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DISSERTATION

Local existence results in algebras of generalised functions

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Preface

In September of 1995, Michael Grosser and Michael Kunzinger visited Lyon to meet J. F. Colombeau. During that stay, on September 16th, they spent some time in the *Café de la Ficelle* and discussed the possibility of obtaining an inverse function theorem for Colombeau functions. After one hour, they had arrived at two essential conclusions: First, the question certainly is not an easy one; and second, for quite a number of reasons, it would definitely be desirable to have such a theorem at one's disposal.

In the following years, the task of developing analogues of the classical local existence results (including the Inverse and Implicit Function Theorems) was put on the agenda of the research group DIANA (DIfferential Algebras and Nonlinear Analysis), among whose members are Michael Grosser and Michael Kunzinger. Yet for some time, other topics being more urgent, this question was not tackled.

The issue received a fresh impetus, however, from a completely independent line of research, namely an application of generalised functions in general relativity: In 1998, Roland Steinbauer (another member of DIANA) studied distributional descriptions of the geometry of impulsive gravitational waves. In particular, he set out to give rigorous mathematical meaning to the "discontinuous coordinate transformation" introduced by Roger Penrose in [Pen68], which relates a continuous representation of the corresponding metric to a discontinuous one. He and Michael Kunzinger succeeded (among other things) in regularising the metric as well as the relevant geodesic equations, solving them in an appropriate Colombeau algebra and relating the solutions to the associated distributions. The question to what extent the regularised version of the transformation in fact represents an "invertible" generalised function in the sense of Colombeau has already been addressed by Roland Steinbauer in his doctoral thesis. Some aspects of it have also been mentioned in his joint work with Michael Kunzinger (cp. Chapter 5 of [GKOS01]). Yet these partial results cannot be said to give a complete and formally satisfactory answer to the question of inversion, mainly due to the lack of a notion resp. a theory of inversion of generalised functions.

Taking a closer look at the difficulties arising in this context, we make the following simple observation: Classically, inverting some function $f : X \to Y$ in a (set-theoretic) category is at least conceptionally easy. As soon as some subset U of X is found on which f is injective, then $f|_U : U \to V := f(U)$ has at least a set-theoretic inverse and f can be said to be "invertible on U", provided the required categorical properties of $f(U)$ and the set-theoretic inverse of $f|_U$ are guaranteed by appropriate theorems given those for U and $f|_U$. For generalised functions in the Colombeau setting, however, we face a serious problem when trying to emulate the above approach: Precisely at the innocently looking step $V := f(U)$ we run into difficulties since, for $u \in \mathcal{G}(U)$, all we have at hand is the family of image sets $u_\varepsilon(U)$, which, a priori, are not in any way related to each other, due to the generality of the notion of moderate families $(u_\varepsilon)_\varepsilon$. From a conceptional point of view as well as from the point of view of important applications, it is clear that a limitation to the case where $u_\varepsilon(U) = V$ holds independently of ε would be highly insufficient. Therefore, the task of finding a suitable substitute for the notion of "image set" as well as corresponding proofs of existence of such have to constitute a central part of any inversion theory of generalised functions. The definitions of invertibility introduced in Chapter 3 of the present work reflect this particular feature.

Of course, in autumn 2003, when I was turning to Michael Grosser for a topic for my thesis, I did not know any of that. The members of the DIANA research group invited me to join a seminar on the special Colombeau algebra to give me an idea of (part of) their field of research. Sceptical at first, since my diploma thesis was more of the algebraic and number theoretic persuasion, I soon discovered that I rather enjoyed entering the analytic world of distributions and generalised functions. Michael Grosser, Michael Kunzinger and Roland Steinbauer then proposed that I undertake the business of transferring the classical local existence results to a generalised setting, with emphasis on developing an inversion theory for generalised functions that is (hopefully) applicable to, and consistent with, the work already done by Roland Steinbauer and Michael Kunzinger concerning the two descriptions of impulsive gravitational waves in general relativity. Needless to say, I accepted their offer.

This work is organised in the following way: In Chapter 1, we start with a detailed review of four classical local existence results, namely the Inverse Function Theorem, the Implicit Function Theorem, the Existence and Uniqueness Theorem for Ordinary Differential Equations and Frobenius' Theorem, studying especially their interrelations. Chapter 2 gives a condensed introduction to the special Colombeau algebra, providing the basic vocabulary and tools for the following chapters. An inversion theory for generalised functions is developed in the third chapter, including several notions of invertibility and a number of generalised inverse function theorems. Chapter 4 is devoted to applying the previously obtained results to the generalised functions modelling the "discontinuous coordinate transformation" outlined above. Finally, in Chapter 5, we present several variants of an ODE theorem in the Colombeau algebra as well as a generalised Frobenius theorem.

Many people contributed in one way or another to the success of this work, and at this point I would like to thank them all. In particular they are: First and foremost, my supervisor Michael Grosser, who expertly guided my first steps as a researcher, teaching me the subtleties of scientific work. His imagination, his intuition and his incredible insight into the workings and deeper meanings of mathematics never cease to amaze me. I am most grateful for the lot of time, effort and energy he devoted to this project. Particularly, I want to thank him for providing me with such a detailed history of the topic of my thesis, for keeping a cool head in the last stages of the writing of this work, and last but not least, for the constant supply of pastries during our working sessions.

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Furthermore, I am much obliged to Andreas Kriegl for the time he took

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Apart from my colleagues and friends at the faculty of mathematics, I am indebted to Christine Brunner who was always there when I needed a friend. Her friendship means more to me than words can say.
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Contents

Chapter 1

Classical local existence results

In classical analysis, four important local existence results are proved: the Inverse Function Theorem, the Implicit Function Theorem, the Existence and Uniqueness Theorem for Ordinary Differential Equations and Frobenius' Theorem. Since the aim of this work is to develop a theory capable of reproducing corresponding results in the setting of generalised functions (cf. Chapter 2), we will start by studying the aforesaid (classical) theorems. This approach appears—and will turn out to actually be—all the more promising taking into account that a generalised function is an equivalence class of nets of smooth maps.

The main focus of this thesis being the development of an inversion theory in the setting of the special Colombeau algebra, we will start in Section 1.1 with the proof of a "quantified" version of the classical Inverse Function Theorem (cp. [AMR83]). Section 1.2 is devoted to the study of the fact that the four main local existence results mentioned above can be, in turn, derived from each other if they are formulated for Banach spaces and C^k-functions ($k \geq 2$) acting on these. In the literature one frequently finds one of the "big four" being used to prove another (cp. e.g. [Die85] for a proof of Frobenius' Theorem employing the Existence and Uniqueness Theorem for ODEs, [KP02] for a presentation of several methods to obtain the Implicit Function Theorem, or [Kri04] and [Tes04] for a proof of the Existence and Uniqueness Theorem for ODEs using the Implicit Function Theorem). However, it seems that a complete presentation of the whole cycle of proofs does not exist so far. For this reason, and since we will take a part of this cycle as a model for similar results in the generalised setting, we will present the proofs of the equivalence of the four classical results in full detail.

1.1 The Inverse Function Theorem

In the proof the Inverse Function Theorem as stated below we will use the following two lemmata.

1.1. Lemma: *Let A be a Banach Algebra with unit e. Let a be an element of A with $\|a\| < 1$. Then the series $\sum_{k=0}^{\infty} a^k$ converges and $\sum_{k=0}^{\infty} a^k \cdot (e-a) = (e-a) \cdot \sum_{k=0}^{\infty} a^k = e$.*

Proof: We know that $\|a^k\| \leq \|a\|^k$. Since $\|a\| < 1$, it follows that $\sum_{k=0}^{\infty} \|a^k\|$ converges and, therefore, $\sum_{k=0}^{\infty} a^k$ converges. Then $(\sum_{k=0}^{\infty} a^k)(e-a) = \sum_{k=0}^{\infty} a^k - (\sum_{k=0}^{\infty} a^k)a = e$. □

1.2. Lemma: *Let A be a Banach algebra with unit e. Let $a, b \in A$ with a invertible and b such that $\|a^{-1}\| \, \|a-b\| < 1$. Then b is invertible and*

$$\|b^{-1}\| \leq \frac{\|a^{-1}\|}{1 - \|a^{-1}\| \, \|a-b\|} \qquad \text{and} \qquad \|a^{-1} - b^{-1}\| \leq \frac{\|a^{-1}\|^2 \|a-b\|}{1 - \|a^{-1}\| \, \|a-b\|}.$$

Proof: We write b as $b = a - (a-b) = a(e - a^{-1}(a-b))$. Since $\|a^{-1}(a-b)\| < 1$, we know by Lemma 1.1 that $e - a^{-1}(a-b)$ is invertible with inverse $\sum_{k=0}^{\infty}(a^{-1}(a-b))^k$. Therefore, b is invertible with inverse $b^{-1} = \sum_{k=0}^{\infty}(a^{-1}(a-b))^k \cdot a^{-1}$. Then we have

$$\|b^{-1}\| \leq \|a^{-1}\| \cdot \sum_{k=0}^{\infty}(\|a^{-1}\| \, \|(a-b)\|)^k\| = \frac{\|a^{-1}\|}{1 - \|a^{-1}\| \, \|a-b\|}.$$

Observing $a^{-1} - b^{-1} = b^{-1}(b-a)a^{-1}$, we obtain

$$\|a^{-1} - b^{-1}\| \leq \|b^{-1}\| \, \|b-a\| \, \|a^{-1}\| \leq \frac{\|a^{-1}\|^2 \|a-b\|}{1 - \|a^{-1}\| \, \|a-b\|}.$$

□

1.3. Theorem (Inverse Function Theorem): *Let X and Y be Banach spaces and U an open subset of X. Let $f \in \mathrm{C}^k(U, Y)$ for $k \in \mathbb{N} \cup \{\infty\}$ and $x_0 \in U$. If $\mathrm{D}f(x_0)$ is invertible in $\mathrm{L}(X, Y)$, then there exist open neighbourhoods W of x_0 in U and V of $y_0 := f(x_0)$ and a function $g \in \mathrm{C}^k(V, W)$ such that g is the inverse of $f|_W$.*

More precisely, let $a := \|\mathrm{D}f(x_0)^{-1}\|$. Let $b > 0$ with $ab < 1$ and $r > 0$ with $\overline{B_r(x_0)} \subseteq U$ such that

$$\|\mathrm{D}f(x_0) - \mathrm{D}f(x)\| \leq b \tag{1.1}$$

for all $x \in B_r(x_0)$. Setting $c := \frac{a}{1-ab}$, the following hold:

(1) $|x_1 - x_2| \leq c \cdot |f(x_1) - f(x_2)|$ *for all* $x_1, x_2 \in \overline{B_r(x_0)}$.

(2) $\mathrm{D}f(x)$ *is invertible and* $\|\mathrm{D}f(x)^{-1}\| \leq c$ *for all* $x \in \overline{B_r(x_0)}$.

(3) $V := f(B_r(x_0))$ *is open.*

(4) $f|_W : W \to V$ *is a* C^k*-diffeomorphism for* $W := B_r(x_0)$.

(5) $\overline{B_{\frac{r}{c}}(y_0)} \subseteq f(\overline{B_r(x_0)})$ *and* $B_{\frac{r}{c}}(y_0) \subseteq f(B_r(x_0))$.

Proof: For the sake of clarity, we establish a number of claims.

Claim 1: For all $x_1, x_2 \in \overline{B_r(x_0)}$

$$|(\mathrm{D}f(x_0)(x_1) - f(x_1)) - (\mathrm{D}f(x_0)(x_2) - f(x_2))| \leq b \cdot |x_1 - x_2|$$

holds.

Proof: Let $x_1, x_2 \in \overline{B_r(x_0)}$. By the Mean Value Theorem, we have

$$\begin{aligned} |(\mathrm{D}f(x_0)(x_1) - f(x_1)) - (\mathrm{D}f(x_0)(x_2) - f(x_2))| \leq & \\ &\leq \sup_{z \in \overline{B_r(x_0)}} \|\mathrm{D}f(x_0) - \mathrm{D}f(z)\| \cdot |x_1 - x_2| \\ &\leq b \cdot |x_1 - x_2|. \end{aligned}$$

qed.

Let $y \in Y$. Define $g^y : \overline{B_r(x_0)} \to Y$ by

$$\begin{aligned} g^y(x) :&= x + \mathrm{D}f(x_0)^{-1}\big(y - f(x)\big) \\ &= \mathrm{D}f(x_0)^{-1}(y) + \mathrm{D}f(x_0)^{-1}\big(\mathrm{D}f(x_0)(x) - f(x)\big). \end{aligned}$$

Claim 2: g^y is a contraction with Lipschitz constant ab. (Note that, at present, y is an arbitrary element of Y.)

Proof: Let $x_1, x_2 \in \overline{B_r(x_0)}$. Then, by Claim 1, we obtain

$$\begin{aligned} |g^y(x_1) - g^y(x_2)| &\leq \|\mathrm{D}f(x_0)^{-1}\| |(\mathrm{D}f(x_0)(x_1) - f(x_1)) - (\mathrm{D}f(x_0)(x_2) - f(x_2))| \\ &\leq ab \cdot |x_1 - x_2|. \end{aligned}$$

qed.

Claim 3: For $y \in \overline{B_{\frac{r}{c}}(y_0)}$ the function g^y maps $\overline{B_r(x_0)}$ into $\overline{B_r(x_0)}$.

Proof: Let $y \in \overline{B_{\frac{r}{c}}(y_0)}$ and $x \in \overline{B_r(x_0)}$. Then, by Claim 1, it follows that

$$\begin{aligned}
|g^y(x) - x_0| &\leq \|\mathrm{D}f(x_0)^{-1}\| \cdot |y - f(x_0)| + \|\mathrm{D}f(x_0)^{-1}\| \\
&\quad \cdot |(\mathrm{D}f(x_0)(x) - f(x)) - (\mathrm{D}f(x_0)(x_0) - f(x_0))| \\
&\leq a \cdot \frac{r}{c} + a \cdot b \cdot |x - x_0| \\
&\leq ar \cdot \frac{1-ab}{a} + a \cdot br \\
&= r.
\end{aligned}$$

qed.

Claim 4: For all $x_1, x_2 \in \overline{B_r(x_0)}$

$$|x_1 - x_2| \leq c \cdot |f(x_1) - f(x_2)|$$

holds.

Proof: Let $x_1, x_2 \in \overline{B_r(x_0)}$. By Claim 2, we obtain

$$\begin{aligned}
ab \cdot |x_1 - x_2| &\geq |g^y(x_1) - g^y(x_2)| \\
&= |x_1 - x_2 - \mathrm{D}f(x_0)^{-1}(f(x_1) - f(x_2))| \\
&\geq |x_1 - x_2| - \|\mathrm{D}f(x_0)^{-1}\| \cdot |(f(x_1) - f(x_2))| \\
&= |x_1 - x_2| - a \cdot |(f(x_1) - f(x_2))|.
\end{aligned}$$

Therefore, it follows from

$$a \cdot |f(x_1) - f(x_2)| \geq (1 - ab) \cdot |x_1 - x_2|$$

that

$$|x_1 - x_2| \leq \frac{a}{1-ab}|f(x_1) - f(x_2)| = c \cdot |f(x_1) - f(x_2)|.$$

qed.

Claim 5: $f|_{\overline{B_r(x_0)}} : \overline{B_r(x_0)} \to f(\overline{B_r(x_0)})$ is a homeomorphism.

Proof: The inequality of Claim 4 implies, in particular, that the restriction of f to $\overline{B_r(x_0)}$ is injective. Hence, $f|_{\overline{B_r(x_0)}} : \overline{B_r(x_0)} \to f(\overline{B_r(x_0)})$ is a bijection. From now on, we will denote $f|_{\overline{B_r(x_0)}}^{-1} : f(\overline{B_r(x_0)}) \to \overline{B_r(x_0)}$ simply by f^{-1}.

Let $y_1, y_2 \in f(\overline{B_r(x_0)})$. Then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are in $\overline{B_r(x_0)}$, and Claim 4 yields

$$|f^{-1}(y_1) - f^{-1}(y_2)| \leq c \cdot |y_1 - y_2|.$$

This shows f^{-1} to be continuous on $f(\overline{B_r(x_0)})$. *qed.*

Claim 6: $\mathrm{D}f(x)$ is invertible and $\|\mathrm{D}f(x)^{-1}\| \leq c$ for all $x \in \overline{B_r(x_0)}$.

Proof: By assumption, $\mathrm{D}f(x_0)$ is invertible. Let $x \in \overline{B_r(x_0)}$. Then

$$\|\mathrm{D}f(x_0)^{-1}\|\|\mathrm{D}f(x_0) - \mathrm{D}f(x)\| \leq ab < 1.$$

By Lemma 1.2, $\mathrm{D}f(x)$ is invertible. Moreover, Lemma 1.2 yields

$$\|\mathrm{D}f(x)^{-1}\| \leq \frac{\|\mathrm{D}f(x_0)^{-1}\|}{1 - \|\mathrm{D}f(x_0)^{-1}\|\|\mathrm{D}f(x_0) - \mathrm{D}f(x)\|} \leq \frac{a}{1-ab} = c.$$

qed.

Claim 7: $\overline{B_{\frac{r}{c}}(y_0)} \subseteq f(\overline{B_r(x_0)})$. In particular, $f(x_0) = y_0$ is interior to $f(U)$.

Proof: Writing g^y as

$$g^y(x) = x + \mathrm{D}f(x_0)^{-1}\big(y - f(x)\big),$$

where y is an arbitrary element of $\overline{B_{\frac{r}{c}}(y_0)}$, it is obvious that x is a fixed point of g^y if and only if $y = f(x)$. We already showed that g^y maps $\overline{B_r(x_0)}$ into $\overline{B_r(x_0)}$ (Claim 3). We also proved that g^y is a contraction with Lipschitz constant ab (Claim 2). From Banach's Fixed Point Theorem, it now follows that for all $y \in \overline{B_{\frac{r}{c}}(y_0)}$ there exists a unique $x \in \overline{B_r(x_0)}$ such that $f(x) = y$. Therefore, $\overline{B_{\frac{r}{c}}(y_0)}$ is contained in $f(\overline{B_r(x_0)})$. *qed.*

Claim 8: $f(B_r(x_0))$ is open in Y.

Proof: Let $x \in B_r(x_0)$. Choose $\eta > 0$ such that $B_\eta(x) \subseteq B_r(x_0)$ and

$$\|\mathrm{D}f(x) - \mathrm{D}f(z)\| \leq \frac{1}{2c}$$

for all $z \in B_\eta(x)$. Note that, by Claim 6, $\mathrm{D}f(x)^{-1}$ exists and $\|\mathrm{D}f(x)^{-1}\| \leq c$. Now apply Claim 7 with $B_\eta(x)$, $f|_{B_\eta(x)}$, x and $\frac{\eta}{2}$ replacing U, f, x_0 and r, respectively, to obtain that $f(x)$ is in the interior of $f(B_\eta(x))$ and, hence, in the interior of $f(B_r(x_0))$. *qed.*

Now let $W := B_r(x_0)$ and $V := f(B_r(x_0))$. We define $g : V \to W$ by $g(y) := f^{-1}(y)$.

Claim 9: g is differentiable and $\mathrm{D}g(y) = \mathrm{D}f(g(y))^{-1}$ for all $y \in V$.

Proof: Let $y, y_1 \in V$. Then $x := g(y)$ and $x_1 := g(y_1)$ are elements of $B_r(x_0)$. Using Claim 4 and Claim 6, we obtain

$$\begin{aligned}
&\frac{|g(y) - g(y_1) - \mathrm{D}f(g(y_1))^{-1}(y - y_1)|}{|y - y_1|} \leq \\
&\quad \leq \frac{\|\mathrm{D}f(g(y_1))^{-1}\| |y - y_1 - \mathrm{D}f(g(y_1))(g(y) - g(y_1))|}{|y - y_1|} \\
&\quad \leq c \cdot \|\mathrm{D}f(x_1)^{-1}\| \cdot \frac{|f(x) - f(x_1) - \mathrm{D}f(x_1)(x - x_1)|}{|x - x_1|} \\
&\quad \leq c^2 \cdot \frac{|f(x) - f(x_1) - \mathrm{D}f(x_1)(x - x_1)|}{|x - x_1|}.
\end{aligned}$$

Since, by Claim 5, $f|_{B_r(x_0)}$ is a homeomorphism, x converges to x_1 if and only if y converges to y_1. Hence, the last quotient tends to 0 for y converging to y_1, and the claim follows. *qed.*

Since $\mathrm{D}g = \mathrm{inv} \circ \mathrm{D}f \circ g$ (where $\mathrm{inv} : \mathrm{GL}(X, Y) \to \mathrm{GL}(Y, X)$, $\varphi \mapsto \varphi^{-1}$), it follows by the chain rule and by induction that g is k times differetiable.

Claim 10: $B_{\frac{r}{c}}(y_0) \subseteq f(B_r(x_0))$.

Proof: Let $y \in B_{\frac{r}{c}}(y_0)$. By Claim 4, we obtain

$$\begin{aligned}
|g(y) - g(f(x_0))| &\leq c \cdot |y - f(x_0)| \\
&< c \cdot \frac{r}{c} \\
&= r.
\end{aligned}$$

Thus, $g(y) \in B_r(x_0)$ and, hence, $y = f(g(y)) \in f(B_r(x_0))$. *qed.*

□

1.4. Remark: Given U, f, k, x_0, a and b as in Theorem 1.3 then, by continuity of $\mathrm{D}f$, there always exists $r > 0$ satisfying (1.1). Furthermore, note that all statements of Theorem 1.3 remain true if only $\|\mathrm{D}f(x_0)^{-1}\| \leq a$ is assumed to hold, and b and r are chosen accordingly.

The following proposition will come in handy in Chapter 3.

