha_0)$. Thus $n(\alpha_1) > n(\alpha_0)$ in the lexicographic order, which is sufficient to prove that there are no closed orbits. QED

11. Further Thoughts

We close this paper with some additional thoughts on the subjects discussed in this paper.

I would like to begin by encouraging the reader to take a look at the papers [24], [25], and [3]. In these papers, discrete Morse Theory is used to investigate quite interesting problems. These references were not mentioned earlier only because they did not easily fit into any of the previous sections of this paper.

There are a number of directions in which discrete Morse Theory can be extended and generalized. Here we mention a few such possibilities. In [16] we show how one can recover the ring structure of the cohomology of a simplicial complex from the point of view of discrete Morse Theory (this follows work of Betz and Cohen [4] and Fukaya [17, 18] in the smooth setting). In [34], Novikov presents a generalization of standard smooth Morse Theory in which the role of the Morse function is now played by a closed 1-form (the classical case arises when the closed 1-form is exact). In [15] we present the analogous generalization for discrete Morse Theory. In [45], Witten shows how smooth Morse Theory can be seen as arising from considerations of supersymmetry in quantum physics. In [11] we present a combinatorial version of Witten's derivation. We believe that this latter work may have greater significance. At crucial points in [45], Witten appeals to path integral arguments which are rather standard in quantum physics, but are ill-defined mathematically. In the corresponding moments in [11] what arises is a well-defined discrete sum. Perhaps the approach in [11] can find uses in the analysis of other quantum field theories.

One topic which we have only touched upon is the study of the dynamics associated to flowing along the gradient vector field associated to a discrete Morse function. In fact, an understanding of the dynamics is crucial to the proof of theorem 7.3, for example. The relevant study takes place in Section 6 of [10]. In [12] we study the dynamical properties of the flow associated to a general discrete vector field.

One area in which much work remains to be done is the investigation of discrete Morse Theory for infinite simplicial complexes. The theory as described in this paper can be applied without change to an infinite simplicial complex K endowed with a discrete Morse function f which is *proper*, i.e., one in which for each real number c the level subcomplex $K(c)$ is a finite complex. Unfortunately, properness is often an unnatural requirement when considering the infinite simplicial complexes which arise in practice. In the interesting paper [29], discrete Morse Theory is applied to the investigation of infinite simplicial complexes K which arise as a covering space of a finite simplicial complex K'. In this case, the authors restrict attention to discrete Morse functions which are lifts of a Morse function on K', and compare the number of critical simplices to the L^2-Betti numbers of K. While it appears to be too much to hope that one can develop a useful theory that applies to all infinite simplicial complexes with no restrictions on the discrete Morse function, it seems likely that there is room for very useful investigations of large classes of complexes and functions with restrictions different than those already considered.

I will close these notes with some comments of a less rigorous nature. Whether in the smooth category or the combinatorial category, Morse Theory is not essential to any problem, it is usually "only" a convenient and efficient language. Anything that can be done with Morse Theory can be done without it. It seems to me that Morse Theory takes on a special significance in three different cases. First are the cases in which Morse Theory is not intrinsic to the problem, but where the existence of such an efficient language may make the difference between whether or not one is able to see the way to the end of a problem. The best example of this in the smooth setting, I think, is the proof of the higher dimensional Poincaré conjecture ([39], [31]). Most of the applications of discrete Morse Theory mentioned in Section 5, for example, seem to fall into this category. Second are the cases in which the space one is studying comes naturally endowed with a Morse function, or a gradient vector field. Here the prime example is Bott's proof of Bott periodicity ([5], see also Part IV of [30]), resting on the fact that the loop space of a Riemannian manifold is endowed with a natural Morse function. In the combinatorial setting, I would place the Morse-theoretic examination of evasiveness of the previous section in this category. Third are the cases in which the objects under investigation can be naturally identified as the critical points of a Morse function on a larger space. Examples of this phenomenon abound in differential geometry, where one often studies extremals of energy functionals. In particular, Morse's first great triumph with Morse Theory was his investigation of the set of geodesics between two points in a Riemannian manifold ([32], see also Part III of [30]). The geodesics are precisely the critical points of the natural Morse function on the path space, and Morse used the Morse inequalities, along with a knowledge of the topology of the path space, to deduce the existence of many critical points. It is intriguing to this author that there are as yet no corresponding examples in the combinatorial setting. I know of no examples in which a collection of classically studied objects in combinatorics can be naturally identified with the critical simplices of a Morse function on some larger complex. Indeed, I believe that soon combinatorial examples of interest will be found that fit into this third category. I wonder if applications of discrete Morse theory will be found that approach the beauty, depth and fundamental significance of the applications of smooth Morse Theory mentioned in this paragraph.

On a broader note, I believe that discrete Morse Theory is only a small part of what someday will be a more complete theory of "combinatorial differential topology", although I hesitate to predict (at least in print) what form such a theory will take.

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