1.5. Proposition: *In the situation of the Inverse Function Theorem 1.3 for $X = Y = \mathbb{R}^n$ the following also hold: Let $0 < \beta < 1$ and $y_1 \in \mathbb{R}^n$ such that*

$$|y_0 - y_1| \leq (1 - \beta)\frac{r}{c}.$$

Then g^y maps $\overline{B_r(x_0)}$ into $\overline{B_r(x_0)}$ for all $y \in \overline{B_{\beta\frac{r}{c}}(y_1)}$, and $B_{\beta\frac{r}{c}}(y_1) \subseteq f(B_r(x_0))$.

Proof: The assertions follow immediately from Claims 3 and 10 of the proof of the Inverse Function Theorem 1.3 and the fact that $B_{\beta\frac{r}{c}}(y_1)$ is contained in $B_{\frac{r}{c}}(y_0)$. □

1.2 The equivalence of four local existence results

In this section we will show that the Implicit Function Theorem, the Existence and Uniqueness Theorem for ODEs, Frobenius' Theorem and the Inverse Function Theorem (all as stated below) are equivalent in the sense that each can be derived from any other. More precisely, we will prove the following circle of implications:

Implicit Function Theorem	$\overset{1.11}{\Rightarrow}$	Existence and Uniqueness Theorem for ODEs
$\Uparrow$ 1.14		$\Downarrow$ 1.12
Inverse Function Theorem	$\overset{1.13}{\Leftarrow}$	Frobenius' Theorem

Note that in order to obtain a completely closed circle of implications, all four theorems are stated below for k times differentiable functions where $k \geq 2$, the reason being that, in the proofs of (2) $\Rightarrow$ (1) in Frobenius' Theorem and of the Inverse Function Theorem, second order derivatives occur. For a more detailed discussion of the sufficiency of C^1 we refer to Remark 1.15 at the end of this section.

1.6. Theorem (Implicit Function Theorem): *Let X, Y and Z be Banach spaces and let U and V be open subsets of X resp. Y. Let $F \in \mathrm{C}^k(U \times V, Z)$ for $k \in (\mathbb{N}\backslash\{1\}) \cup \{\infty\}$ and $(x_0, y_0) \in U \times V$. If $\partial_2 F(x_0, y_0) \in \mathrm{L}(Y, Z)$ is an isomorphism, then there exist an open neighbourhood $U_1 \times V_1 \subseteq U \times V$ of (x_0, y_0) (we may suppose U_1 and V_1 to be open balls with centres x_0 resp. y_0) and a unique function $f : U_1 \to V_1$ such that $F(x, f(x)) = F(x_0, y_0)$ for all $x \in U_1$. The map f is in $\mathrm{C}^k(U_1, V_1)$ and satisfies*

$$\mathrm{D}f(x) = -(\partial_2 F(x, f(x)))^{-1} \circ \partial_1 F(x, f(x)).$$

1.7. Theorem (Existence and Uniqueness Theorem for ODEs): *Let I be an open interval, U an open subset of a Banach space X and P an open subset of another Banach space. Suppose $F \in \mathrm{C}^k(I \times U \times P, X)$ for $k \in (\mathbb{N}\backslash\{1\}) \cup \{\infty\}$ and $(t_0, x_0, p_0) \in I \times U \times P$. Then the initial value problem*

$$x'(t) = F(t, x(t), p_0), \quad x(t_0) = x_0,$$

has a $k+1$ times differentiable solution $x(t_0, x_0, p_0) : I_1 \to U$ which is unique in $\mathrm{C}^1(I_1, U)$, where $I_1 = [t_0 - a, t_0 + a]$ ($a > 0$) is contained in I. Furthermore, there exist an interval $J = [t_0 - b, t_0 + b]$ ($b > 0$) in I and an open neighbourhood $J_1 \times U_1 \times P_1 \subseteq J \times U \times P$ of (t_0, x_0, p_0) such that the map $(t_1, x_1, p_1, t) \mapsto x(t_1, x_1, p_1)(t)$ is in $\mathrm{C}^k(J_1 \times U_1 \times P_1 \times J, U)$ and $x(t_1, x_1, p_1)$ is the unique solution of the corresponding initial value problem.

1.8. Theorem (Frobenius' Theorem): *Let X and Y be Banach spaces and let U and V be open subsets of X resp. Y. Let $F : U \times V \to \mathrm{L}(X, Y)$ be k times differentiable for $k \in (\mathbb{N} \backslash \{1\}) \cup \{\infty\}$. The following are equivalent:*

(1) For all $(x_0, y_0) \in U \times V$ the initial value problem

$$\mathrm{D}f(x) = F(x, f(x)), \quad f(x_0) = y_0, \tag{1.2}$$

has a $k+1$ times differentiable solution $f(x_0, y_0) : U(x_0, y_0) \to V$ which is unique in $\mathrm{C}^1(U(x_0, y_0), V)$, where $U(x_0, y_0)$ is an open neighbourhood of x_0 in U.

(2) The integrability condition for the solvability of (1.2) is satisfied, i.e.

$$\mathrm{D}F(z)(v_1, F(z) \cdot v_1) \cdot v_2$$

is symmetric in $v_1, v_2 \in X$ for all $z \in U \times V$.

If these equivalent conditions are satisfied, then we additionally have: For fixed $(x_0, y_0) \in U \times V$ there exist an open subset W of U containing x_0 and an open neighbourhood $W_1 \times V_1 \subseteq W \times V$ of (x_0, y_0) such that the mapping $(x_1, y_1, x) \mapsto f(x_1, y_1)(x)$ is in $\mathrm{C}^k(W_1 \times V_1 \times W, V)$ and $f(x_1, y_1)$ is the unique solution of the corresponding initial value problem.

1.9. Theorem (Inverse Function Theorem): *Let X and Y be Banach spaces and U an open subset of X. Let $f \in \mathrm{C}^k(U, Y)$ for $k \in (\mathbb{N} \backslash \{1\}) \cup \{\infty\}$ and $x_0 \in U$. If $\mathrm{D}f(x_0)$ is invertible in $\mathrm{L}(X, Y)$, then there exist open neighbourhoods W of x_0 in U and V of $y_0 := f(x_0)$ and a function $g \in \mathrm{C}^k(V, W)$ such that g is the inverse of $f|_W$. Furthermore, the map g satisfies*

$$\mathrm{D}g(x) = \mathrm{D}f(g(x))^{-1}.$$

We will start with the proof of the Implicit Function Theorem implying the Existence and Uniqueness Theorem for ODEs. For this purpose we need the following

1.10. Lemma: *Let I be a compact interval, X and Y Banach spaces, U an open subset of X and $f \in \mathrm{C}^k(U, Y)$ where $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the map $f_* : \mathrm{C}(I, U) \to \mathrm{C}(I, Y)$ defined by $f_*(g) := f \circ g$ is in $\mathrm{C}^k(\mathrm{C}(I, U), \mathrm{C}(I, Y))$.*

Proof: We first show that f_* is continuous: Let $g_0 \in \mathrm{C}(I,U)$ and $\varepsilon > 0$. The point $g_0(t)$ is an element of U for all $t \in I$. Since f is continuous, for each $t \in I$ there exists some $\delta(t) > 0$ such that $|f(x) - f(g_0(t))| < \frac{\varepsilon}{2}$ for all $x \in U$ with $|x - g_0(t)| < 2\delta(t)$. The open balls $B_{\delta(t)}(g_0(t))$, $t \in I$, cover the set $g_0(I)$. Since I is compact and g_0 is continuous, the set $g_0(I)$ is compact. Hence, there exists a finite subcover $\{B_{\delta(t_j)}(g_0(t_j)) \mid 1 \le j \le n\}$ of $g_0(I)$. Define $\delta := \min_{1\le j\le n} \delta(t_j)$ and let $\|g - g_0\|_\infty < \delta$. Observe that for each $t \in I$ there is a t_j such that $|g_0(t_j) - g_0(t)| < \delta(t_j)$. Also note that

$$|g(t) - g_0(t_j)| \le |g(t) - g_0(t)| + |g_0(t) - g_0(t_j)| < 2\delta(t_j).$$

Then we have

$$\begin{aligned} |f(g(t)) - f(g_0(t))| &\le |f(g(t)) - f(g_0(t_j))| + |f(g_0(t_j)) - f(g_0(t))| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore,

$$\|f_*(g) - f_*(g_0)\|_\infty = \sup_{t\in I} |f(g(t)) - f(g_0(t))| \le \varepsilon,$$

which settles the case $k = 0$.

Next, we show that (for $k > 0$) f_* is differentiable: Let $g_0 \in \mathrm{C}(I,U)$ and $\varepsilon > 0$. We claim that the derivative $\mathrm{D}f_* : \mathrm{C}(I,U) \to \mathrm{L}(\mathrm{C}(I,X), \mathrm{C}(I,Y))$ at g_0 is given by

$$(\mathrm{D}f_*(g_0)(h))(t) = \mathrm{D}f(g_0(t))(h(t)).$$

By assumption, $\mathrm{D}f$ is continuous, and we just showed that in this case $(\mathrm{D}f)_*$ is continuous, too. Now choose $\delta > 0$ such that $\|(\mathrm{D}f)_*(h) - (\mathrm{D}f)_*(g_0)\| < \varepsilon$ for all $h \in \mathrm{C}(I,U)$ with $\|h - g_0\|_\infty < \delta$. Let $\|g - g_0\|_\infty < \delta$. Then, by the Mean Value Theorem,

$$\begin{aligned} &|f(g(t)) - f(g_0(t)) - \mathrm{D}f(g_0(t))(g(t) - g_0(t))| = \\ &\qquad = \Big| \int_0^1 \mathrm{D}f\big(g_0(t) + \sigma(g(t) - g_0(t))\big)\, d\sigma \cdot (g(t) - g_0(t)) \\ &\qquad\qquad - \mathrm{D}f(g_0(t))(g(t) - g_0(t))\Big| \\ &\qquad \le \int_0^1 |\mathrm{D}f\big(g_0(t) + \sigma(g(t) - g_0(t))\big) - \mathrm{D}f(g_0(t))|\, d\sigma \cdot |g(t) - g_0(t)|. \end{aligned}$$

For all $t \in I$ and $\sigma \in [0,1]$ we have

$$\begin{aligned}|g_0(t) + \sigma(g(t) - g_0(t)) - g_0(t)| &\leq |\sigma|\,|g(t) - g_0(t)| \\ &\leq \|g - g_0\|_\infty \\ &< \delta\end{aligned}$$

and, therefore,

$$\int_0^1 \|\mathrm{D}f\big(g_0(t) + \sigma(g(t) - g_0(t))\big) - \mathrm{D}f(g_0(t))\|\,d\sigma < \int_0^1 \varepsilon\,d\sigma = \varepsilon.$$

It follows that

$$\begin{aligned}&\frac{\|f_*(g) - f_*(g_0) - \mathrm{D}f(g_0(.))(g(.) - g_0(.))\|_\infty}{\|g - g_0\|_\infty} = \\ &\qquad = \frac{\sup_{t\in I}|f(g(t)) - f(g_0(t)) - \mathrm{D}f(g_0(t))(g(t) - g_0(t))|}{\|g - g_0\|_\infty} \\ &\qquad \leq \frac{\varepsilon \cdot \sup_{t\in I}|g(t) - g_0(t)|}{\|g - g_0\|_\infty} \\ &\qquad = \varepsilon.\end{aligned}$$

To conclude the case $k = 1$ it remains to be shown that $\mathrm{D}f_*$ is continuous: Consider the linear map $\lambda : \mathrm{C}(I, \mathrm{L}(X, Y)) \to \mathrm{L}(\mathrm{C}(I, X), \mathrm{C}(I, Y))$ defined by

$$(\lambda(T) \cdot g)(t) := T(t) \cdot g(t).$$

For $T \in \mathrm{C}(I, \mathrm{L}(X, Y))$ and $g \in \mathrm{C}(I, X)$ we have

$$\|\lambda(T) \cdot g\|_\infty = \sup_{t\in I}|T(t) \cdot g(t)| \leq \sup_{t\in I}|T(t)| \cdot |g(t)| \leq \|T\|_\infty \cdot \|g\|_\infty.$$

It follows that

$$\|\lambda\| = \sup_{T\neq 0}\ \sup_{g\neq 0} \frac{\|\lambda(T) \cdot g\|_\infty}{\|T\|_\infty \cdot \|g\|_\infty} \leq 1$$

and, therefore, λ is continuous. Now, observing that $\mathrm{D}f_* = \lambda \circ (\mathrm{D}f)_*$, the claim for $k = 1$ follows.

Finally, the general case $k > 1$ follows by induction. □

1.11. Proof that the Implicit Function Theorem 1.6 implies the Existence and Uniqueness Theorem for ODEs 1.7.

We prove the theorem in four steps. The bulk of the work will be done in the first step where we apply the Implicit Function Theorem 1.6.

Step 1: For the time being, we assume that F is independent of t and consider the initial value problem

$$x'(t) = F(x(t), p), \quad x(0) = 0, \tag{1.3}$$

for $p \in P$. Let $p_0 \in P$. We claim the existence of an open neighbourhood $P_1 \subseteq P$ of p_0, an interval $I_1 = [-a, a] \subseteq I$ with $a > 0$ and for every $p \in P_1$ a function $x(p) \in \mathrm{C}^{k+1}(I_1, U)$ which is a solution (unique in $\mathrm{C}^1(I_1, U)$) of (1.3).
Existence: We introduce a second parameter $\eta \in \mathbb{R}$ and consider the initial value problem

$$x'(t) = \eta\, F(x(t), p), \quad x(0) = 0. \tag{1.4}$$

For $\eta = 0$ the differential equation becomes trivial and we know the (unique) C^1-solution of (1.4) to be $g_0 : x \mapsto 0$. We now define

$$\begin{array}{rccc} G: & (P \times \mathbb{R}) \times \mathrm{C}^1([-1,1]\,, U) & \to & \mathrm{C}([-1,1]\,, X) \times X \\ & (p, \eta; g) & \mapsto & (g' - \eta\, F_*(g,p), \mathrm{ev}_0(g)) \end{array},$$

where $\mathrm{ev}_0 : \mathrm{C}^1([-1,1]\,, U) \to U \subseteq X$, $\mathrm{ev}_0(g) := g(0)$, is the evaluation at 0. ev_0 is smooth since it is linear and continuous. By Lemma 1.10, the function G is k times differentiable. Obviously, finding solutions of (1.4) is equivalent to finding zeros of G. For $g_0 : x \mapsto 0$ we have $G(p_0, 0; g_0) = (0,0)$ and $\partial_2 G(p_0, 0; g_0) = (\mathrm{D}, \mathrm{ev}_0)$ where $\mathrm{D}g = g'$, since both differentiation and the evaluation ev_0 are linear and continuous in g. By the Fundamental Theorem of Calculus, $\partial_2 G(p_0, 0; g_0)$ is an isomorphism in

$$\mathrm{L}(\mathrm{C}^1([-1,1]\,, X), \mathrm{C}([-1,1]\,, X) \times X)$$

with inverse

$$(h, y_0) \mapsto \left(t \mapsto \int_0^t h(s)ds + y_0 \right).$$

Applying the Implicit Function Theorem 1.6, we know there exist an open neighbourhood $(P_1 \times (-\eta_1, \eta_1)) \times A \subseteq (P \times \mathbb{R}) \times \mathrm{C}^1([-1,1]\,, U)$ of $(p_0, 0; g_0)$ and a function $f \in \mathrm{C}^k(P_1 \times (-\eta_1, \eta_1), A)$ such that

$$G(p, \eta; f(p, \eta)) = (0,0) \tag{1.5}$$

for all $(p, \eta) \in P_1 \times (-\eta_1, \eta_1)$. We may assume that A is an open ball with centre g_0, i.e. that there exists some $\varepsilon > 0$ such that

$$A = \{g \in \mathrm{C}^1([-1,1], U) \mid \max(\|g(t)\|_\infty, \|g'(t)\|_\infty) < \varepsilon\}.$$

Equation (1.5) is equivalent to

$$f(p,\eta)'(t) = \eta\, F(f(p,\eta), p)(t), \quad f(p,\eta)(0) = 0.$$

Hence, $f(p,\eta) \in A \subseteq \mathrm{C}^1([-1,1], U)$ is a solution of (1.4). To derive from that a solution of (1.3) we have to do some scaling. Fix some $a \in (0,\eta_1)$ and set $I_1 := [-a,a]$. For $p \in P_1$ we define $x(p) : I_1 \to U$ by

$$x(p)(t) := f(p,a)\left(\frac{t}{a}\right).$$

Then

$$x(p)(0) = f(p,a)(0) = 0$$

and

$$\begin{aligned} x(p)'(t) &= \frac{\partial}{\partial t}\left(f(p,a)\left(\frac{t}{a}\right)\right) \\ &= f(p,a)'\left(\frac{t}{a}\right) \cdot \frac{1}{a} \\ &= a\,F\left(f(p,a)\left(\frac{t}{a}\right), p\right) \cdot \frac{1}{a} \\ &= F(x(p)(t), p). \end{aligned}$$

So, for every $p \in P_1$ we found a solution $x(p) \in \mathrm{C}^1(I_1, U)$ of (1.3). By induction, it follows from the differential equation (1.3) that for fixed $p \in P_1$ the solution $x(p)$ is even $k+1$ times differentiable.
Uniqueness: For $p \in P_1$ let $y(p) \in \mathrm{C}^1(I_1, U)$ be another solution of (1.3). We prove uniqueness in two steps. First, we show that there exists a neighbourhood $[-c,c]$ of 0 such that $y(p) = x(p)$ on $[-c,c]$: Since $x(p)$, $y(p)$ and F are continuous and $I_1 = [-a,a]$ is compact, there exists some $c \in (0,a]$ such that

$$\begin{aligned} \|x(p)\|_{\infty,[-c,c]} &< \varepsilon, \\ \|y(p)\|_{\infty,[-c,c]} &< \varepsilon, \\ c \cdot \|F(x(p)(\,.\,), p)\|_{\infty,I_1} &< \varepsilon, \\ c \cdot \|F(y(p)(\,.\,), p)\|_{\infty,I_1} &< \varepsilon. \end{aligned} \tag{1.6}$$

Setting $f_p(t) := x(p)(c\,t)$ and $g_p(t) := y(p)(c\,t)$, we obtain, by (1.6), that f_p and g_p are elements of A. Moreover, both f_p and g_p are solutions of the implicit equation

$$G(p,a;g) = (0,0). \tag{1.7}$$

By the Implicit Function Theorem 1.6, for every $(p,c) \in P_1 \times (0,a] \subseteq P_1 \times (-\eta_1,\eta_1)$ there exists only one function in A such that the implicit equation

(1.7) holds. Therefore, $g_p = f_p$ and $y(p)(t) = g_p(\frac{t}{c}) = f_p(\frac{t}{c}) = x(p)(t)$ for all $t \in [-c, c]$.
Now, suppose that there exists $s \in I_1$ (w.l.o.g. $s > 0$) such that $x(p)(s) \neq y(p)(s)$. We set

$$\tilde{t} := \inf\{t \in (0, a] \,|\, x(p)(t) \neq y(p)(t)\} \in (0, a).$$

By the continuity of $x(p)$ and $y(p)$, we have $\tilde{z} := x(p)(\tilde{t}) = y(p)(\tilde{t})$. Setting $\tilde{x}_p(t) := x(p)(t + \tilde{t}) - \tilde{z}$ and $\tilde{y}_p(t) := y(p)(t + \tilde{t}) - \tilde{z}$, we obtain that both $\tilde{x}_p$ and $\tilde{y}_p$ are solutions of the initial value problem

$$z'(t) = F(z(t) + \tilde{z}, p), \quad z(0) = 0. \tag{1.8}$$

However, we proved above that solutions of initial value problems like (1.8) are unique on a neighbourhood of 0, yielding $\tilde{x}_p(t) = \tilde{y}_p(t)$ for t close to 0. Therefore, also $x(p)$ and $y(p)$ coincide on a neighbourhood of $\tilde{t}$ which is a contradiction to the definition of $\tilde{t}$. Hence, $x(p)(t) = y(p)(t)$ for all $t \in I_1$.
Finally, note that, since a was an arbitrary value in $(0, \eta_1)$, the restriction of $x(p)$ to any interval $\tilde{I}$ contained in I_1 with $0 \in \tilde{I}^\circ$ is the unique solution of (1.3) in $\mathrm{C}^1(\tilde{I}, U)$.

Step 2: We now claim that the mapping $(p, t) \mapsto x(p)(t)$ is in $\mathrm{C}^k(P_1 \times I_1, U)$.
For $|c| \leq 1$ we define $\bar{c} : t \mapsto c \cdot t$. Note that

$$\begin{aligned} G(p, c\,\eta; g \circ \bar{c})(t) &= \big((g \circ \bar{c})'(t) - c\,\eta\, F((g \circ \bar{c})(t), p), \mathrm{ev}_0(g \circ \bar{c})\big) \\ &= \big(c\, g'(c\,t) - c\,\eta\, F(g(c\,t), p), g(c \cdot 0)\big) \\ &= c\, G(p, \eta; g)(c\,t) \end{aligned}$$

and, therefore,

$$f(p, \eta)(c\,t) = f(p, c\,\eta)(t),$$

by the uniqueness of solutions of (1.5). Hence,

$$x(p)(t) = f(p, a)\left(\frac{t}{a}\right) = f(p, t)(1) = (\mathrm{ev}_1 \circ f)(p, t)$$

and, thus, $(p, t) \mapsto x(p)(t)$ is k times differentiable since $\mathrm{ev}_1 \circ f$ has this property.

Step 3: Now we consider the case where F is not independent of t, i.e. we look for solutions of

$$x'(t) = F(t, x(t), p), \quad x(0) = 0. \tag{1.9}$$

For $\tilde{F} := (1, F)$ and $\tilde{x}(s) := (t(s), x(s))$ the time-independent initial value problem

$$\tilde{x}'(s) = \tilde{F}(\tilde{x}(s), p), \quad \tilde{x}(0) = (0, 0), \tag{1.10}$$

is equivalent to (1.9). By Step 1, there exist an open neighbourhood $P_1 \subseteq P$ of p_0, an interval $I_1 = [-a, a] \subseteq I$ with $a > 0$ and for every $p \in P_1$ a function $\tilde{x}(p) \in \mathrm{C}^{k+1}(I_1, I \times U)$ which is the unique solution of (1.10) in $\mathrm{C}^1(I_1, I \times U)$. From Step 2, it follows that the map $(p, s) \mapsto \tilde{x}(p)(s)$ is k times differentiable. The first component of $\tilde{x}(p)$ is the identity. Hence, to obtain a solution $x(p) \in \mathrm{C}^{k+1}(I_1, U)$ of (1.9), we define $x(p)$ to be the second component of $\tilde{x}(p)$. Clearly, also, the mapping $(p, s) \mapsto x(p)(s)$ is k times differentiable.

Uniqueness: Let $\tilde{I}$ be an arbitrary interval contained in I_1 with $0 \in \tilde{I}^\circ$. For $p \in P_1$ let $y(p) \in \mathrm{C}^1(\tilde{I}, U)$ be another solution of (1.9). Then the function $\tilde{y}_p : \tilde{I} \to I \times U$ defined by $\tilde{y}_p(s) := (s, y(s))$ is continuously differentiable and a solution of (1.10). Since solutions of (1.10) are unique in $\mathrm{C}^1(\tilde{I}, I \times U)$, it follows that $\tilde{y}_p(t) = \tilde{x}(p)(t)$ and, hence, $y(p)(t) = x(p)(t)$ for all $t \in \tilde{I}$.

Step 4: Finally, we look at the initial value problem

$$x'(t) = F(t, x(t), p_0), \quad x(t_0) = x_0, \tag{1.11}$$

for some $(t_0, x_0, p_0) \in I \times U \times P$. Let $\alpha, \beta > 0$ such that $B_\alpha(t_0) \subseteq I$ and $B_\beta(x_0) \subseteq U$. Choose $\lambda \in (0, 1)$ and $\mu \in (0, \frac{\beta}{2})$ and set $\gamma := \beta - \mu$. We reduce (1.11) to a differential equation with initial condition $\tilde{x}(0) = 0$ by defining $\tilde{F} : B_{\lambda\alpha}(0) \times B_{\gamma-\mu}(0) \times (B_{(1-\lambda)\alpha}(t_0) \times B_\mu(x_0) \times P) \to X$ by

$$\tilde{F}(t, x, (t_1, x_1, p)) := F(t + t_1, x + x_1, p).$$

By Step 3, there exist an open neighbourhood $\tilde{J}_1 \times U_1 \times P_1 \subseteq B_{(1-\lambda)\alpha}(t_0) \times B_\mu(x_0) \times P$ of (t_0, x_0, p_0), an interval $\tilde{J} = [-\tilde{b}, \tilde{b}] \subseteq (-\lambda\alpha, \lambda\alpha)$ with $\tilde{b} > 0$ and for every $(t_1, x_1, p) \in \tilde{J}_1 \times U_1 \times P_1$ a function $\tilde{x}(t_1, x_1, p) \in \mathrm{C}^{k+1}(\tilde{J}, B_{\gamma-\mu}(0))$ which is a solution (unique in $\mathrm{C}^1(\tilde{J}, B_{\gamma-\mu}(0))$) of the initial value problem

$$\tilde{x}'(t) = \tilde{F}(t, \tilde{x}(t), (t_1, x_1, p)), \quad \tilde{x}(0) = 0. \tag{1.12}$$

Moreover, the mapping $(t_1, x_1, p, t) \mapsto \tilde{x}(t_1, x_1, p)(t)$ is k times differentiable. Set $b := \frac{\tilde{b}}{2}$ and let $b_1 \leq b$ such that $B_{b_1}(t_0) \subseteq \tilde{J}_1$. Set $J := [t_0 - b, t_0 + b]$ and $J_1 := (t_0 - b_1, t_0 + b_1)$. Then $J_1 \times U_1 \times P_1$ is an open neighbourhood of (t_0, x_0, p_0) in $J \times U \times P$. Now define $x : J_1 \times U_1 \times P_1 \to \mathrm{C}^{k+1}(J, B_\gamma(x_0))$ by

$$x(t_1, x_1, p)(t) := \tilde{x}(t_1, x_1, p)(t - t_1) + x_1.$$

The map is well-defined since for $t \in J$ and $t_1 \in J_1$ we have $t - t_1 \in \tilde{J}$ and for $x_1 \in U_1 \subseteq B_\mu(x_0)$ the inclusion

$$\tilde{x}(t_1, x_1, p)(\tilde{J}) + x_1 \subseteq B_{\gamma-\mu}(0) + B_\mu(x_0) = B_\gamma(x_0)$$

holds. Moreover, x is k times differentiable and for $(t_1, x_1, p) \in J_1 \times U_1 \times P_1$ we have

$$\begin{aligned}
x(t_1, x_1, p)'(t) &= \frac{\partial}{\partial t}\left(\tilde{x}(t_1, x_1, p)(t - t_1) + x_1\right) \\
&= \tilde{x}(t_1, x_1, p)'(t - t_1) \\
&= \tilde{F}(t - t_1, \tilde{x}(t_1, x_1, p)(t - t_1), (t_1, x_1, p)) \\
&= F(t - t_1 + t_1, \tilde{x}(t_1, x_1, p)(t - t_1) + x_1, p) \\
&= F(t, x(t_1, x_1, p)(t), p)
\end{aligned}$$

and further

$$x(t_1, x_1, p)(t_1) = \tilde{x}((t_1, x_1, p))(t_1 - t_1) + x_1 = x_1.$$

Thus, $x(t_1, x_1, p)$ is a solution of

$$x'(t) = F(t, x(t), p), \quad x(t_1) = x_1. \tag{1.13}$$

Uniqueness: For $(t_1, x_1, p) \in J_1 \times U_1 \times P_1$ let $y(t_1, x_1, p) \in \mathrm{C}^1(J, U)$ be another solution of (1.13). For better readability we will denote $x(t_1, x_1, p)$, $\tilde{x}(t_1, x_1, p)$ and $y(t_1, x_1, p)$ simply by x, $\tilde{x}$ resp. y. Again, we prove uniqueness in two steps. First, we show that there exists a neighbourhood $\tilde{I}$ of t_1 such that $y = x$ on $\tilde{I}$: By the continuity of y, there exists some $c \in (0, b]$ such that $\sup_{t \in \tilde{I}} |y(t) - x_1| < \gamma - \mu$ where $\tilde{I} := \overline{B_c(t_1)}$. Then the function $\tilde{y} : (\tilde{I} - t_1) \to B_{\gamma-\mu}(0)$ defined by $\tilde{y}(t) := y(t + t_1) - x_1$ is continuously differentiable and a solution of (1.12). Since solutions of (1.12) are unique in $\mathrm{C}^1(\tilde{J}, B_{\gamma-\mu}(0))$ and $\tilde{I} - t_1$ is contained in $\tilde{J}$, it follows that $\tilde{y}(t) = \tilde{x}(t)$ for $t \in \tilde{I} - t_1$ and, hence, $y(t) = \tilde{y}(t - t_1) + x_1 = \tilde{x}(t - t_1) + x_1 = x(t)$ for all $t \in \tilde{I}$.

Finally, reasoning as at the end of Step 1, we conclude that $x(t) = y(t)$ even for all $t \in J$. $\square$

1.12. Proof that the Existence and Uniqueness Theorem for ODEs 1.7 implies Frobenius' Theorem 1.8.

(1) $\Rightarrow$ (2): Let $(x_0, y_0) \in U \times V$ and let f be the (unique) solution of (1.2). Then $\mathrm{D}f = F \circ (\mathrm{id}, f)$ and $f(x_0) = y_0$. For $v_1, v_2 \in X$ we obtain

$$\begin{aligned}
\mathrm{D}^2 f(x_0)(v_1, v_2) &= (\mathrm{D}^2 f(x_0) \cdot v_1) \cdot v_2 \\
&= \mathrm{ev}_{v_2}\left(\mathrm{D}(\mathrm{D}f)(x_0) \cdot v_1\right) \\
&= \mathrm{ev}_{v_2}\left(\mathrm{D}(F \circ (\mathrm{id}, f))(x_0) \cdot v_1\right) \\
&= \mathrm{ev}_{v_2}\Big(\big(\mathrm{D}F(x_0, f(x_0)) \circ (\mathrm{id}, \mathrm{D}f(x_0))\big) \cdot v_1\Big) \\
&= \mathrm{ev}_{v_2}\Big(\mathrm{D}F\big(x_0, f(x_0)\big)\big(v_1, F(x_0, f(x_0)) \cdot v_1\big)\Big) \\
&= \mathrm{D}F(x_0, y_0)\big(v_1, F(x_0, y_0) \cdot v_1\big) \cdot v_2.
\end{aligned}$$

The last expression is symmetric in v_1 and v_2 since, by Schwarz's Theorem, $\mathrm{D}^2 f(x_0)$ has this property.

(2) $\Rightarrow$ (1): Fix $(x_0, y_0) \in U \times V$.

Existence: The idea is to reduce the "total" differential equation to an "ordinary" one with parameter, in the sense of Theorem 1.7. Then we use property (2) to show that we can construct a solution of the initial value problem (1.2) out of the solutions of the ordinary one.

Let $\eta > 0$ such that $B_\eta(x_0) \subseteq U$. Consider the initial value problem we get by studying the behaviour along lines through x_0:

$$g'(t) = F(x_0 + tv, g(t)) \cdot v, \quad g(0) = y_0, \tag{1.14}$$

where $|t| < \eta$ and $v \in B_1(0) \subseteq X$. By the Existence and Uniqueness Theorem for ODEs 1.7, there exist $\eta_1 \in (0, \eta)$ and an open neighbourhood $B_s(0) \subseteq B_1(0)$ of 0 such that the map $(v, t) \mapsto g(v, t)$ is in $\mathrm{C}^k(B_s(0) \times (-\eta_1, \eta_1), V)$, where $g(v, .) \in \mathrm{C}^{k+1}((-\eta_1, \eta_1), V)$ is a solution (unique in $\mathrm{C}^1((-\eta_1, \eta_1), V)$) of (1.14) for $v \in B_s(0)$. Now fix some $a \in (0, \eta_1)$ and set $U(x_0, y_0) := B_{as}(x_0)$. Then define $f(x_0, y_0) : U(x_0, y_0) \to V$ by

$$f(x_0, y_0)(x) := g\left(\frac{x - x_0}{a}, a\right).$$

Clearly, $f(x_0, y_0)$ is k times differentiable. In the following, we will denote $f(x_0, y_0)$ simply by f.

To prove that f is indeed a solution of (1.2), we will use the equality of $\partial_1 g(v, t) \cdot w$ and $F(x_0 + tv, g(v, t)) \cdot (tw)$. Therefore, we will show first that the map $h : (-\eta_1, \eta_1) \to Y$, defined by

$$h(t) := \partial_1 g(v, t) \cdot w - F(x_0 + tv, g(v, t)) \cdot (tw),$$

is the zero function for all $(v, w) \in B_s(0) \times X$. Since $v \mapsto g(v, 0) = y_0$ is constant and F maps to a space of linear functions, we have

$$h(0) = \partial_1 g(v, 0) \cdot w - F(x_0 + 0 \cdot v, g(v, 0)) \cdot (0 \cdot w) = 0.$$

By Schwarz's Theorem, the chain rule and the integrability condition (2), we obtain

$$\begin{aligned}
h'(t) &= \\
&= \frac{\partial}{\partial t}\Big(\partial_1 g(v,t) \cdot w - F(x_0 + tv, g(v,t)) \cdot (tw)\Big) \\
&= \frac{\partial}{\partial v}\Big(\underbrace{\frac{\partial}{\partial t} g(v,t)}_{=F(x_0+tv,g(v,t))\cdot v}\Big) \cdot w \\
&\qquad\qquad - \Big(\partial_1 F(z) \cdot v \cdot tw + \partial_2 F(z) \cdot \Big(\underbrace{\frac{\partial}{\partial t} g(v,t)}_{=F(z)\cdot v}\Big) \cdot tw + F(z) \cdot w\Big) \\
&= \frac{\partial}{\partial v}\Big(F(x_0 + tv, g(v,t)) \cdot v\Big) \cdot w - \Big(\mathrm{D}F(z) \cdot (v, F(z) \cdot v) \cdot tw + F(z) \cdot w\Big) \\
&\overset{(2)}{=} \Big(\partial_1 F(z) \cdot tw \cdot v + \partial_2 F(z) \cdot (\partial_1 g(v,t) \cdot w) \cdot v + F(z) \cdot w\Big) \\
&\qquad\qquad - \Big(\mathrm{D}F(z) \cdot (tw, F(z) \cdot tw) \cdot v + F(z) \cdot w\Big) \\
&= \partial_1 F(z) \cdot tw \cdot v + \partial_2 F(z) \cdot (\partial_1 g(v,t) \cdot w) \cdot v \\
&\qquad\qquad - \partial_1 F(z) \cdot tw \cdot v - \partial_2 F(z) \cdot (F(z) \cdot tw) \cdot v \\
&= \partial_2 F(z) \cdot (\partial_1 g(v,t) \cdot w - F(z) \cdot tw) \cdot v \\
&= \partial_2 F(z) \cdot k(t) \cdot v \\
&= \Big(\mathrm{ev}_v \circ \partial_2 F(x_0 + tv, g(v,t))\Big) \cdot h(t)
\end{aligned}$$

for all $(v, w) \in B_s(0) \times X$, where $z = (x_0 + tv, g(v, t))$. Therefore, h is a solution of a linear differential equation (with nonconstant coefficients) with initial condition $h(0) = 0$ and, thus, it follows that $h = 0$ for all $(v, w) \in B_s(0) \times X$. Observe that for $v = 0$ the initial value problem (1.14) is reduced to

$$g'(t) = 0, \quad g(0) = \tilde{y}_0.$$

Therefore, $g(0, .)$ is the constant function $t \mapsto \tilde{y}_0$. Thus, by the definition of f, we obtain

$$f(x_0) = g\Big(\frac{1}{a}(x_0 - x_0), a\Big) = y_0.$$

Finally, we have

$$\begin{aligned}
\mathrm{D}f(x)\cdot w &= \frac{\partial}{\partial x}\left(g\Big(\frac{x-x_0}{a},a\Big)\right)\cdot w\\
&= \partial_1 g\Big(\frac{x-x_0}{a},a\Big)\cdot\frac{1}{a}\cdot w\\
&= F\Big(x_0+a\cdot\frac{x-x_0}{a},g\Big(\frac{x-x_0}{a},a\Big)\Big)\cdot a\frac{1}{a}w\\
&= F(x,f(x))\cdot w
\end{aligned}$$

for all $w\in X$, which proves that f is a solution of (1.2). At last, by induction, it follows from the differential equation (1.2) that f is even $k+1$ times differentiable.

Uniqueness: Let $\tilde f\in \mathrm{C}^1(U(x_0,y_0),V)$ be another solution of (1.2). Then the function $\tilde g_v:(-a,a)\to V$ defined by $\tilde g_v(t):=\tilde f(x_0+tv)$ is continuously differentiable and a solution of (1.14) for all $v\in B_s(0)$. Since solutions of (1.14) are unique in $\mathrm{C}^1((-\eta_1,\eta_1),V)$ and $a<\eta_1$, it follows that $\tilde g_v = g(v,\,.\,)|_{(-a,a)}$ for all $v\in B_s(0)$. Hence, $\tilde f(x)=\tilde f(x_0+a\cdot\frac{x-x_0}{a})=\tilde g_{\frac{x-x_0}{a}}(a)= g\left(\frac{x-x_0}{a},a\right)=f(x)$ for all $x\in U(x_0,y_0)$.

Proof of the last statement: By the Existence and Uniqueness Theorem for ODEs 1.7, there exist $\delta>0$ and an open neighbourhood $U_1\times V_1\times B_s(0)\subseteq U\times V\times B_1(0)$ of $(x_0,y_0,0)$ such that the map $(x_1,y_1,v,t)\mapsto g(x_1,y_1,v)(t)$ is in $\mathrm{C}^k(U_1\times V_1\times B_s(0)\times(-\delta,\delta),V)$, where $g(x_1,y_1,v)$ a the solution (unique in $\mathrm{C}^1((-\delta,\delta),V)$) of the initial value problem

$$g'(t)=F(x_1+tv,g(t))\cdot v,\quad g(0)=y_1.$$

Fix $a\in(0,\delta)$ such that $B_{as}(x_0)\subseteq U_1$. Choose $\lambda\in(\frac12,1)$ and set $W:=B_{\lambda as}(x_0)$ and $W_1:=B_{(1-\lambda)as}(x_0)$. For $(x_1,y_1)\in W_1\times V_1$ we define $f(x_1,y_1):W\to V$ by

$$f(x_1,y_1)(x):=g\left(x_1,y_1,\frac{x-x_1}{a}\right)(a).$$

Then, by the same line of argument as above, $f(x_1,y_1)$ is $k+1$ times differentiable and the unique solution of (1.2) in $\mathrm{C}^1(W,V)$. The mapping $(x_1,y_1,x)\mapsto f(x_1,y_1)(x)$ is in $\mathrm{C}^k(W_1\times V_1\times W,V)$ since $(x_1,y_1,v_1,a)\mapsto g(x_1,y_1,v_1)(a)$ is in $\mathrm{C}^k(W_1\times V_1\times B_s(0),V)$. □

1.13. Proof that Frobenius' Theorem 1.8 implies the Inverse Function Theorem 1.9.

The (k times differentiable) inverse g—if it exists—satisfies

$$f(g(x))=x.$$

Differentiation with respect to x yields

$$\mathrm{D}f(g(x)) \circ \mathrm{D}g(x) = I,$$

where I denotes the identity matrix. Hence,

$$\mathrm{D}g(x) = \mathrm{D}f(g(x))^{-1}.$$

Therefore, g is a solution of above differential equation with initial condition $g(y_0) = x_0$. Thus motivated, we choose $\alpha > 0$ such that $\mathrm{D}f(x) \in \mathrm{L}(X, Y)$ is an isomorphism and

$$\|\mathrm{D}f(x) - A\| \leq \frac{1}{2\|A^{-1}\|} \tag{1.15}$$

for all $x \in B_\alpha(x_0)$ where $A := \mathrm{D}f(x_0)$. Now define

$$G : \; Y \times B_\alpha(x_0) \to \mathrm{L}(Y, X)$$

by $G(x, y) := \mathrm{D}f(y)^{-1}$. By the assumption on f, the map G is $k-1$ times differentiable. Consider the initial value problem

$$\mathrm{D}g(x) = G(x, g(x)), \quad g(y_0) = x_0. \tag{1.16}$$

We now show that the integrability condition for the solvability of (1.16) is satisfied. For $(x, y) \in Y \times B_\alpha(x_0)$ and $(v, w) \in Y \times X$ we have

$$\begin{aligned} \mathrm{D}G(x, y) \cdot (v, w) &= \big(\partial_1 G(x, y), \partial_2 G(x, y)\big) \cdot (v, w) \\ &= \partial_2 G(x, y) \cdot w \\ &= \mathrm{D}(\mathrm{inv} \circ \mathrm{D}f)(y) \cdot w \\ &= \mathrm{D}\,\mathrm{inv}\,\big(\mathrm{D}f(y)\big)\big(\mathrm{D}^2 f(y) \cdot w\big) \\ &= -\mathrm{D}f(y)^{-1} \circ \big(\mathrm{D}^2 f(y) \cdot w\big) \circ \mathrm{D}f(y)^{-1} \\ &= -G(x, y) \circ \big(\mathrm{D}^2 f(y) \cdot w\big) \circ G(x, y), \end{aligned}$$

where $\mathrm{inv} : \mathrm{GL}(X, Y) \to \mathrm{GL}(Y, X)$, $\varphi \mapsto \varphi^{-1}$. Hence, we obtain, by the bilinearity of $\mathrm{D}^2 f(y)$,

$$\begin{aligned} &\mathrm{D}G(x, y)(v_1, G(x, y) \cdot v_1) \cdot v_2 \\ &\qquad = \Big(- G(x, y) \circ \big(\mathrm{D}^2 f(y) \cdot (G(x, y) \cdot v_1)\big) \circ G(x, y)\Big) \cdot v_2 \\ &\qquad = -G(x, y)\Big(\big(\mathrm{D}^2 f(y) \cdot (G(x, y) \cdot v_1)\big) \cdot (G(x, y) \cdot v_2)\Big) \\ &\qquad = -G(x, y)\Big(\big(\mathrm{D}^2 f(y) \cdot (G(x, y) \cdot v_2)\big) \cdot (G(x, y) \cdot v_1)\Big) \\ &\qquad = \mathrm{D}G(x, y)(v_2, G(x, y) \cdot v_2) \cdot v_1 \end{aligned}$$

for all $v_1, v_2 \in Y$. Therefore, by Frobenius' Theorem, the initial value problem (1.16) has a k times differentiable solution $g : \tilde{V} \to B_\alpha(x_0)$ where $\tilde{V} \subseteq Y$ is an open neighbourhood of y_0. Let $\beta > 0$ such that $V := B_\beta(y_0)$ is contained in $\tilde{V}$. Then, for $t \in (-\beta, \beta)$ and $v \in B_1(0)$, we calculate

$$\begin{aligned}\frac{\partial}{\partial t}(f \circ g)(y_0 + tv) &= \Big(\mathrm{D}f\big(g(y_0 + tv)\big) \circ \mathrm{D}g(y_0 + tv)\Big) \cdot v \\ &= \Big(\mathrm{D}f\big(g(y_0 + tv)\big) \circ \mathrm{D}f\big(g(y_0 + tv)\big)^{-1}\Big) \cdot v \\ &= v.\end{aligned}$$

It follows that

$$(f\circ g)(y_0+tv) = f\circ g(y_0+0\cdot v)+\int_0^t \frac{\partial}{\partial s}(f\circ g)(y_0+sv)ds = y_0+v\int_0^t 1ds = y_0+tv$$

for all $t \in (-\beta, \beta)$ and $v \in B_1(0)$, establishing

$$f(g(y)) = y \tag{1.17}$$

for all $y \in V$. Set $W := B_\alpha(0) \cap f^{-1}(V)$. By the continuity of f, the set $f^{-1}(V)$ is open in the open set U and, therefore, open in X. As an intersection of open sets W is also open.
We now show that f maps W onto V: Let y be an element of V. Then $g(y) \in B_\alpha(x_0)$ and, by (1.17), also $g(y) \in f^{-1}(V)$. Hence, $g(y)$ is an element of W whose image under f is y.
Finally, we prove that f is injective on $B_\alpha(x_0)$—and, therefore, also on W: Assume that there exist $x_1 \neq x_2$ in $B_\alpha(x_0)$ such that $f(x_1) = f(x_2)$. Then, by (1.15),

$$\begin{aligned}|A \cdot (x_1 - x_2)| &= |f(x_1) - f(x_2) - A \cdot (x_1 - x_2)| \\ &\leq \int_0^1 \|\mathrm{D}f(\underbrace{x_2 + t(x_1 - x_2)}_{\in B_\alpha(x_0)}) - A\|\, dt \cdot |x_1 - x_2| \\ &\leq \frac{1}{2\|A^{-1}\|} \cdot \|A^{-1}\| \cdot |A \cdot (x_1 - x_2)| \\ &= \frac{1}{2} \cdot |A \cdot (x_1 - x_2)|.\end{aligned}$$

A being an isomorphism, the above inequality can be satisfied only if $x_1 = x_2$.
Summing up, f maps W bijectively to V and g is the (k times differentiable) inverse of $f|_W$. ∎

1.14. Proof that the Inverse Function Theorem 1.9 implies the Implicit Function Theorem 1.6.

Let $\tilde{U} \times \tilde{V} \subseteq U \times V$ be an open neighbourhood of (x_0, y_0) such that $\partial_2 F(x, y)$ is an isomorphism for all $(x, y) \in \tilde{U} \times \tilde{V}$. We define $g : \tilde{U} \times \tilde{V} \to X \times Z$ by

$$g(x, y) := (x, F(x, y)).$$

Obviously, g is k times differentiable and its derivative at (x_0, y_0) is given by

$$\mathrm{D}g(x, y) = \begin{pmatrix} \mathrm{id} & 0 \\ \partial_1 F(x_0, y_0) & \partial_2 F(x_0, y_0) \end{pmatrix}.$$

By assumption, $\partial_2 F(x_0, y_0)$ is invertible and, hence, also $\mathrm{D}g(x_0, y_0)$ has an inverse, namely

$$\mathrm{D}g(x, y)^{-1} = \begin{pmatrix} \mathrm{id} & 0 \\ -\partial_2 F(x_0, y_0) \circ \partial_1 F(x_0, y_0) & \partial_2 F(x_0, y_0)^{-1} \end{pmatrix}.$$

By the Inverse Function Theorem 1.9, there exist open neighbourhoods $\tilde{U}_1 \subseteq \tilde{U}$ of x_0 and $V_1 \subseteq \tilde{V}$ of y_0, an open neighbourhood $W \subseteq X \times Z$ of $g(x_0, y_0) = (x_0, F(x_0, y_0))$ and a k times differentiable function $h = (h_1, h_2) : W \to \tilde{U}_1 \times V_1$ such that h is the inverse of $g|_{\tilde{U}_1 \times V_1}$. We may assume that V_1 is an open ball with centre y_0. Now, set $z_0 := F(x_0, y_0)$ and choose an open neighbourhood U_1 of x_0 (e.g. an open ball with centre x_0) such that $U_1 \times \{z_0\}$ is contained in W. Let $x \in U_1$. Since g maps $\tilde{U}_1 \times V_1$ bijectively to W, there exists a unique point $(u, y) \in \tilde{U}_1 \times V_1$ such that $(u, F(u, y)) = g(u, y) = (x, z_0)$. Hence, we have $u = x$ and, therefore, $F(x, y) = z_0$. We denote the map from U_1 to V_1 that assigns y to x by f and obtain

$$F(x, f(x)) = z_0 = F(x_0, y_0)$$

for all $x \in U_1$. Since g was a bijection from $\tilde{U}_1 \times V_1$ to W, the map f is the only function from U_1 to V_1 to have this property. From

$$(x, f(x)) = g^{-1}(x, z_0) = h(x, z_0) = (h_1(x, z_0), h_2(x, z_0))$$

for $x \in U_1$, it follows that f is the map h_2 restricted to $U_1 \times \{z_0\}$ and, therefore, f is k times differentiable.

Differentiating $F(x, f(x)) = F(x_0, y_0)$ with respect to x yields

$$\partial_1 F(x, f(x)) + \partial_2 F(x, f(x)) \circ \mathrm{D}f(x) = 0$$

and, thus, we obtain the differentiation rule

$$\mathrm{D}f(x) = -\partial_2 F(x, f(x))^{-1} \circ \partial_1 F(x, f(x)).$$

for all $x \in U_1$. □

1.15. Remark: As to the question of C^1 vs. C^2, there are two more levels of interest: Which of the theorems, on the one hand, in fact hold assuming only C^1, and what, on the other hand, the proofs given above actually do show.

- The Implicit Function Theorem, the Existence and Uniqueness Theorem for ODEs and the Inverse Function Theorem hold true also for C^1-functions. As to Frobenius' Theorem, only (1) $\Rightarrow$ (2) requires C^2. For (2) $\Rightarrow$ (1) and the uniqueness statement, C^1 is sufficient.
- Our proofs given in this section are capable of handling also the C^1 case as outlined above with the one exception of the proof of the Inverse Function Theorem: Although we only require C^1 in order to apply the direction (2) $\Rightarrow$ (1) of Frobenius' Theorem, the checking of the integrability condition for the relevant differential equation forces us to use second derivatives of the function to be inverted.

Chapter 2

The special Colombeau algebra

In this chapter we will give a short description of the so-called special Colombeau algebra (cf. Definition 2.1). For the convenience of the reader, we state all the propositions and theorems we will use in the following chapters. If not stated otherwise, they are taken from [GKOS01] (Chapter 1) where proofs can also be found. However, in some exceptional cases explicit proofs are provided in this chapter. We will do so if the respective results are either slightly upgraded versions of already published theorems (in this case there is a reference to the original theorem), or if they are entirely new (auxiliary) theorems for later use.

In the following, $\mathrm{C}^k(U)$ resp. $\mathcal{D}'(U)$ denote the space of k-times continuously differentiable functions ($k \in \mathbb{N}_0 \cup \{\infty\}$) resp. of distributions on U with values in $\mathbb{K}$ where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$. For subsets A, B of a topological space $(X, \mathcal{T})$, the relation $A \subset\subset B$ is shorthand for the statement that A is a compact subset of the interior of B.

2.1 Definition of $\mathcal{G}(U)$ and embedding of $\mathcal{D}'(U)$

The theory of distributions was developed in order to handle singular (e.g. delta-like) objects in linear partial differential equations, obeying rigorous mathematical standards. However, the limitations of a purely linear theory soon became apparent (cf. [Lew57]). Unfortunately, there is no way to define a "reasonable" product on all of $\mathcal{D}'$ which still has values in $\mathcal{D}'$. For some examples on this subject consult [GKOS01]. Nonetheless, there exist various approaches to defining a multiplication of distributions that avoid these difficulties. They can be divided into two main categories (also cp. [Obe92]):

1. Intrinsic products: A product of distributions valued in $\mathcal{D}'$ is defined only for certain subsets of $\mathcal{D}'$.

2. Extrinsic products: In this case the vector space of distributions is embedded into an algebra.

We are interested in 2. More precisely, if U is an open subset of $\mathbb{R}^n$, we are looking for an associative and commutative algebra $(\mathcal{A}(U), +, \circ)$ satisfying the following:

(i) $\mathcal{D}'(U)$ is linearly embedded into $\mathcal{A}(U)$ and $f(x) \equiv 1$ is the unit in $\mathcal{A}(U)$.

(ii) There exist derivation operators $\partial_i : \mathcal{A}(U) \to \mathcal{A}(U)$ which are linear and satisfy the Leibniz rule, for $i = 1, \ldots, n$.

(iii) $\partial_i|_{\mathcal{D}'(U)}$ is the usual partial derivative.

(iv) $\circ|_{\boxed{?} \times \boxed{?}}$ is the usual product.

Condition (ii) is the statement that $\mathcal{A}(U)$ is a *differential algebra*. The impossibility result of L. Schwartz (cf. [Sch54]) shows that there exists no algebra satisfying (i)–(iv) if $\boxed{?}$ is set equal to $\mathrm{C}(U)$ in (iv). From a slight variation of his proof, it follows that the same is true if $\boxed{?}$ is replaced by $\mathrm{C}^k(U)$ for any $k \in \mathbb{N}$. However, in the 1980s, J. F. Colombeau introduced a method to construct associative, commutative differential algebras whose product coincides with the pointwise product of smooth functions (i.e. $\boxed{?} = \mathrm{C}^\infty$) and which contain the space of distributions. One of those is the special Colombeau algebra which is defined as follows:

2.1. Definition: *Let U be an open subset of $\mathbb{R}^n$. Set*

$$\begin{aligned}
\mathcal{E}(U) &:= \mathrm{C}^\infty(U)^{(0,1]}, \\
\mathcal{E}_M(U) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \mid \forall\, K \subset\subset U\ \forall\, \alpha \in \mathbb{N}_0^n\ \exists\, N \in \mathbb{N} : \\
&\qquad\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}, \\
\mathcal{N}(U) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \mid \forall\, K \subset\subset U\ \forall\, \alpha \in \mathbb{N}_0^n\ \forall\, m \in \mathbb{N} : \\
&\qquad\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.
\end{aligned}$$

Elements of $\mathcal{E}_M(U)$ resp. $\mathcal{N}(U)$ are called moderate *resp.* negligible *functions. $\mathcal{E}_M$ is a subalgebra of $\mathcal{E}(U)$, $\mathcal{N}(U)$ is an ideal in $\mathcal{E}_M(U)$. The special Colombeau algebra on U is defined as*

$$\mathcal{G}(U) := \mathcal{E}_M(U)/\mathcal{N}(U).$$

Operations on $\mathcal{E}_M(U)$, the algebra of all moderate nets of smooth functions, are defined for each ε separately. Differentiation is carried out componentwise, i.e. $\partial^\alpha(u_\varepsilon)_\varepsilon := (\partial^\alpha u_\varepsilon)_\varepsilon$. The set of all negligible nets of smooth functions $\mathcal{N}(U)$ is a differential ideal of $\mathcal{E}_M(U)$, turning $\mathcal{G}(U)$ into an associative, commutative differential algebra. Throughout this work, the term "generalised functions" refers to elements of the special Colombeau algebra.

If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ and V is an open subset of U, the *restriction* $u|_V \in \mathcal{G}(V)$ is defined as $(u_\varepsilon|_V)_\varepsilon + \mathcal{N}(V)$. We say that u vanishes on V if $u|_V = 0$ in $\mathcal{G}(V)$. The *support* of u is defined as

$$\operatorname{supp} u := \Big(\bigcup\{V \subseteq U \mid V \text{ open}, \, u|_V = 0\}\Big)^c .$$

The algebra $\mathrm{C}^\infty(U)$ can be embedded into $\mathcal{G}(U)$ via the obvious map $\sigma : f \mapsto (f)_\varepsilon + \mathcal{N}(U)$. For the embedding of $\mathcal{D}'(U)$ we will use

2.2. Theorem: *$U \mapsto \mathcal{G}(U)$ is a fine sheaf of differential algebras on $\mathbb{R}^n$.*

The main idea for embedding $\mathcal{D}'(U)$ is to regularise the distributions via convolution with a so-called mollifier:

2.3. Definition: *The space of Schwartz functions on $\mathbb{R}^n$ is defined by*

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in \mathrm{C}^\infty(\mathbb{R}^n) \mid \forall\, \alpha \in \mathbb{N}_0^n \ \forall\, p \in \mathbb{N}_0 : \sup_{x \in \mathbb{R}^n} (1 + |x|)^p \, \partial^\alpha \varphi(x) < \infty\}.$$

A mollifier is an element $\rho \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\int \rho(x)\, dx = 1,$$
$$\int x^\alpha \rho(x)\, dx = 0 \qquad \forall\, |\alpha| \geq 1.$$

We always set

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\, \rho\left(\frac{x}{\varepsilon}\right).$$

Since the convolution $w * \rho_\varepsilon$ is not defined for arbitrary $w \in \mathcal{D}'(U)$, the embedding is constructed in three steps. First, we restrict our attention to compactly supported distributions for which the convolution with ρ_ε is, in fact, defined.

2.4. Proposition: *For any open subset U of $\mathbb{R}^n$ the map*

$$\begin{array}{rcl} \iota_0 : \mathcal{E}'(U) & \to & \mathcal{G}(U) \\ w & \mapsto & ((w * \rho_\varepsilon)|_U)_\varepsilon + \mathcal{N}(U)) \end{array}$$

is a linear embedding.

2.5. Remark: In the above convolution formula, as well as in all comparable identities to follow, we tacitly assume that w is extended to all of $\mathbb{R}^n$ by setting it equal to zero outside of U.

It can be shown that on $\mathcal{D}(U)$ the embedding ι_0 coincides with σ:

2.6. Proposition: *$\iota_0|_{\mathcal{D}(U)} = \sigma$. Consequently, ι_0 is an injective homomorphism of algebras on $\mathcal{D}(U)$.*

Next, we choose an open covering $(U_\lambda)_{\lambda\in\Lambda}$ of U such that each $\overline{U_\lambda}$ is a compact subset of U, a family $(\psi_\lambda)_\lambda$ of elements of $\mathcal{D}(U)$ with $\psi_\lambda \equiv 1$ in some neighbourhood of $\overline{U_\lambda}$ and a mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$. Multiplying $w \in \mathcal{D}'(U)$ with the cut-off function ψ_λ gives a distribution with compact support. Therefore, for each $\lambda \in \Lambda$ we may apply the previously constructed $\mathcal{E}'$-embedding. Hence, for every $\lambda \in \Lambda$ we define the partial embedding

$$\begin{array}{rcccl}\iota_\lambda & : & \mathcal{D}'(U) & \to & \mathcal{G}(U_\lambda) \\ & & w & \mapsto & \Big(\big((\psi_\lambda w) * \rho_\varepsilon\big)|_{U_\lambda}\Big)_\varepsilon + \mathcal{N}(U_\lambda).\end{array}$$

Finally, the following proposition opens the way to the definition of the embedding of $\mathcal{D}'(U)$.

2.7. Proposition: *For any $w \in \mathcal{D}'(U)$, $(\iota_\lambda(w))_{\lambda\in\Lambda}$ is a coherent familiy, i.e.*

$$\iota_\lambda(w)|_{U_\lambda\cap U_\mu} = \iota_\mu(w)|_{U_\lambda\cap U_\mu}$$

for all $\lambda, \mu \in \Lambda$.

Since $\mathcal{G}$ is a sheaf, for any $w \in \mathcal{D}'(U)$ there exists a unique $u \in \mathcal{G}(U)$ with $u|_{U_\lambda} = \iota_\lambda(w)$ for all $\lambda \in \Lambda$. We will denote this u by $\iota(w)$. Then it is easy to show

2.8. Theorem: *The map $\iota : \mathcal{D}'(U) \hookrightarrow \mathcal{G}(U)$ is a linear embedding.*

Given a smooth partition of unity $(\chi_j)_{j\in\mathbb{N}}$ subordinate to $(U_\lambda)_\lambda$ (where $\operatorname{supp}\chi_j \subseteq U_{\lambda_j}$) we can even give an explicit formula for the embedding $\iota : \mathcal{D}'(U) \to \mathcal{G}(U)$:

$$\iota(w) = \left(\sum_{j=1}^{\infty} \chi_j\big((\psi_{\lambda_j} w) * \rho_\varepsilon\big)\right)_\varepsilon + \mathcal{N}(U). \tag{2.1}$$

$\mathcal{G}(U)$ indeed satisfies properties (iii) and (iv) for $\boxed{?} = \mathrm{C}^\infty(U)$:

2.9. Theorem: *If $\alpha \in \mathbb{N}_0^n$ and $w \in \mathcal{D}'(U)$, then $\partial^\alpha(\iota(w)) = \iota(\partial^\alpha w)$.*

2.10. Proposition: *$\iota|_{\mathrm{C}^\infty(U)} = \sigma$, turning $\mathrm{C}^\infty(U)$ into a subalgebra of $\mathcal{G}(U)$.*

The embedding ι is consistent with our previous construction of ι_0:

2.11. Proposition: $\iota|_{\mathcal{E}'(U)} = \iota_0$.

The embedding ι depends on the choice of the mollifier ρ. However, it does neither depend on the open covering of U nor on the family of cut-off functions nor the partition of unity:

2.12. Theorem: *The embedding $\iota : \mathcal{D}'(U) \hookrightarrow \mathcal{G}(U)$ does not depend on the particular choice of $(U_\lambda)_\lambda$, $(\psi_\lambda)_\lambda$ and $(\chi_j)_j$.*

We denote by $\hat{\iota}$ the entirety of all $\iota = \iota_U : \mathcal{D}'(U) \to \mathcal{G}(U)$, U an open subset of $\mathbb{R}^n$. Then we may state

2.13. Proposition: *$\hat{\iota} : \mathcal{D}' \to \mathcal{G}$ is a sheaf morphism (in the category of real resp. complex vector spaces), i.e. for open sets $V \subseteq U \subseteq \mathbb{R}^n$ and $w \in \mathcal{D}'(U)$ we have*

$$\iota_U(w)|_V = \iota_V(w|_V).$$

In short: $\hat{\iota}$ commutes with restrictions.

For certain types of functions and distributions a simpler embedding formula holds.

2.14. Proposition: *If $f \in \mathrm{L}^1_{loc}(U)$ is polynomially bounded (i.e. if there exist $C > 0$ and $r \in \mathbb{N}$ with $|f(x)| \leq C(1+|x|)^r$ a.e.), then*

$$\iota(f) = \big((f * \rho_\varepsilon)|_U\big)_\varepsilon + \mathcal{N}(U)$$

holds.

For any open subeset U of $\mathbb{R}^n$ we set

$$\mathcal{S}'(U) := \{w \in \mathcal{D}'(U) \,|\, \exists\, \tilde{w} \in \mathcal{S}'(\mathbb{R}^n) \text{ such that } \tilde{w}|_U = w \text{ in } \mathcal{D}'(U)\}.$$

2.15. Proposition: *Let $w \in \mathcal{S}'(U)$ and take any extension $\tilde{w} \in \mathcal{S}'(\mathbb{R}^n)$ of w. Then $\iota(w) = \big((\tilde{w} * \rho_\varepsilon)|_U\big)_\varepsilon + \mathcal{N}(U)$.*

2.16. Example: By Proposition 2.11, the image of the Dirac measure ("delta function") under the embedding ι is given by

$$\iota(\delta) = (\rho_\varepsilon)_\varepsilon + \mathcal{N}(\mathbb{R}^n).$$

According to Proposition 2.15, the Heaviside function H embedded into $\mathcal{G}(\mathbb{R})$ has the form

$$\iota(H)(x) = (H * \rho_\varepsilon(x))_\varepsilon + \mathcal{N}(\mathbb{R}) = \left(\int_{-\infty}^{x} \rho_\varepsilon(y)\,dy\right)_\varepsilon + \mathcal{N}(\mathbb{R}).$$

Finally, the following theorem provides a useful characterisation of $\mathcal{N}(U)$ as a subspace of $\mathcal{E}_M(U)$. We will apply it quite often without referring to the theorem in every instance.

2.17. Theorem: *$(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)$ is negligible if and only if the following condition is satisfied:*

$$\forall K \subset\subset U \ \forall m \in \mathbb{N}: \ \sup_{x \in K} |u_\varepsilon(x)| = O(\varepsilon^m) \text{ as } \varepsilon \to 0.$$

2.2 Composition of generalised functions

Generalised functions can be composed with smooth classical functions provided they grow not "too fast":

2.18. Definition: *The space of slowly increasing smooth functions is given by*

$$\mathcal{O}_M(\mathbb{K}^n) := \{f \in \mathrm{C}^\infty(\mathbb{K}^n) \,|\, \forall \alpha \in \mathbb{N}_0^n \ \exists N \in \mathbb{N}_0 \ \exists C > 0 : \\ |\partial^\alpha f(x)| \le C(1+|x|)^N \ \forall x \in \mathbb{K}^n\}.$$

2.19. Proposition: *If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)^m$ and $v \in \mathcal{O}_M(\mathbb{K}^m)$, then*

$$v \circ u := [(v \circ u_\varepsilon)_\varepsilon]$$

is a well-defined element of $\mathcal{G}(U)$, i.e. $(v \circ u_\varepsilon)_\varepsilon$ is moderate and $v \circ u$ is independent of the choice of the representative $(u_\varepsilon)_\varepsilon$ of u.

The composition of two arbitrary generalised functions is not defined. For instance, consider the moderate nets $(e^x)_\varepsilon$ and $(\frac{1}{\varepsilon})_\varepsilon$. Composing these two componentwise gives $\left(e^{\frac{1}{\varepsilon}}\right)_\varepsilon$, a net that no longer satisfies the $\mathcal{E}_M$-estimates. However, if, loosely speaking, the "image" of any compact subset K of U under the first "function" (note that we rather have to deal with the collection of all $u_\varepsilon(K)$, $\varepsilon \in (0,1]$) is always contained in a compact set, the composition works out fine. We will call this property "compactly bounded" or short "c-bounded". Since, plainly, an invertible generalised function must be capable of being composed with its inverse, the notion of c-boundedness will play a crucial role in this work (cf. [GKOS01] resp. below). However, there is a certain inconsistency in [GKOS01] as to the precise meaning of "c-boundedness from Ω into Ω'" of moderate nets $(u_\varepsilon)_\varepsilon$:

- Firstly, considering Ω and Ω' simply as open subsets of $\mathbb{R}^n$ resp. $\mathbb{R}^m$, Definition 1.2.7 of [GKOS01] does not require that any u_ε actually maps Ω into Ω'; only the corresponding compactness condition is stipulated ((1.1) in [GKOS01]).

- Alternatively, viewing Ω and Ω' as smooth manifolds of dimensions n resp. m in the natural way, Definition 3.2.45 of [GKOS01] can also be applied requiring—this time—that, in addition, each u_ε maps Ω into Ω'.

It seems not to be known, in general, whether these two definitions ([GKOS01] 1.2.7 resp. 3.2.50) lead to the same notion of c-bounded generalised functions from Ω into Ω'. As an additional mishap, at both places in [GKOS01] the resulting spaces of c-bounded generalised functions are denoted by $\mathcal{G}^s[\Omega, \Omega']$. Partial results on the equality of these notions have been obtained in unpublished work by M. Grosser and H. Vernaeve.
Since in the present work range spaces are focused upon in many places, we will include the requirement $u_\varepsilon(\Omega) \subseteq \Omega'$ in our definition of c-boundedness. Moreover, this leaves the door open for a "smooth" generalisation to the manifold setting.

2.20. Definition: *Let U and V be open subsets of $\mathbb{R}^n$ resp. $\mathbb{R}^m$. An element $(u_\varepsilon)_\varepsilon = (u_\varepsilon^1, \dots, u_\varepsilon^m) \in \mathcal{E}_M(U)^m$ is called compactly bounded (c-bounded) from U into V if*

(1) $\exists\, \varepsilon_0 \in (0,1]$ such that $\forall\, \varepsilon \leq \varepsilon_0 :\ u_\varepsilon(U) \subseteq V$ and

(2) $\forall\, K \subset\subset U\ \exists\, L \subset\subset V\ \exists\, \varepsilon_0 \in (0,1]$ such that $\forall\, \varepsilon \leq \varepsilon_0 :\ u_\varepsilon(K) \subseteq L$

are satisfied. The collection of c-bounded moderate functions from U into V is denoted by $\mathcal{E}_M[U, V]$.

An element of $\mathcal{G}(U)^m$ is called compactly bounded (c-bounded) if all representatives satisfy (1) *and* (2)*. The space of c-bounded generalised functions from U into V is denoted by $\mathcal{G}[U, V]$.*

2.21. Proposition: *Let $u \in \mathcal{G}(U)^m$ be c-bounded into V and let $v \in \mathcal{G}(V)$, with representatives $(u_\varepsilon)_\varepsilon$ resp. $(v_\varepsilon)_\varepsilon$. Then the composition*

$$v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon]$$

is a well-defined generalised function in $\mathcal{G}(U)$.

2.3 Point values and generalised numbers

2.22. Definition: *We set*

$$\mathcal{E}_M := \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \exists\, N \in \mathbb{N} :\ |r_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},$$
$$\mathcal{N} := \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \forall\, m \in \mathbb{N} :\ |r_\varepsilon| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.$$

$\mathcal{K} := \mathcal{E}_M/\mathcal{N}$ is called the ring of generalised numbers. In case $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{K} = \mathbb{C}$ we set $\mathcal{K} = \mathcal{R}$ resp. $\mathcal{K} = \mathcal{C}$.

$\mathcal{K}$ is embedded into every $\mathcal{G}(U)$ in the obvious way.

2.23. Definition: *For $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ and $x_0 \in U$ the point value of u at x_0 is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\mathcal{K}$.*

$\mathcal{K}$ is the ring of "constants" of $\mathcal{G}(U)$:

2.24. Proposition: *Let U be a connected open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}(U)$. Then $\mathrm{D}u = 0$ if and only if $u \in \mathcal{K}$.*

We now give a characterisation of the (multiplicatively) invertible elements of the ring $\mathcal{K}$.

2.25. Definition: *An element $r \in \mathcal{K}$ is called strictly non-zero if there exist some representative $(r_\varepsilon)_\varepsilon$ of r and an $N \in \mathbb{N}$ with $|r_\varepsilon| \geq \varepsilon^N$ for ε sufficiently small.*

2.26. Theorem: *Let $r \in \mathcal{K}$. The following are equivalent:*

(1) r is invertible.

(2) r is strictly non-zero.

In order to obtain a point value characterisation of generalised functions the definition of point values has to be extended.

2.27. Definition: *On*

$$U_M := \{(x_\varepsilon)_\varepsilon \in U^{(0,1]} \mid \exists\, N \in \mathbb{N} : \ |x_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}$$

we introduce an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \ \Leftrightarrow \ \forall\, m \in \mathbb{N} : \ |x_\varepsilon - y_\varepsilon| = O(\varepsilon^m) \text{ as } \varepsilon \to 0$$

and denote by $\tilde{U} := U_M/\sim$ the set of generalised points. The set of compactly supported points is

$$\tilde{U}_c := \{\tilde{x} = [(\tilde{x}_\varepsilon)_\varepsilon] \in \tilde{U} \mid \exists\, K \subset\subset U \ \exists\, \varepsilon_0 \in (0,1] \text{ such that } \forall\, \varepsilon \leq \varepsilon_0 : \ x_\varepsilon \in K\}.$$

A point $\tilde{x} \in \tilde{U}_c$ is called near-standard if there exists $x \in U$ such that $x_\varepsilon \to x$ as $\varepsilon \to 0$ for every representative $(x_\varepsilon)_\varepsilon$ of x.

For $U = \mathbb{K}$ we have $\tilde{\mathbb{K}} = \mathcal{K}$. Thus, we have the canonical identification $\widetilde{\mathbb{K}^n} = \tilde{\mathbb{K}}^n = \mathcal{K}^n$. For $\tilde{\mathbb{K}}_c$ we write $\mathcal{K}_c$.

2.28. Proposition: *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U \times V)$ and $\tilde{y} = [(\tilde{y}_\varepsilon)_\varepsilon] \in \tilde{V}_c$. Then the net $(u_\varepsilon(\,.\,,\tilde{y}_\varepsilon))_\varepsilon$ is in $\mathcal{E}_M(U)$ and $u(\,.\,,\tilde{y}) := [(u_\varepsilon(\,.\,,\tilde{y}_\varepsilon))_\varepsilon]$ is a well-defined element of $\mathcal{G}(U)$.*

Proof: $(u_\varepsilon(\,.\,,\tilde{y}_\varepsilon))_\varepsilon$ is the composition of $(u_\varepsilon)_\varepsilon$ with the moderate and c-bounded net $(x \mapsto (x,\tilde{y}_\varepsilon))_\varepsilon$. The proposition follows immediately from Proposition 2.21. □

Obviously, for $u \in \mathcal{G}(U)$ and $\tilde{x} \in \tilde{U}_c$, $u(\tilde{x})$ is a generalised number, the *generalised point value of u at $\tilde{x}$*. In Chapter 5 we will use the following

2.29. Corollary: *If $\tilde{v} = [(\tilde{v}_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}^n_c$, then the evaluation $\mathrm{ev}_{\tilde{v}} := [(\mathrm{ev}_{\tilde{v}_\varepsilon})_\varepsilon]$ at $\tilde{v}$ given by $\mathrm{ev}_{\tilde{v}_\varepsilon} : \mathrm{L}(\mathbb{R}^n,\mathbb{R}^m) \to \mathbb{R}^m$, $\mathrm{ev}_{\tilde{v}_\varepsilon}(A) = A \cdot v_\varepsilon$, is a well-defined element of $\mathcal{G}(\mathbb{R}^{nm})^m$.*

Proof: Apply Proposition 2.28 to $\mathrm{ev} : \mathrm{L}(\mathbb{R}^n,\mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{ev}(A,v) := A \cdot v$, and $\tilde{v}$. □

In [GKOS01], it is proved that two generalised functions are equal in the Colombeau algebra if and only if their generalised point values coincide (in the ring of generalised numbers) at all compactly supported points. S. Konjik and M. Kunzinger improved this result by showing that it is sufficient to check the values at all near-standard points (cf. [KK06]). We will need a slightly extended result:

2.30. Proposition: *Let $u \in \mathcal{G}(U \times V)$. Then*

$$u = 0 \text{ in } \mathcal{G}(U \times V) \quad \Leftrightarrow \quad u(\,.\,,\tilde{y}) = 0 \text{ in } \mathcal{G}(U) \text{ for all near-standard points } \tilde{y} \in \tilde{V}_c.$$

Proof: ($\Rightarrow$) Let $\tilde{y}$ be a near-standard point in $\tilde{V}_c$ and $L \subset\subset V$ such that $\tilde{y}_\varepsilon \in L$ for all $\varepsilon \le \varepsilon_1$ for some $\varepsilon_1 \in (0,1]$. Let $K \subset\subset U$. From

$$\sup_{x \in K} |u_\varepsilon(x,\tilde{y}_\varepsilon)| \le \sup_{x\in K, y \in L} |u_\varepsilon(x,y)| \le C\varepsilon^m,$$

it follows that $(u_\varepsilon(\,.\,,\tilde{y}_\varepsilon))_\varepsilon$ is in $\mathcal{N}(U)$.
($\Leftarrow$) If $u \neq 0$ in $\mathcal{G}(U \times V)$, then, by Theorem 2.17, we have

$$\exists\, K \subset\subset U \times V \;\exists\, m \in \mathbb{N}\; \forall\, \eta > 0\; \exists\, \varepsilon \in (0,\eta) : \sup_{(x,y)\in K} |u_\varepsilon(x,y)| > \varepsilon^m. \quad (2.2)$$

Expression (2.2) yields the existence of sequences $\varepsilon_k \searrow 0$ and $(x_k,y_k) \in K$ such that $|u_{\varepsilon_k}(x_k,y_k)| \ge \varepsilon_k^m$ for all $k \in \mathbb{N}$. Since K is compact, there exists a subsequence $(x_{k_l},y_{k_l})_{l\in\mathbb{N}}$ which converges to some $(x,y) \in K$. For $\varepsilon > 0$ we set $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) := (x_{k_l}, y_{k_l})$ for $\varepsilon \in (\varepsilon_{k_{l+1}}, \varepsilon_{k_l}]$, $l \in \mathbb{N}$. Then $\tilde{x}$ resp. $\tilde{y}$ is a near-standard point in $\tilde{U}_c$ resp. $\tilde{V}_c$. Let $\alpha, \beta > 0$ such that $\overline{B_\alpha(x)} \times \overline{B_\beta(y)} \subseteq U \times V$. For sufficiently small ε the points $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)$ are contained in $\overline{B_\alpha(x)} \times \overline{B_\beta(y)}$. Therefore, we obtain

$$\sup_{x \in \overline{B_\alpha(x)}} |u_\varepsilon(x, \tilde{y}_\varepsilon)| \geq |u_\varepsilon(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)| \geq \varepsilon^m,$$

implying $u(\,.\,, \tilde{y}) \neq 0$ in $\mathcal{G}(U)$, contradiction. $\square$

Finally, we prove some results that we will use in Chapters 3 and 5. Before doing so, a remark on notation is in order: By $\mathcal{K}^{mn}$ we denote the space of generalised $(m \times n)$-matrices over $\mathcal{K}$. $\mathcal{G}(U)^{mn}$ denotes the algebra of generalised functions u with point values in $\mathcal{K}^{mn}$. Obviously, for any $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)^m$ the derivative $\mathrm{D}u$ has $(\mathrm{D}u_\varepsilon)_\varepsilon$ as representative and, therefore, can be regarded as an element of $\mathcal{G}(U)^{mn}$.

2.31. Proposition: *Let A be a square matrix in $\mathcal{K}^{n^2}$ such that* $\det(A)$ *is strictly non-zero. Let $(A_\varepsilon)_\varepsilon$ and $(\bar{A}_\varepsilon)_\varepsilon$ be two representatives of A. Then $(A_\varepsilon^{-1})_\varepsilon$ is moderate, $(\|A_\varepsilon^{-1}\|)_\varepsilon$ is strictly non-zero and $(A_\varepsilon^{-1} - \bar{A}_\varepsilon^{-1})_\varepsilon$ is negligible.*

Proof: Let a_ε^{ij} resp. b_ε^{ij} denote the entries of A_ε resp. A_ε^{-1}. Then

$$|b_\varepsilon^{ij}| = \frac{1}{|\det(A_\varepsilon)|} |R_{ij}((a_\varepsilon^{rs})_{r,s})|,$$

where R_{ij} is a polynomial of degree $n-1$ in n^2 variables. Since $\det(A)$ is strictly non-zero, and by the moderateness of the $(a_\varepsilon^{ij})_\varepsilon$, the net $(b_\varepsilon^{ij})_\varepsilon$, and therefore $(A_\varepsilon^{-1})_\varepsilon$, is moderate.

Next, we show that $(A_\varepsilon^{-1})_\varepsilon$ is strictly non-zero: By the moderateness of $(A_\varepsilon)_\varepsilon$, there exist $C > 0$ and $N \in \mathbb{N}$ such that $\|A_\varepsilon\| \leq C\varepsilon^{-N}$ for ε sufficiently small. Therefore,

$$\frac{1}{C}\,\varepsilon^N \leq \frac{1}{\|A_\varepsilon\|} \leq \|A_\varepsilon^{-1}\|$$

yields the desired estimate.

Finally, let $(N_\varepsilon)_\varepsilon$ be an element of $\mathcal{N}^{n^2}$ such that $\bar{A}_\varepsilon = A_\varepsilon + N_\varepsilon$. Choose $C_1 > 0$, $N_1 \in \mathbb{N}$ and ε' such that

$$\|A_\varepsilon^{-1}\| \|A_\varepsilon - \bar{A}_\varepsilon\| \leq C_1 \varepsilon^{-N_1} \cdot \|N_\varepsilon\| < 1$$

for all $\varepsilon \leq \varepsilon'$. Applying Lemma 1.2, we obtain

$$\begin{aligned}\|A_\varepsilon^{-1} - \bar{A}_\varepsilon^{-1}\| &\leq \frac{\|A_\varepsilon^{-1}\|^2 \|A_\varepsilon - \bar{A}_\varepsilon\|}{1 - \|A_\varepsilon^{-1}\| \, \|A_\varepsilon - \bar{A}_\varepsilon\|} \\ &\leq \frac{C_1^2 \varepsilon^{-2N_1} \cdot C_2 \varepsilon^m}{1 - C_1^2 \varepsilon^{-2N_1} \cdot C_2 \varepsilon^m} \\ &\leq C_3 \varepsilon^{m+2N_1}\end{aligned}$$

for constants $C_2, C_3 > 0$, arbitrary $m \in \mathbb{N}$ and sufficiently small ε. This establishes the negligibility of $(A_\varepsilon^{-1} - \bar{A}_\varepsilon^{-1})_\varepsilon$. □

2.32. Proposition: *Let U be an open subset of $\mathbb{R}^l$ and $a = [(a_\varepsilon)_\varepsilon] \in \mathcal{G}(U)^{tm}$ and $b = [(b_\varepsilon)_\varepsilon] \in \mathcal{G}(U)^{mn}$. We define $c_\varepsilon : U \to \mathrm{L}(\mathbb{K}^n, \mathbb{K}^t)$ by $c_\varepsilon(x) := a_\varepsilon(x) \circ b_\varepsilon(x)$. Then the net $(c_\varepsilon)_\varepsilon$ is moderate and $c := [(c_\varepsilon)_\varepsilon]$ is a well-defined element of $\mathcal{G}(U)^{tn}$.*

Proof: The composition $\mathrm{comp} : \mathrm{L}(\mathbb{K}^m, \mathbb{K}^t) \times \mathrm{L}(\mathbb{K}^n, \mathbb{K}^m) \to \mathrm{L}(\mathbb{K}^n, \mathbb{K}^t)$ defined by $\mathrm{comp}(A, B) := A \circ B$ is smooth and bilinear. Thus, comp is an element of $\mathcal{O}_M \left(\mathbb{K}^{tm} \times \mathbb{K}^{mn}\right)^{tn}$. By Proposition 2.19, the composition $c = \mathrm{comp} \circ (a, b)$ is a well-defined element of $\mathcal{G}(U)^{tn}$. □

The next result presents an exponential law for generalised functions with values in the space of generalised matrices over $\mathbb{R}$.

2.33. Proposition: *Let U be an open subset of $\mathbb{R}^l$. If $u := [(u_\varepsilon)_\varepsilon]$ is in $\mathcal{G}(U)^{mn}$, then $\hat{u} := [(\hat{u}_\varepsilon)_\varepsilon]$ defined by $\hat{u}_\varepsilon : U \times \mathbb{R}^n \to \mathbb{R}^m$, $\hat{u}_\varepsilon(x, v) := u_\varepsilon(x) \cdot v$ is in $\mathcal{G}(U \times \mathbb{R}^n)^m$. Conversely, if $w \in \mathcal{G}(U \times \mathbb{R}^n)^m$ such that there exists a representative $(w_\varepsilon)_\varepsilon$ with w_ε linear in the second component for all $\varepsilon \in (0, 1]$, then $\check{w} := [(\check{w}_\varepsilon)_\varepsilon]$ defined by $\check{w}_\varepsilon : U \to \mathbb{R}^{mn}$, $\check{w}_\varepsilon(x) := w_\varepsilon(x, \,.\,)$ is in $\mathcal{G}(U)^{mn}$.*

Proof: Let $u := [(u_\varepsilon)_\varepsilon]$ be in $\mathcal{G}(U)^{mn}$. Define $\tilde{u}_\varepsilon : U \times \mathbb{R}^n \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$, $\tilde{u}_\varepsilon(x, v) := u_\varepsilon(x)$, and $g : U \times \mathbb{R}^n \to \mathrm{L}(\mathbb{R}, \mathbb{R}^n)$, $g(x, v) := v$. By Proposition 2.32, it follows that $\hat{u}$, given by $\hat{u}_\varepsilon(x, v) = \tilde{u}_\varepsilon(x, v) \circ g_\varepsilon(x, v) = u_\varepsilon(x) \cdot v$, is a well-defined element of $\mathcal{G}(U \times \mathbb{R}^n)^m$.

Conversely, let $w \in \mathcal{G}(U \times \mathbb{R}^n)^m$ such that there exists a representative $(w_\varepsilon)_\varepsilon$ with w_ε linear in the second component for all $\varepsilon \in (0, 1]$. By the classical exponential law, the functions $\check{w}_\varepsilon : U \to \mathbb{R}^{mn}$, $\check{w}_\varepsilon(x) := w_\varepsilon(x, \,.\,)$, are smooth for all ε. Let $K \subset\subset U$. By the moderateness of $(w_\varepsilon)_\varepsilon$, it follows that

$$\sup_{x \in K} |\partial^\alpha \check{w}_\varepsilon(x)| = \sup_{x \in K} |\partial_1^\alpha w_\varepsilon(x, \,.\,)| = \sup_{\substack{x \in K \\ |v| \leq 1}} |\partial_1^\alpha w_\varepsilon(x, v)| \leq C \cdot \varepsilon^{-N}$$

for all $\alpha \in \mathbb{N}_0^l$. For any $(n_\varepsilon)_\varepsilon \in \mathcal{N}(U \times \mathbb{R}^n)^m$ that is linear in the second component we also have $(\check{n}_\varepsilon)_\varepsilon \in \mathcal{N}(U)^{mn}$, so $\check{u}$ is well-defined. □

2.4 Association

The terms "associated" and "distributional shadow" (to be defined below) will be used in Chapter 4.

2.34. Definition: *Two elements u and v of $\mathcal{G}(U)$ are called associated (denoted by $u \approx v$) if*

$$\lim_{\varepsilon\to 0} \int\limits_U (u_\varepsilon(x) - v_\varepsilon(x))\, \varphi(x)\, dx = 0 \qquad \forall\, \varphi \in \mathcal{D}(U)$$

for some (and therefore all) representative(s) $(u_\varepsilon)_\varepsilon$ of u resp. $(v_\varepsilon)_\varepsilon$ of v.

Let $u \in \mathcal{G}(U)$ and $w \in \mathcal{D}'(U)$ and suppose that $u \approx \iota(w)$. Then u is said to admit w as associated distribution and w is called distributional shadow of u. In this case we simply write $u \approx w$.

The distributional shadow of u is uniquely determined (if it exists):

2.35. Proposition: *If $w \in \mathcal{D}'(U)$ and $\iota(w) \approx 0$, then $w = 0$.*

On $\mathcal{K}$, the ring of constants in $\mathcal{G}(U)$, $\approx$ induces an equivalence relation we also denote by $\approx$. We explicitly rephrase this in

2.36. Definition: *Two elements r and s of $\mathcal{K}$ are called associated (denoted by $r \approx s$) if $(r_\varepsilon - s_\varepsilon) \to 0$ as $\varepsilon \to 0$ for some (and therefore all) representative(s) $(r_\varepsilon)_\varepsilon$ of r resp. $(s_\varepsilon)_\varepsilon$ of s.*

If there exists some $a \in \mathbb{K}$ with $r \approx a$, then a is called associated number or shadow of r.

Finally, we study the relation between $f \in \mathrm{C}^k(U)$ and $\iota(f)$.

2.37. Definition: *Let $u \in \mathcal{G}(U)$ and $f \in \mathrm{C}^k(U)$ for $k \in \mathbb{N}_0 \cup \{\infty\}$. The generalised function u is called C^k-associated with f (denoted by $u \approx_k f$) if for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ and one (hence any) representative $(u_\varepsilon)_\varepsilon$ of u*

$$\partial^\alpha u_\varepsilon \to \partial^\alpha f$$

for $\varepsilon \to 0$ uniformly on compact subsets of U.

2.38. Lemma: *Let $g \in \mathrm{C}(\mathbb{R}^n)$ be bounded, $\rho \in \mathrm{L}^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$. Then, for $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$,*

$$g * \rho_\varepsilon \to g$$

for $\varepsilon \to 0$ uniformly on compact sets.

Proof: Let $K \subset\subset \mathbb{R}^n$ and $\eta > 0$. Choose N such that

$$\int\limits_{|z|>N} |\rho(z)|\, dz < \frac{\eta}{4\|g\|_\infty}.$$

The function g is uniformly continuous on the compact set $K + \overline{B_N(0)}$. Hence, there exists some $\varepsilon_0 \in (0, 1]$ such that

$$|g(x - \varepsilon z) - g(x)| < \frac{\eta}{2\|\rho\|_1}$$

for all $x \in K$, $z \in \overline{B_N(0)}$ and $\varepsilon \leq \varepsilon_0$. Then, for $x \in K$ and substituting z for $\frac{y}{\varepsilon}$, we obtain

$$\begin{aligned}
&|(g*\rho_\varepsilon)(x) - g(x)| \leq \\
&\quad \leq \int\limits_{\mathbb{R}^n} |g(x-y) - g(x)||\rho_\varepsilon(y)|\, dy \\
&\quad = \int\limits_{|z|\leq N} \underbrace{|g(x-\varepsilon z) - g(x)|}_{\leq \frac{\eta}{2\|\rho\|_1}} |\rho(z)|\, dz + \int\limits_{|z|>N} \underbrace{|g(x-\varepsilon z) - g(x)|}_{\leq 2\,\|g\|_\infty} |\rho(z)|\, dz \\
&\quad < \eta
\end{aligned}$$

for all $\varepsilon \leq \varepsilon_0$. □

2.39. Proposition: *Let $f \in \mathrm{C}^k(U)$ for $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\iota(f)$ is C^k-associated with f.*

Proof: We will show the convergence for the representative occurring in (2.1), i.e.

$$f_\varepsilon := \sum_{j=1}^{\infty} \chi_j \cdot \big((\psi_{\lambda_j} f) * \rho_\varepsilon\big).$$

Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. The function $\partial^\alpha(\psi_{\lambda_j} f)$ is defined on $\mathbb{R}^n$ and continuous. Since ψ_{λ_j} has compact support, $\partial^\alpha(\psi_{\lambda_j} f)$ is also bounded. From Lemma 2.38, it follows that

$$\partial^\alpha(\psi_{\lambda_j} f) * \rho_\varepsilon \to \partial^\alpha(\psi_{\lambda_j} f)$$

for $\varepsilon \to 0$ uniformly on compact sets. Now let K be a compact subset of U. Then for only a finite number of values of j, say $j = 1, \ldots, M$, the intersection $K \cap \operatorname{supp} \chi_j$ is non-empty. Therefore, on K we have

$$
\begin{aligned}
\partial^\alpha f_\varepsilon &= \sum_{j=1}^{M} \partial^\alpha \Big(\chi_j \cdot \big((\psi_{\lambda_j} f) * \rho_\varepsilon \big) \Big) \\
&= \sum_{j=1}^{M} \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \cdot \partial^\beta \chi_j \cdot \Big(\partial^{\alpha-\beta} (\psi_{\lambda_j} f) * \rho_\varepsilon \Big) \\
&\to \sum_{j=1}^{M} \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \cdot \partial^\beta \chi_j \cdot \partial^{\alpha-\beta} (\psi_{\lambda_j} f) \\
&= \sum_{j=1}^{M} \partial^\alpha \Big(\chi_j \psi_{\lambda_j} f \Big) \\
&= \partial^\alpha \Big(f \cdot \sum_{j=1}^{M} \chi_j \Big) \\
&= \partial^\alpha f
\end{aligned}
$$

as $\varepsilon \to 0$. This concludes the proof. $\square$

Chapter 3

Inversion of generalised functions

In the setting of generalised functions the question of inversion of functions has, so far, not been addressed. Part of the reason for this may be the considerable technical problems caused by the lack of a reasonable notion of range or image of a set under a generalised function. However, in certain applications "discontinuous coordinate transformations"—which can be modelled by a generalised function—have already been employed successfully, though on a rather informal level (see Chapter 4).
In this chapter we present and discuss several notions of invertibility of generalised functions. In Section 3.1, we give definitions of left resp. right invertibility, invertibility and strict invertibility, followed by a discussion of the immediate implications. Motivated by several questions arising naturally when trying to invert a net of smooth functions, we find several necessary conditions for (left, right) invertibility (Section 3.2). In Section 3.3, we analyse to which extent the properties "ca-injective" and "ca-surjective" ("ca" being shorthand for "asymptotically on compact sets") defined in the preceding section are sufficient to guarantee the existence of a (left, right) inverse of a generalised function. Finally, in Section 3.4, we prove some generalised inverse function theorems and study their relation to the classical Inverse Function Theorem 1.3 in Chapter 1.

At this point two remarks are in order: First, since generalised functions are defined on open subsets of $\mathbb{R}^n$ and we are interested in inverting such functions, we consider only generalised functions with (generalised) values in $\mathbb{R}$. Hence, more specifically than in Chapter 2, in this (and the following) chapter(s) $\mathrm{C}^k(U)$ (for $k \in \mathbb{N}_0 \cup \{\infty\}$), $\mathcal{E}_M(U)$, $\mathcal{N}(U)$ and $\mathcal{G}(U)$ denote the spaces of functions, nets resp. generalised functions with (generalised) values in $\mathbb{R}$.

Second, this chapter contains several graphics of nets of smooth functions. To give an idea of the behaviour of a net $(f_\varepsilon)_\varepsilon$ each graphic consists of five plots of f_ε for five different values of ε where the curves are shaded differently; the plots of f_ε become darker for ε tending to 0.

3.1 Invertibility of generalised functions

We start right away with a definition of invertibility of a generalised function on an open set.

3.1. Definition (Invertibility): *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}(U)^n$. Let A be an open subset of U.*

(LI) *u is called left invertible on A if there exist some $v \in \mathcal{G}(V)^n$ with V an open subset of $\mathbb{R}^n$ and an open set $B \subseteq V$ such that $u|_A$ is c-bounded into B and $v \circ u|_A = \mathrm{id}_A$. Then v is called a left inverse of u on A. Notation: u is left invertible (on A) with left inversion data $[A, V, v, B]$.*

(RI) *u is called right invertible on A if there exist some $v \in \mathcal{G}(V)^n$ with V an open subset of $\mathbb{R}^n$ and an open set $B \subseteq V$ such that $v|_B$ is c-bounded into A and $u \circ v|_B = \mathrm{id}_B$. Then v is called a right inverse of u on A. Notation: u is right invertible (on A) with right inversion data $[A, V, v, B]$.*

(I) *u is called invertible on A if it is both right and left invertible on A with right inversion data $[A, V, v, B_r]$ and left inversion data $[A, V, v, B_l]$. Then v is called an inverse of u on A. Notation: u is invertible (on A) with inversion data $[A, V, v, B_l, B_r]$.*

(SI) *u is called strictly invertible on A if it is invertible on A with inversion data $[A, V, v, B, B]$ for an open subset B of V. Then v is called a strict inverse of u on A. Notation: u is strictly invertible (on A) with inversion data $[A, V, v, B]$.*

Throughout this work we will also use the formulations "u is invertible (on A) by $[A, V, v, B_l, B_r]$" and "$[A, V, v, B_l, B_r]$ is an inverse of u (on A)". If we do not specify a set on which a given $u \in \mathcal{G}(U)^n$ is invertible, we always refer to invertibility on U, i.e. on its domain. The same rules of language apply to the cases of "left invertible", "right invertible" or "strictly invertible".

3.2. Remark:

(1) Note that u need not to be a c-bounded function on U. Only the restriction to the set A where it is composed with a left inverse must have this property.

(2) The notion of invertiblity of a generalised function u is more than the combination of left and right invertibility with respect to the same v yet possibly different sets A_l (for left) and A_r (for right).

(3) If a smooth function $f : U \to V$ (with U and V open subsets of $\mathbb{R}^n$) is classically invertible with smooth inverse $g : V \to U$, then, obviously, $\sigma(f) = \iota(f)$ is strictly invertible on U with inversion data $[U, V, \sigma(g), V]$.

Since the discontinuity in the "discontinuous coordinate transformation" in Chapter 4 consists of a jump, one type of functions we are interested in inverting are jump functions. Therefore, let us consider

3.3. Example: Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ be defined by $u_\varepsilon(x) := x + \arctan \frac{x}{\varepsilon}$ (Figure 3.1). Then u models a function with a jump of height π at 0.

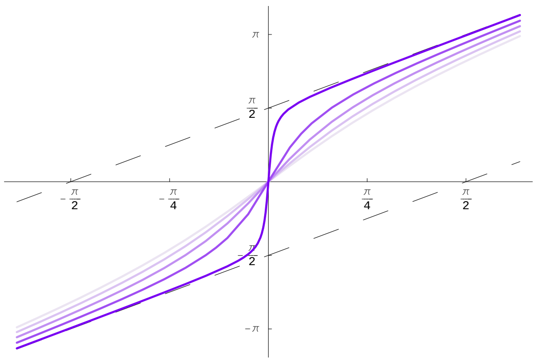

Figure 3.1: $u_\varepsilon(x) = x + \arctan \frac{x}{\varepsilon}$

We are interested in inverting u "around the jump", i.e. we want to find an inverse in the sense of Definition 3.1 (I) on an open set $A \subseteq U$ containing 0. For every ε the function u_ε is (classically) invertible by some C^∞-map $v_\varepsilon : u_\varepsilon(U) = (u_\varepsilon(-\alpha), u_\varepsilon(\alpha)) \to U$. In the following, we will successively specify sets V, A, B_l and B_r, showing that, in fact, u is invertible in the sense of Definition 3.1 (I).
To this end, first note that $u_\varepsilon(x) \nearrow x + \frac{\pi}{2}$ for every $x > 0$. Setting $x = \alpha$ and choosing $\beta \in (0, \alpha)$, we see that for ε small, say $\varepsilon \leq \varepsilon_0$, $u_\varepsilon(U)$ contains $(-(\beta + \frac{\pi}{2}), \beta + \frac{\pi}{2})$. So $V := (-(\beta + \frac{\pi}{2}), \beta + \frac{\pi}{2})$ is a suitable choice for a

common domain for all v_ε ($\varepsilon \le \varepsilon_0$). Defining $A := (-\alpha_1, \alpha_1)$ for some fixed α_1 with $0 < \alpha_1 < \beta$ and using $u_\varepsilon(\alpha_1) \nearrow \alpha_1 + \frac{\pi}{2}$, we obtain that $u_\varepsilon(A) = u_\varepsilon((-\alpha_1, \alpha_1)) \subseteq [-(\alpha_1 + \frac{\pi}{2}), \alpha_1 + \frac{\pi}{2}]$ for ε small (say $\varepsilon \le \varepsilon_1 \le \varepsilon_0$). Therefore, for B_l we may take any open subset of V containing $[-(\alpha_1 + \frac{\pi}{2}), \alpha_1 + \frac{\pi}{2}]$, e.g. $B_l := (-(\beta_l + \frac{\pi}{2}), \beta_l + \frac{\pi}{2})$ for $\beta_l \in (\alpha_1, \beta)$, guaranteeing that $(u_\varepsilon)_\varepsilon$ be c-bounded from A into B_l. Finally, to have $(v_\varepsilon)_\varepsilon$ c-bounded on a suitable set B_r, pick β_r with $0 < \beta_r < \alpha_1$ and set $B_r := (-(\beta_r + \frac{\pi}{2}), \beta_r + \frac{\pi}{2})$ to complete the inversion data set $[A, V, v, B_l, B_r]$ where $v := [(v_\varepsilon)_\varepsilon]$ (the moderateness of $(v_\varepsilon)_\varepsilon$ will be obvious). Summing up, we have the following inequalities and inclusions:

$$\begin{array}{ccccccccccc} 0 & < & \beta_r & < & \alpha_1 & < & \beta_l & < & \beta & < & \alpha, \\ & & & & A & & & & & \subseteq & U, \\ & & B_r & & & \subseteq & B_l & \subseteq & V. & & \end{array}$$

In the preceding example the set B_r is contained in B_l. The following proposition shows that this is no coincidence.

3.4. Proposition: *Let $u \in \mathcal{G}(U)^n$ be invertible on A with inversion data $[A, V, v, B_l, B_r]$. Then $B_r \subseteq B_l$.*

Proof: Let $x \in B_r$ and let $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ be representatives of u resp. v. Since $v|_{B_r}$ is c-bounded into A, there exists some $K \subset\subset A$ such that $v_\varepsilon(x) \subseteq K$ for small ε. By the c-boundedness of $u|_A$ into B_l, on the other hand, there exists some $K' \subset\subset B_l$ such that $u_\varepsilon(K) \subseteq K'$ for small ε. Therefore, $u_\varepsilon \circ v_\varepsilon(x)$ is an element of K'. Since v is a right inverse of u on A and $x \in B_r$, there exists a negligible net $(n_\varepsilon)_\varepsilon$ on B_r such that $u_\varepsilon \circ v_\varepsilon(x) = x + n_\varepsilon(x)$, yielding $x + n_\varepsilon(x) \to x$ for $\varepsilon \to 0$ where $x + n_\varepsilon(x) \in K'$ for small ε. Since K' is compact, the limit x is also in K' and, hence, in B_l. $\Box$

From the definition of invertibility and the preceding proposition, it follows

3.5. Proposition:

(1) If $u \in \mathcal{G}(U)^n$ is left resp. right invertible on A with left resp. right inversion data $[A, V, v, B]$, then v is right resp. left invertible on B with right resp. left inversion data $[B, U, u, A]$.

(2) If $u \in \mathcal{G}(U)^n$ is invertible on A with inversion data $[A, V, v, B_l, B_r]$, then v is left invertible on B_r with left inversion data $[B_r, U, u, A]$ and right invertible on B_l with right inversion data $[B_l, U, u, A]$.

(3) The inverse is unique in the following sense: If u is invertible on A with inversion data $[A, V^1, v^1, B_l^1, B_r^1]$ and $[A, V^2, v^2, B_l^2, B_r^2]$, then $v^1|_{B_r} = v^2|_{B_r}$ where $B_r := B_r^1 \cap B_r^2$.

(4) *If $u \in \mathcal{G}(U)^n$ is strictly invertible on A with inversion data $[A, V, v, B]$, then v is strictly invertible on B with inversion data $[B, U, u, A]$.*

(5) *The strict inverse is unique in the following sense: If u is strictly invertible on A with inversion data $[A, V^1, v^1, B^1]$ and $[A, V^2, v^2, B^2]$, then $v^1|_B = v^2|_B$ where $B := B^1 \cap B^2$.*

Proof: (1), (2) and (4) follow directly from the definition.
(3): By Proposition 3.4, we obtain

$$\begin{aligned} v^1|_{B_r} &= \mathrm{id}_A \circ v^1|_{B_r} = (v^2|_{B_l^2} \circ u|_A) \circ v^1|_{B_r} \\ &= v^2|_{B_l^2} \circ (u|_A \circ v^1|_{B_r}) = v^2|_{B_l^2} \circ \mathrm{id}_{B_r} = v^2|_{B_r} \end{aligned}$$

since $v^1|_{B_r}$ is c-bounded into A and $u|_A$ is c-bounded into B_r^2.
(5): This is a special case of (3). □

In the remainder of this section we will discuss various aspects of the notions of invertibility introduced above.

In classical inversion theory we are used to the fact that if a function is invertible (as a function) on some set A, this is still true for any subset of A. Taking a closer look at the definition, it becomes obvious that in the case of generalised functions we have to be more careful: For some left invertible $u \in \mathcal{G}(U)^n$ with left inversion data $[A, V, v, B_l]$ everything turns out fine. We can decrease the size of A without losing left invertibility. On the other hand, if u is right invertible with right inversion data $[A, V, v, B_r]$, shrinking A may not be possible, even (and here is the difference to the classical case) if B_r is shrunk as well. We illustrate this with an

3.6. Example: Consider v from Example 3.3. By Proposition 3.5 (1), it is right invertible with right inversion data $[B_l, U, u, A]$. (When discussing the right invertibility of v be careful to observe the reversed roles of U and V resp. A and B_l compared to the (original) notation in Definition 3.1 (RI).) Let B be an open subset of B_l. v is right invertible on B provided B contains the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and A is shrunk accordingly (while still containing 0). If B fails to satisfy this condition, then no open subset A' of A is small enough such that $(u_\varepsilon|_{A'})_\varepsilon$ is c-bounded into B.

The example shows that right invertibility on some set is not a local property in the usual sense. However, in the preceding example it is "local around the jump": The interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that has to be contained in B is exactly the "gap" in the image of the jump function modelled by u.
The issue of shrinking B_l resp. B_r is settled by the symmetry of the relation between the left resp. right invertible function and a left resp. right inverse

(cf. Proposition 3.5 (1)). Anyway, there are situations where we can safely reduce the size of B_l resp. B_r as the following remark shows.

3.7. Remark: Let $u \in \mathcal{G}(U)^n$ and A an open subset of U. For $i = 1, 2$ the sets V^i, B_l^i and B_r^i are open subsets of $\mathbb{R}^n$ with $B_l^i, B_r^i \subseteq V^i$ and $v^i \in \mathcal{G}(V^i)^n$.

(1) If u is left invertible on A with left inversion data both $[A, V^1, v^1, B_l^1]$ and $[A, V^2, v^2, B_l^2]$, then u is also left invertible with left inversion data $[A, V^i, v^i, B_l]$ for $i = 1, 2$, where $B_l := B_l^1 \cap B_l^2$: Let $K \subset\subset A$. Then there exists some $K^i \subset\subset B_l^i$ such that $u_\varepsilon(K) \subseteq K^i$ for small ε and $i = 1, 2$ and, thus, $u_\varepsilon(K) \subseteq K^1 \cap K^2 \subset\subset B_l^1 \cap B_l^2 = B_l$. Therefore, v^i together with $[A, V^i, v^i, B_l]$ is a left inverse of u on A.

(2) If u is right invertible on A with right inversion data both $[A, V^1, v^1, B_r^1]$ and $[A, V^2, v^2, B_r^2]$, then u is also right invertible with right inversion data $[A, V^i, v^i, B_r]$ for $i = 1, 2$ where $B_r := B_r^1 \cap B_r^2$: Since v^i restricted to B_r^i composed with u gives the identity in $\mathcal{G}(B_r^i)^n$ and $B_r \subseteq B_r^i$, also $u \circ v^i|_{B_r} = \mathrm{id}_{B_r}$ holds. Hence, v^i with $[A, V^i, v^i, B_r]$ is a right inverse of u on A.

(3) Combining the two preceding results, we obtain: If u is invertible on A with inversion data both $[A, V^1, v^1, B_l^1, B_r^1]$ and $[A, V^2, v^2, B_l^2, B_r^2]$, then u is also invertible with inversion data $[A, V^i, v^i, B_l, B_r]$ for $i = 1, 2$ where $B_l := B_l^1 \cap B_l^2$ and $B_r := B_r^1 \cap B_r^2$.

Next, we address the question of enlarging sets. Obviously, for a *left* invertible $u \in \mathcal{G}(U)^n$ with left inversion data $[A, V, v, B_l]$, enlarging A is not possible without further information on u—as is the case in classical theory. In contrast, let $[A, V, v, B_r]$ be a *right* inverse of u. Replacing A by a larger set (that is still contained in U) poses no problem at all since $(v_\varepsilon|_{B_r})_\varepsilon$ is c-bounded into any superset of A.
Again, the question of modifying B_l resp. B_r is answered by referring to Proposition 3.5 (1).

Combining the preceding results for the left and right case, we conclude that for an invertible u with inversion data $[A, V, v, B_l, B_r]$, without further specific information, A may neither be enlarged nor shrinked; B_r can safely be made smaller, B_l larger.

As to strict invertibility, there is no tolerance left for changing the size of either A or B.

These results reflect the fact that in the case of invertibility of u on A the set A has a double role: It has to be big enough such that $(v_\varepsilon|_{B_r})_\varepsilon$ is c-bounded into it and at the same time it has to be small enough such that the composition of $(u_\varepsilon|_A)_\varepsilon$ with $(v_\varepsilon|_{B_l})_\varepsilon$ still gives the identity in $\mathcal{G}(A)^n$. So,

the size of A has to be carefully balanced between the requirements of left and right invertibility. Generally, such a balance might be hard to achieve. Nevertheless, it turns out that it is possible in more cases than one might expect.
At first sight, a convenient way to circumvent the difficulty of balancing the size of A might consist in introducing a notion of "weak invertibility" using "weak inversion data sets" $[A_l, A_r, V, v, B_l, B_r]$. This choice, however, would make it difficult, if not impossible, to prove uniqueness of the inverse (cf. Proposition 3.5 (3) and (5)).

The notion of strict invertibility is the one that comes closest to a generalised equivalent of classical invertibility. However, in most cases we are interested in, it will be too much to ask for, as shall be demonstrated in the following

3.8. Example: Consider again the u modelling a jump function from Example 3.3. We attempt to find open sets A and B such that u is strictly invertible with strict inversion data $[A, V, v, B]$. W.l.o.g. we may assume that A and B are open intervals.
We already discussed in Example 3.6 that B has to contain the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, $B := (-(\gamma + \frac{\pi}{2}), \gamma + \frac{\pi}{2})$ for some $\gamma > 0$. For $(v_\varepsilon|_B)_\varepsilon$ to be c-bounded into A, the set A has to contain the closed interval $[-\gamma, \gamma]$. Let $A := (-(\gamma + \delta), \gamma + \delta)$ for some $\delta > 0$. For any $0 < \eta < \delta$ we eventually have $B = (-(\gamma + \frac{\pi}{2}), \gamma + \frac{\pi}{2}) \subseteq u_\varepsilon([-(\gamma + \eta), \gamma + \eta])$, thereby destroying any hope for c-boundedness into B. Thus, u is not strictly invertible on any open set A containing 0.

3.2 Necessary conditions for invertibility

In this section we will work out some aspects of what "being (left, right) invertible" entails. To this end, we start with a few (rather heuristic) questions that arise when attempting to invert a given $u \in \mathcal{G}(U)^n$.

Let $(u_\varepsilon)_\varepsilon$ be a representative of u. The obvious idea to invert u, of course, is to invert u_ε for each ε separately. For this to be possible every u_ε has to be injective. If this is not the case, we may ask

Question 1: If u_ε is not injective for every ε, is it possible for another representative of u to have this property (so that an inverse of u still may be found by inverting smooth functions)?

For now, let us assume that every u_ε is injective on U. By inverting every u_ε, we obtain a net of inverses v_ε. This gives rise to

Question 2: Does there exist an open set that is contained in all the—possibly different!—domains of the inverses v_ε, so that we can indeed speak of a net of functions on some fixed domain V?

If the last question is answered affirmatively, we still have to determine if the inverse net $(v_\varepsilon)_\varepsilon$ is moderate on this common domain. More precisely,

Question 3: Are all v_ε smooth? If yes, is $(v_\varepsilon|_V)_\varepsilon$ in $\mathcal{E}_M(V)^n$?

Concerning Question 1, we consider

3.9. Example: Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ be given by $u_\varepsilon(x) := \sin\frac{x}{\varepsilon}$ (Figure 3.2). No matter how small we choose a

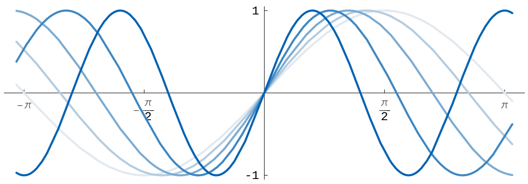

Figure 3.2: $u_\varepsilon(x) = \sin\frac{x}{\varepsilon}$

subset of U, eventually u_ε becomes non-injective on this set.

Do we have to check other representatives of u in Example 3.9 for injectivity (to construct a left inverse of u around 0)? The answer to that (and hence to Question 1) is no. To see this we need the following proposition and corollary.

3.10. Proposition: *Let U be an open subset of $\mathbb{R}^n$ and $f, m \in \mathrm{C}^1(U, \mathbb{R}^n)$ such that $f = \mathrm{id}_U + m$. Then f is injective on any compact convex subset K of U for which $\max_{x\in K} \|\mathrm{D}m(x)\| < 1$ is satisfied.*

Proof: Let $K \subset\subset U$ be as required in the proposition. Let $x, y \in K$ and set $\alpha := \max_{z\in K} \|\mathrm{D}m(z)\| < 1$. Then, by the Mean Value Theorem,

$$\begin{aligned} |f(x) - f(y)| &= |x - y + m(x) - m(y)| \\ &\geq |x - y| - \sup_{z\in K} \|\mathrm{D}m(z)\| \cdot |x - y| \\ &\geq (1 - \alpha) \cdot |x - y|, \end{aligned}$$

yielding the injectivity of f on K. □

In the application of Proposition 3.10 to generalised functions, we want to get rid of the convexity condition on the compact sets. To this end, we will use the following lemma ([GKOS01], Lemma 3.2.47).

3.11. Lemma: *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a continuously differentiable map. Let $K \subset\subset U$. Then there exists $C > 0$ such that*

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in K$.
C can be chosen as $C_1 \cdot \sup_{z \in L}(|f(z)| + \|\mathrm{D}f(z)\|)$, where L is any fixed compact neighbourhood of K in U and C_1 only depends on L.

3.12. Remark: If in Lemma 3.11, U and K are subsets of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n$, $x = (t, u)$ and $y = (t, v)$ (for $t \in \mathbb{R}^k$ and $u, v \in \mathbb{R}^l$), then an inspection of the proof of [GKOS01], 3.2.47 shows that $\mathrm{D}f$ can be replaced by $\partial_2 f$ when estimating $|f(t, u) - f(t, v)|$. If, in addition, K has the form $K_1 \times K_2$ ($K_1 \times K_2 \subset\subset U$), then L can be replaced by $K_1 \times L_2$, where L_2 is any fixed compact neighbourhood of K_2 with $K_1 \times L_2 \subset\subset U$ (we will meet this situation twice in the proof of Theorem 5.2 in Chapter 5).

The following results from Proposition 3.10.

3.13. Corollary: *Let U be an open subset of $\mathbb{R}^n$. Then for every representative $(u_\varepsilon)_\varepsilon$ of $\mathrm{id}_U \in \mathcal{G}(U)^n$ and for every compact subset K of U there exists some $\varepsilon_0 \in (0, 1]$ such that $u_\varepsilon|_K$ is injective for all $\varepsilon \leq \varepsilon_0$.*

Proof: Let $m_\varepsilon := \mathrm{id}_U - u_\varepsilon$. Then $(m_\varepsilon)_\varepsilon$ is an element of $\mathcal{N}(U)^n$. By Lemma 3.11, for all ε there exists a constant $C_\varepsilon > 0$ such that

$$|m_\varepsilon(x) - m_\varepsilon(y)| \leq C_\varepsilon \cdot |x - y| \tag{3.1}$$

for all $x, y \in K$. Since m is negligible and because of the form of the C_ε, we may find some ε_0 such that $C_\varepsilon < 1$ for all $\varepsilon \leq \varepsilon_0$. Now the assertion follows as in the proof of Proposition 3.10 by applying (3.1) in place of the Mean Value Theorem. □

If u is left invertible on A by $[A, V, v, B_l]$, then, for every representative $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v, the composition $(v_\varepsilon \circ u_\varepsilon|_A)_\varepsilon$ is a representative of the identity in $\mathcal{G}(A)^n$. Therefore, $v_\varepsilon \circ u_\varepsilon$ and consequently u_ε is injective on any compact subset of A for sufficiently small ε. In particular, this implies that the generalised function in Example 3.9 has no chance of being left invertible. This result motivates the following

3.14. Definition: *A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$ is called compactly asymptotically injective (ca-injective) if for every compact subset K of U there exists some $\varepsilon_0 \in (0, 1]$ such that $u_\varepsilon|_K$ is injective for all $\varepsilon \leq \varepsilon_0$.*

An element u of $\mathcal{G}(U)^n$ is called compactly asymptotically injective (ca-injective) if all representatives have this property.

3.15. Remark: Note that if one representative of a generalised function is ca-injective, this is not necessarily true for every other: Consider $n_\varepsilon(x) := e^{-\frac{1}{\varepsilon}}x$ and $\tilde{n}_\varepsilon(x) := 0$. $(n_\varepsilon)_\varepsilon$ is injective for all ε (even on $\mathbb{R}$) while $(\tilde{n}_\varepsilon)_\varepsilon$ is not. Yet both they are representatives of the same generalised function.
However, in the next section we will prove that the ca-injectivity of one representative implies ca-injectivity of all representatives provided $\det \circ \mathrm{D}u$ is strictly non-zero (a property to be defined later in this section) (cf. Corollary 3.36).

With the terminology of Definition 3.14 we have

3.16. Proposition: *If $u \in \mathcal{G}(U)^n$ is left invertible, then u is ca-injective.*

Question 2, though only a matter of manipulating sets, is not as trivial as it may seem. It can happen that the domains of the inverses v_ε shrink to a point with decreasing ε, so that there is no common open domain on which to define the inverse net. To illustrate this we consider the simple

3.17. Example: Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-1, 1)$ given by $u_\varepsilon(x) := \varepsilon x$ (Figure 3.3). Of course, for each ε there exists a smooth inverse

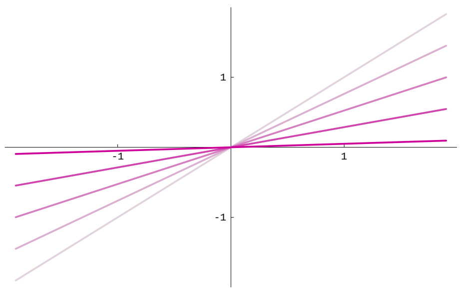

Figure 3.3: $u_\varepsilon(x) = \varepsilon x$

of u_ε—we denote it by v_ε. Since the image of the interval $(-1, 1)$ under u_ε gets ever smaller with decreasing ε, so do the domains of the inverses v_ε. So the intersection of all these domains contains only one point, namely 0.

In the one-dimensional case, a property that guarantees a common domain for the inverses is the following: Let u_ε be injective on an open interval U in $\mathbb{R}$ for all ε. Suppose that two different points x and y in U (w.l.o.g. $x < y$) can be found such that $u_\varepsilon(x)$ and $u_\varepsilon(y)$ converge to different limits a and b (w.l.o.g. $a < b$). Then the Intermediate Value Theorem ensures that for all $\delta > 0$ there exists some ε_0 such that $[a + \delta, b - \delta] \subseteq u_\varepsilon((x, y))$ for all $\varepsilon \leq \varepsilon_0$.
In Example 3.17, the values of u_ε at any given point converge to 0, so we are lacking a and b as above. In contrast, the net of functions modelling a

jump in Example 3.3 converges pointwise to an injective (if discontinuous) function, thereby allowing for a non-empty common domain for the inverses. Next, we want show a theorem that represents a generalisation of the previous observation to the n-dimensional case. The first idea for a proof would be to try and apply the Intermediate Value Theorem coordinatewise. However, this leads to considerable trouble. So, to prove the "n-dimensional" theorem, we will take another approach. To this end, we need a definitely nontrivial topological result of Brouwer on injective continuous maps in $\mathbb{R}^n$. A proof can be found on page 52 of [MT97].

3.18. Theorem (Brouwer): *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ an injective continuous map. Then the image $f(U)$ is open in $\mathbb{R}^n$ and f maps U homeomorphically to $f(U)$.*

3.19. Remark: In the proof of the next theorem and at some places in the next section we will do calculations involving the distances between sets. We will use the following definition and (easy to prove) facts: Let A and B be non-empty subsets of a normed space. The *distance of A and B* is defined by

$$\operatorname{dist}(A, B) := \inf_{x \in A,\, y \in B} |x - y|.$$

Note that $\operatorname{dist}(A, B) = \operatorname{dist}(\overline{A}, \overline{B})$. If $A \cap B = \emptyset$ and $\partial A, \partial B \neq \emptyset$, then $\operatorname{dist}(A, B) = \operatorname{dist}(\partial A, \partial B)$. Furthermore, if $A_\gamma \cap B = \emptyset$ for $A_\gamma := A + B_\gamma(0)$, then $\operatorname{dist}(A, B) = \operatorname{dist}(A_\gamma, B) + \gamma$. In particular, $\operatorname{dist}(A, A_\gamma^c) = \gamma$.

Now we may state the announced theorem. Roughly speaking, it establishes a kind of continuous dependence of connected parts $f(A)$ of the image set $f(U)$ on the function f.

3.20. Theorem: *Let U be an open subset of $\mathbb{R}^n$, $f, g \in \mathrm{C}(U, \mathbb{R}^n)$ both injective and W a connected open subset of $\mathbb{R}^n$ with $\overline{W} \subset\subset f(U)$. Choose an open ball $B_\delta(y)$ ($y \in W$, $\delta > 0$) inside W such that the closure of $W_\delta := W + B_\delta(0)$ is still a subset of $f(U)$, i.e. let $\delta > 0$ such that $B_\delta(y) \subseteq W$ and $\overline{W_\delta} \subseteq f(U)$. If, for $A := f^{-1}(\overline{W_\delta})$,*

$$\|g - f\|_{\infty, A} < \delta$$

holds, then

$$\overline{W} \subseteq g(A)^\circ.$$

Proof: By Theorem 3.18, both $f(U)$ and $g(U)$ are open and f and g map U homeomorphically to $f(U)$ resp. $g(U)$. Clearly, $\overline{W}$ is the disjoint union of the three sets

$$\begin{aligned} G_1 &:= \overline{W} \cap g(A)^\circ, \\ G_2 &:= \overline{W} \cap \partial g(A), \\ G_3 &:= \overline{W} \cap \operatorname{ext} g(A). \end{aligned}$$

We will show that $G_1 \neq \emptyset$ and $G_2 = \emptyset$. By the connectedness of $\overline{W}$, it follows that $\overline{W} = G_1$ (note that G_1 and G_3 are open in the relative topology of $\overline{W}$), that is

$$\overline{W} \subseteq g(A)^\circ.$$

Observe that, by Theorem 3.18, we do not have to distinguish either between $\operatorname{int}_V C$ and $C^\circ = \operatorname{int}_{\mathbb{R}^n} C$ (for $V = U, f(U), g(U)$ and C any subset of V) or between $\partial_V C$ and $\partial C = \partial_{\mathbb{R}^n} C$ (for V as before and C any *compact* subset of V; note that A, $f(A)$ and $g(A)$ are compact).

$G_1 \neq \emptyset$: Let $x := f^{-1}(y)$. Then x is an element of A. Since f and g are homeomorphisms and y is an element of the interior of $\overline{W_\delta}$, it follows $x \in A^\circ$ and $g(x) \in g(A)^\circ$. By

$$|g(x) - y| = |g(x) - f(x)| \leq \|g - f\|_{\infty,A} < \delta,$$

we obtain

$$g(x) \in B_\delta(y) \cap g(A)^\circ \subseteq \overline{W} \cap g(A)^\circ.$$

$G_2 = \emptyset$: Assume that there exists $a \in \overline{W} \cap \partial g(A)$. By $\partial g(A) = g(\partial A)$, the point $x := g^{-1}(a)$ is an element of ∂A. Moreover, $f(x) \in \partial f(A) = \partial \overline{W_\delta}$. On the one hand,

$$|a - f(x)| = |g(x) - f(x)| \leq \|g - f\|_{\infty,A} < \delta. \tag{3.2}$$

On the other hand, a being an element of $\overline{W}$, we obtain

$$|a - f(x)| \geq \operatorname{dist}(\overline{W}, \partial W_\delta) = \operatorname{dist}(W, W_\delta^c) = \delta,$$

which is a contradiction to (3.2). Hence, $\overline{W} \cap \partial g(A) = \emptyset$. □

With respect to generalised functions the above theorem implies

3.21. Corollary: *Let U be an open subset of $\mathbb{R}^n$. Then for every representative $(u_\varepsilon)_\varepsilon$ of $\mathrm{id}_U \in \mathcal{G}(U)^n$ and for every compact subset K of U there exist a compact subset L of U containing K and some $\varepsilon_0 \in (0,1]$ such that $K \subseteq u_\varepsilon(L)$ for all $\varepsilon \leq \varepsilon_0$.*

Proof: We first prove the claim for connected U. Let K be a (non-empty) compact subset of U.
In the first step, we construct a non-empty, open subset W of U with $\overline{W} \subset\subset U$ which contains K: For $\eta_1 := \frac{1}{2}\operatorname{dist}(K, U^c)$ we have $K \subseteq \bigcup_{x\in K} B_{\eta_1}(x)$. By the compactness of K, there exist $x_0, \dots, x_k \in K$ such that $K \subseteq \bigcup_{i=0}^{k} B_{\eta_1}(x_i)$. Since U is connected, we can find k curves $\gamma_1, \dots, \gamma_k : [0,1] \to U$ satisfying $\gamma_i(0) = x_{i-1}$ and $\gamma_i(1) = x_i$. Now, set

$$K' := \Big(\bigcup_{i=0}^{k} \overline{B_{\eta_1}(x_i)}\Big) \cup \Big(\bigcup_{i=1}^{k} \gamma_i([0,1])\Big).$$

K' is a connected compact subset of U. For $\eta_2 = \frac{1}{2}\operatorname{dist}(K', U^c)$ we define $W := K' + B_{\eta_2}(0)$. Then W is a non-empty, open subset of U with $\overline{W} \subset\subset U$ containing K.
In the second step, we prove the claim for connected U by means of Corollary 3.13 and Theorem 3.20: Let $m_\varepsilon := \mathrm{id}_U - u_\varepsilon$. Then $(m_\varepsilon)_\varepsilon$ is an element of $\mathcal{N}(U)^n$. Let $\delta > 0$ such that there exists some $y \in W$ with $B_\delta(y) \subseteq W$ and such that $\overline{W_{2\delta}} \subseteq U$ for $W_{2\delta} := W + B_{2\delta}(0)$. By Corollary 3.13, there exists some $\varepsilon_1 \in (0,1]$ such that u_ε is injective on $\overline{W_{2\delta}}$ for all $\varepsilon \le \varepsilon_1$. Choose $\varepsilon_0 \le \varepsilon_1$ such that $\sup_{x\in\overline{W_\delta}} |m_\varepsilon(x)| < \delta$ for all $\varepsilon \le \varepsilon_0$. Now apply Theorem 3.20 to $W_{2\delta}$, $\mathrm{id}_{W_{2\delta}}$, $u_\varepsilon|_{W_{2\delta}}$, W and δ in place of U, f, g, W and δ for every $\varepsilon \le \varepsilon_0$ and set $L := \overline{W_\delta}$.

Now we prove the claim for arbitrary U. Let K be a compact subset of U. We may write $U = \bigcup_{i\in I} U_i$ where the U_i denote the (open) connected components of U and I is a suitable index set. By the compactness of K, only finitely many of the U_i are needed to cover K, say $K \subseteq \bigcup_{j=1}^{k} U_{i_j}$. Set $K_j := K \cap U_{i_j}$ for $j = 1, \dots, k$. From

$$K_j = K \cap U_{i_j} = K \backslash \big(\bigcup_{i\ne i_j} U_i\big) = K \cap \Big(\mathbb{R}^n \backslash \big(\bigcup_{i\ne i_j} U_i\big)\Big),$$

it follows that all of the K_j are closed and, hence, compact. Therefore, K can be written as a union of compact sets $K_1, \dots, K_k$ where each K_j is contained in only one connected component. By the first part of the proof, for every j there exist a compact subset L_j of U containing K_j and some $\varepsilon_j \in (0,1]$ such that $K_j \subseteq u_\varepsilon(L_j)$ for all $\varepsilon \le \varepsilon_j$. Let $\varepsilon_0 := \min(\varepsilon_1, \dots, \varepsilon_k)$ and $L := \bigcup_{j=1}^{k} K_j$. Then L is a compact subset of U containing K and $K \subseteq u_\varepsilon(L)$ for all $\varepsilon \le \varepsilon_0$. □

For right invertible $u \in \mathcal{G}(U)^n$ with right inversion data $[A, V, v, B_r]$, Corollary 3.21 has the following meaning: For any representatives $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v, the composition $(u_\varepsilon \circ v_\varepsilon|_{B_r})_\varepsilon$ is a representative of the identity in $\mathcal{G}(B_r)^n$. Therefore, for every compact subset K of B_r there exists a compact subset L of B_r with $K \subseteq L$ such that $K \subseteq u_\varepsilon \circ v_\varepsilon(L)$ for ε sufficiently small. Since $(v_\varepsilon|_{B_r})_\varepsilon$ is c-bounded into A, there exists a compact subset L' of A such that $v_\varepsilon(L) \subseteq L'$ for small ε. This entails that $K \subseteq u_\varepsilon(L')$ for ε small enough. This observation motivates the next

3.22. Definition: *Let U and V be open subsets of $\mathbb{R}^n$. A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$ is called compactly asymptotically surjective (ca-surjective) onto V if for every compact subset K of V there exist a compact subset L of U and some $\varepsilon_0 \in (0,1]$ such that $K \subseteq u_\varepsilon(L)$ for all $\varepsilon \leq \varepsilon_0$.*

An element u of $\mathcal{G}(U)^n$ is called compactly asymptotically surjective (ca-surjective) onto V if all representatives have this property.

With the terminology of Definition 3.22 we have

3.23. Proposition: *If $u \in \mathcal{G}(U)^n$ is right invertible on A with right inversion data $[A, V, v, B_r]$, then u is ca-surjective onto B_r.*

Finally, let us turn to Question 3. Given some $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)^n$, suppose that for every ε the function u_ε is invertible as a function on U with inverse v_ε. Moreover, assume that there exists an open subset V of $\mathbb{R}^n$ such that V is contained in all $u_\varepsilon(U)$. Concerning the smoothness of v_ε, we know that v_ε is C^∞ if and only if the determinant of the differential of u_ε is non-zero at all points of U.

But what if one (invertible) representative u_ε of u does not have this property? Is it still possible for another representative of u to have invertible differentials at all points of U and, thus, provide an inverse of u?

3.24. Example: Consider $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R})$ given by $u_\varepsilon(x) := x^3$. u_ε is invertible as a function on $\mathbb{R}$ for every ε but the inverses are not smooth. As the following proposition will show, u cannot be inverted on any open set containing 0.

3.25. Proposition: *Let U be an open subset of $\mathbb{R}^n$, A an open subset of U and $u \in \mathcal{G}(U)^n$ left invertible on A with left inversion data $[A, V, v, B_l]$. Then for every representative $(u_\varepsilon)_\varepsilon$ of u and for every compact subset K of A, there exist $C > 0$, $N \in \mathbb{N}$ and $\varepsilon_0 \in (0,1]$ such that*

$$\inf_{x \in K} |\det(\mathrm{D}u_\varepsilon(x))| \geq C\varepsilon^N \tag{3.3}$$

for all $\varepsilon \leq \varepsilon_0$. In particular, $\det(\mathrm{D}u(x))$ is strictly non-zero for all $x \in A$.

Proof: Let $(v_\varepsilon)_\varepsilon$ be a representative of v. Since u is left invertible on A, we have

$$v_\varepsilon \circ u_\varepsilon|_A = \mathrm{id}_A + n_\varepsilon \tag{3.4}$$

for some $(n_\varepsilon)_\varepsilon \in \mathcal{N}(A)^n$. Differentiating (3.4) yields

$$\mathrm{D}v_\varepsilon(u_\varepsilon(x)) \circ \mathrm{D}u_\varepsilon(x) = I + \mathrm{D}n_\varepsilon(x) \tag{3.5}$$

for all $x \in A$, with I denoting the $(n \times n)$-identity matrix. Since the determinant of a square matrix is a continuous function, we have

$$\det(I + \mathrm{D}n_\varepsilon(x)) \to \det(I) = 1$$

as $\varepsilon \to 0$. Now choose $\varepsilon_1 \in (0,1]$ such that $\sup_{x\in K} \|\mathrm{D}n_\varepsilon(x)\|$ is sufficiently small to ensure that $\det(I+\mathrm{D}n_\varepsilon(x)) \geq \frac{1}{2}$ for all $x \in K$ and $\varepsilon \leq \varepsilon_1$. By (3.5), it now follows that

$$|\det(\mathrm{D}u_\varepsilon(x))| \geq \frac{1}{2 \cdot |\det(\mathrm{D}v_\varepsilon(u_\varepsilon(x)))|} \tag{3.6}$$

for all $x \in K$ and $\varepsilon \leq \varepsilon_1$. Since u is left invertible on A, it is c-bounded into B_l. Therefore, there exist a compact subset L of B_l and some $\varepsilon_2 \leq \varepsilon_1$ such that

$$u_\varepsilon(K) \subseteq L \tag{3.7}$$

for all $\varepsilon \leq \varepsilon_2$. The determinant being an element of $\mathcal{O}_M(\mathbb{R}^n)$, it follows from Theorem 2.19 that $(\det \circ v_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(B_r)$. Together with (3.7) we conclude that there exist $C_1 > 0$, $N \in \mathbb{N}$ and $\varepsilon_0 \leq \varepsilon_2$ such that

$$\sup_{x\in K} |\det(\mathrm{D}v_\varepsilon(u_\varepsilon(x)))| \leq \sup_{y\in L} |\det(\mathrm{D}v_\varepsilon(y))| \leq C_1\varepsilon^{-N}$$

for $\varepsilon \leq \varepsilon_0$. Plugging this inequality into (3.6) yields

$$|\det(\mathrm{D}u_\varepsilon(x))| \geq \frac{1}{2C_1}\varepsilon^N$$

for all $\varepsilon \leq \varepsilon_0$, as desired. □

It turns out that Proposition 3.25 also provides a necessary condition for the moderateness of the inverse net $(v_\varepsilon|_V)_\varepsilon$, the lower bound in property (3.3) being an immediate consequence of the moderateness of the representative $(v_\varepsilon)_\varepsilon$ of the inverse v.

3.26. Definition: *Let U be an open subset of $\mathbb{R}^n$. A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)$ is called strictly non-zero if for every compact subset K of U there exist $C > 0$, a natural number N and some $\varepsilon_0 \in (0,1]$ such that*

$$\inf_{x\in K} |u_\varepsilon(x)| \geq C\varepsilon^N \tag{3.8}$$

for all $\varepsilon \leq \varepsilon_0$.

An element u of $\mathcal{G}(U)$ is called strictly non-zero if it possesses a representative with this property.

Clearly, if one representative satisfies (3.8), then so do all. With the terminology of Definition 3.26, Proposition 3.25 now reads

3.27. Proposition: *If $u \in \mathcal{G}(U)^n$ is left invertible, then* $\det \circ \mathrm{D}u$ *is strictly non-zero.*

To sum up, we have determined three properties that are necessary for a given $u \in \mathcal{G}(U)^n$ to be invertible on some open subset A of U by $[A, V, v, B_l, B_r]$, namely:

1. u has to be ca-injective on A,
2. $u|_A$ has to be ca-surjective onto B_r and
3. $\det \circ \mathrm{D}u_\varepsilon$ has to be strictly non-zero on A.

In the next section we will prove that these three conditions are also sufficient to guarantee at least *local* invertibility of a c-bounded u, in the following sense:

3.28. Definition (Local invertibility): *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}(U)^n$. We call u locally (left, right) invertible if for every point $z \in U$ there exists an open neighbourhood A of z in U such that u is (left, right) invertible on A.*

Obviously, (left, right) invertibility on some open set implies local (left, right) invertibility on that very set but not vice versa.

Note that—contrary to the widespread usage of the term "local" and the intuition based thereupon—for a generalised function u, which is locally (left, right) invertible on some open set U, and some given $x_0 \in U$, the neighbourhood A of x_0 on which u is (left, right) invertible cannot, in general, be chosen either arbitrarily small or arbitrarily large (see also Section 3.1 and cp. Remarks 4.18 and 4.20). Local invertibility only guarantees the *existence* of such a neighbourhood, its (minimum resp. maximum) size depending on the function u and the point x_0 (cp. Example 3.6).

3.3 Sufficient conditions for invertibility

Our first aim in this section is to prove a partial converse to Proposition 3.16, i.e. that compact asymptotic injectivity (ca-injectivity) of a c-bounded $u \in \mathcal{G}[U, \mathbb{R}^n]$, with $\det \circ \mathrm{D}u$ strictly non-zero, implies local left invertibility of u. To this end, some preliminaries are necessary.
Let $u \in \mathcal{G}(U)^n$ and assume that $(u_\varepsilon)_\varepsilon$ is a representative such that u_ε is injective with inverse $v_\varepsilon : u_\varepsilon(U) \to U$, for every ε. If we are interested only in

left inverses of u, it is of no importance whether there is a common nontrivial open set *inside* of all $u_\varepsilon(U)$; rather, we need some open set *containing* all $u_\varepsilon(U)$ to serve as a common domain for the v_ε. So far, each v_ε is only defined on $u_\varepsilon(U)$. Therefore, we somehow have to extend the functions v_ε (in a smooth way!) to a larger set without losing their property of being (left) inverse to the u_ε on some open subset A of U, independent of ε and possibly smaller than U. We will do this by means of two-member partitions of unity $(p_\varepsilon, 1-p_\varepsilon)$, where the plateau functions p_ε serve to retain the values of v_ε on some $K_\varepsilon \subset\subset u_\varepsilon(U)$. New values for v_ε outside K_ε can be chosen from the convex hull of $\operatorname{im} v_\varepsilon$ (or some larger compact set).
We formulate the well-known existence result for plateau functions as a lemma, including the proof as given in [DR84], Ch. I, §2. We will make further use of the technicalities of this very proof in the sequel.

3.29. Lemma: *Let U be an open subset of $\mathbb{R}^n$ and K compact in U. Then there exists a plateau function $p \in \mathcal{D}(U)$ such that $0 \le p \le 1$ and $p|_K = 1$.*

Proof: Let $\tilde\rho : \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde\rho(x) := \begin{cases} e^{-\frac{1}{(1-x)^2(1+x)^2}}, & x \in (-1,1) \\ 0, & \text{otherwise} \end{cases}$$

and $\rho : \mathbb{R} \to \mathbb{R}$ be defined by

$$\rho(x) := \frac{\tilde\rho(x)}{\int_{-\infty}^{\infty} \tilde\rho(t)dt}.$$

Then $\sigma : \mathbb{R} \to \mathbb{R}$,

$$\sigma(x) := \int_{-\infty}^{x} 2\rho(2t-1)dt,$$

is a C^∞-function that is zero for negative x, strictly monotonically increasing for $0 \le x \le 1$ and identically one for $x > 1$. Moreover, it is antisymmetric around $(\frac{1}{2}, \frac{1}{2})$. We define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := \sigma(x+1) - \sigma(x).$$

The support of h is the closed interval $[-1, 1]$, the function is symmetric with respect to the y-axis and it is antisymmetric around $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. Therefore, we get

$$\sum_{j\in\mathbb{Z}}^{\infty} h(x-j) = 1$$

(note that for every x the sum has at most two terms different from zero). Choose a "grid size" $\eta > 0$ such that every n-dimensional cube with side length η having non-empty intersection with K is contained in U. For $(j_1, \ldots, j_n) \in \mathbb{Z}^n$ we now define $\varphi_{(j_1,\ldots,j_n)} : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_{(j_1,\ldots,j_n)}(x_1, \ldots, x_n) := \prod_{i=1}^{n} h\left(\frac{2x_i}{\eta} - j_i\right).$$

The support of $\varphi_{(j_1,\ldots,j_n)}$ is exactly the closed n-dimensional cube with side length η and centre $\frac{\eta}{2}(j_1, \ldots, j_n)$. For $j = (j_1, \ldots, j_n)$ we have

$$\begin{aligned}
\sum_{j \in \mathbb{Z}^n} \varphi_j(x_1, \ldots, x_n) &= \sum_{j_1 \in \mathbb{Z}} \cdots \sum_{j_n \in \mathbb{Z}} \varphi_{(j_1,\ldots,j_n)}(x_1, \ldots, x_n) \\
&= \prod_{i=1}^{n} \sum_{j_i \in \mathbb{Z}} h\left(\frac{2x_i}{\eta} - j_i\right) \\
&= 1.
\end{aligned}$$

Let $J := \{j \in \mathbb{Z}^n \mid \operatorname{supp} \varphi_j \cap K \neq \emptyset\}$. Since K is compact, J is finite. Now define $p : \mathbb{R}^n \to \mathbb{R}$ by

$$p := \sum_{j \in J} \varphi_j.$$

The function p is C^∞, maps $\mathbb{R}^n$ into $[0, 1]$ and has compact support. By the choice of η, the support of p is contained in U. Finally, since $\varphi_j|_K = 0$ for $j \in \mathbb{Z}^n \backslash J$, we have

$$p|_K = \sum_{j \in J} \varphi_j|_K = \sum_{j \in \mathbb{Z}^n} \varphi_j|_K = \left.\left(\sum_{j \in \mathbb{Z}^n} \varphi_j\right)\right|_K = 1.$$

Restricting p to U yields a function in $\mathcal{D}(U)$ with the properties claimed. □

3.30. Proposition: *Let U_ε (for $\varepsilon \in (0, \varepsilon_0]$) be an open subset of $\mathbb{R}^n$ and K_ε compact in U_ε such that $(\operatorname{dist}(K_\varepsilon, U_\varepsilon^c))_\varepsilon$ is strictly non-zero. Let U be another open subset of $\mathbb{R}^n$ such that $U_\varepsilon \subseteq U$ for all ε. Then there exists a net $(p_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)$ of plateau functions such that $p_\varepsilon|_{K_\varepsilon} = 1$ and $\operatorname{supp} p_\varepsilon \subseteq U_\varepsilon$ for ε sufficiently small.*

Proof: By assumption, there exist $C > 0$, $N \in \mathbb{N}$ and $\varepsilon_1 \in (0, \varepsilon_0)$ with $\operatorname{dist}(K_\varepsilon, U_\varepsilon^c) \geq C\varepsilon^N$ for all $\varepsilon \leq \varepsilon_1$. Hence, we can choose η_ε with $C\varepsilon^{N+1} \leq \eta_\varepsilon < \operatorname{dist}(K_\varepsilon, U_\varepsilon^c)$ such that every n-dimensional cube with side length η_ε having non-empty intersection with K_ε is contained in U_ε, for all $\varepsilon \leq \varepsilon_1$. From now on, we always let $\varepsilon \leq \varepsilon_1$. Construct plateau functions $q_\varepsilon : U_\varepsilon \to [0, 1]$ as in the proof of Lemma 3.29 with respect to U_ε and K_ε using grid size η_ε. Let $p_\varepsilon \in \mathcal{D}(U)$ be the smooth extension by 0 of q_ε to U. Then, conferring to the proof of Lemma 3.29, the plateau function p_ε is given by

$$p_\varepsilon = \left(\sum_{j \in J_\varepsilon} \varphi_j^\varepsilon \right)\bigg|_U ,$$

where, for any $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, φ_j^ε maps from $\mathbb{R}^n$ to $\mathbb{R}$ and is given by

$$\varphi_j^\varepsilon(x_1, \ldots, x_n) = \prod_{i=1}^n h\left(\frac{2x_i}{\eta_\varepsilon} - j_i \right),$$

and $J_\varepsilon := \{ j \in \mathbb{Z}^n \mid \operatorname{supp} \varphi_j^\varepsilon \cap K_\varepsilon \neq \emptyset \}$. By our choice of $(\eta_\varepsilon)_\varepsilon$, we know that $\frac{2|y|}{\eta_\varepsilon} \leq \frac{2|y|}{C} \varepsilon^{-(N+1)}$ for all $y \in \mathbb{R}^n$ and, thus, the net of mappings $\left(y \mapsto \frac{2y}{\eta_\varepsilon} - j_i \right)_\varepsilon$ is moderate. The function h, having compact support, is an element of $\mathcal{O}_M(\mathbb{R})$. Hence, by Proposition 2.19, $(\varphi_j^\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)$. Let $\varphi_0^\varepsilon := \varphi_{(0,\ldots,0)}^\varepsilon$. Any φ_j^ε can be written as the composition of the translation $x \mapsto x - \frac{\eta_\varepsilon}{2} j$ and φ_0^ε. Therefore, by the moderateness of $(\varphi_0^\varepsilon)_\varepsilon$, it follows that for all $\alpha \in \mathbb{N}_0^n$ there exist $N_1 \in \mathbb{N}$ and $C_1 > 0$ such that

$$\sup_{x \in \operatorname{supp} \varphi_j^\varepsilon} |\partial^\alpha \varphi_j^\varepsilon(x)| = \sup_{x \in \operatorname{supp} \varphi_0^\varepsilon} |\partial^\alpha \varphi_0^\varepsilon(x)| \leq C_1 \varepsilon^{-N_1} \tag{3.9}$$

for all $j \in \mathbb{Z}^n$. Now set $J_\varepsilon^x := \{ j \in J_\varepsilon \mid x \in (\operatorname{supp} \varphi_j^\varepsilon)^\circ \}$ for $x \in \mathbb{R}^n$. Then $J_\varepsilon^x \subseteq J_\varepsilon$ holds, and $j \notin J_\varepsilon^x$ entails $\partial^\alpha \varphi_j^\varepsilon(x) = 0$ for arbitrary $\alpha \in \mathbb{N}_0^n$. Hence,

$$\partial^\alpha p_\varepsilon(x) = \sum_{j \in J_\varepsilon^x} \partial^\alpha \varphi_j^\varepsilon(x) \tag{3.10}$$

for all $x \in \mathbb{R}^n$. Since the support of φ_j^ε is precisely the closed n-dimensional cube with side length η_ε and centre $\frac{\eta_\varepsilon}{2} j$, we obtain

$$|J_\varepsilon^x| \leq 2^n. \tag{3.11}$$

Now we show the moderateness estimates for $(p_\varepsilon)_\varepsilon$, even globally on U. Let $\alpha \in \mathbb{N}_0^n$. Then, by (3.10), (3.9) and (3.11), it follows for $x \in U$ that

$$\begin{aligned}
|\partial^\alpha p_\varepsilon(x)| &= \Big| \sum_{j \in J_\varepsilon^x} \partial^\alpha \varphi_j^\varepsilon(x) \Big| \\
&\leq \sum_{j \in J_\varepsilon^x} \sup_{y \in \operatorname{supp} \varphi_j^\varepsilon} |\partial^\alpha \varphi_j^\varepsilon(y)| \\
&\leq \sum_{j \in J_\varepsilon^x} C_1 \varepsilon^{-N_1} \\
&\leq 2^n \cdot C_1 \varepsilon^{-N_1} \\
&= C_2 \varepsilon^{-N_1},
\end{aligned}$$

thereby concluding the proof of the proposition. □

Now that we have found a way to extend the inverses v_ε to a common domain, we turn to the question of moderateness. It turns out that if $(\det \circ \mathrm{D}u_\varepsilon)_\varepsilon$ is strictly non-zero, this is already sufficient to guarantee the desired result. The next proposition consists of two parts. Roughly speaking, the first part deals with the "disposition to moderateness" of the inverse net $(v_\varepsilon)_\varepsilon$ (recall, there is still no common domain for the v_ε) while in the second part we take care of the smooth and moderate extension of the v_ε.

3.31. Proposition: *Let U be an open subset of $\mathbb{R}^n$ containing open subsets W_ε for $\varepsilon \in (0, \varepsilon_0]$ such that $W_\varepsilon \subseteq K$ for some $K \subset\subset U$. Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$. For all ε let u_ε be injective on W_ε with inverse $v_\varepsilon : V_\varepsilon \to W_\varepsilon$ where $V_\varepsilon := u_\varepsilon(W_\varepsilon)$. Suppose that*

$$\inf_{x \in W_\varepsilon} |\det(\mathrm{D}u_\varepsilon(x))| \geq C_1 \varepsilon^{N_1} \tag{3.12}$$

for some $C_1 > 0$ and $N_1 \in \mathbb{N}_0$ and for all $\varepsilon \leq \varepsilon_0$. Then the following holds:

(1) The inverses v_ε are smooth, and for all $\alpha \in \mathbb{N}_0^n$ there exist $C > 0$, $N \in \mathbb{N}$ and some $\varepsilon_1 \in (0, \varepsilon_0]$ such that for all $\varepsilon \leq \varepsilon_1$ the estimate

$$\sup_{y \in V_\varepsilon} |\partial^\alpha v_\varepsilon(y)| \leq C \varepsilon^{-N} \tag{3.13}$$

holds. In particular, if there exists a non-empty open subset V of $\mathbb{R}^n$ such that $V \subseteq \bigcap_{\varepsilon \in (0,\varepsilon_0]} V_\varepsilon$, then $(v_\varepsilon|_V)_\varepsilon$ is in $\mathcal{E}_M(V)^n$ and uniformly bounded (the latter following from the inclusion $W_\varepsilon \subseteq K$).

(2) Let $K_\varepsilon \subset\subset V_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0]$ and $[(\tilde{x}_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}_c^n$ such that $\tilde{x}_\varepsilon \in L \subset\subset \mathbb{R}^n$ for all $\varepsilon \leq \varepsilon_0$. If there exist a constant $C_2 > 0$ and a natural number N_2 such that

$$\operatorname{dist}(K_\varepsilon, V_\varepsilon^c) \geq C_2 \varepsilon^{N_2} \tag{3.14}$$

for all $\varepsilon \leq \varepsilon_0$, then there exist smooth functions $\tilde{v}_\varepsilon$ defined on $\mathbb{R}^n$ such that $\tilde{v}_\varepsilon|_{K_\varepsilon} = v_\varepsilon|_{K_\varepsilon}$ and $\tilde{v}_\varepsilon(x) = \tilde{x}_\varepsilon$ for all $x \in \mathbb{R}^n \setminus V_\varepsilon$ and such that $(\tilde{v}_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. Furthermore, the net $(\tilde{v}_\varepsilon)_\varepsilon$ is uniformly bounded. In particular, $(\tilde{v}_\varepsilon)_\varepsilon$ is c-bounded into any open subset of $\mathbb{R}^n$ containing the convex hull of $K \cup L$.

3.32. Remark:

(1) The compact superset K serves a twofold purpose: First, the $\mathcal{E}_M$-estimates of $(v_\varepsilon)_\varepsilon$ (even if we would restrict them to compact subsets of V_ε which, anyway, we do not) are transformed to estimates of $(u_\varepsilon)_\varepsilon$ on K. Second, the inclusion $\operatorname{im} v_\varepsilon \subseteq W_\varepsilon \subseteq K$ is crucial for the c-boundedness of $(\tilde{v}_\varepsilon)_\varepsilon$.

(2) We introduce the following terminology: A function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ is called a (K, y_0)-*extension of* $f : U \to \mathbb{R}^m$ (where $U \subseteq \mathbb{R}^n$ open, $K \subset\subset U$ and $y_0 \in \mathbb{R}^m$) if $\tilde{f}|_K = f|_K$ and $\tilde{f}(x) = y_0$ for all $x \in \mathbb{R}^n \backslash U$. In this sense, $\tilde{v}_\varepsilon$ in the proposition is a $(K_\varepsilon, \tilde{x}_\varepsilon)$-extension of v_ε.

Proof: (1): Since u_ε is smooth and $\det(\mathrm{D}u_\varepsilon(x))$ is non-zero for all $x \in W_\varepsilon$, the inverse v_ε is also smooth for all ε. We note that, since differentiation is done componentwise, we only have to consider $\partial^\alpha v_\varepsilon^{(i)}$, where $v_\varepsilon^{(i)}$ denotes the i-th component of v_ε. Let $y \in V_\varepsilon$.

First, for $\alpha = 0$ we have $\operatorname{im} v_\varepsilon = W_\varepsilon \subseteq K \subset\subset U$. Therefore, $\sup_{y \in V_\varepsilon} |v_\varepsilon(y)|$ is bounded independently of ε. Next, we consider the first partial derivatives of $v_\varepsilon^{(i)}$. Observe

$$\begin{aligned} \frac{\partial v_\varepsilon^{(i)}(y)}{\partial y_j} &= [\mathrm{D}v_\varepsilon(y)]_{ij} \\ &= \left[\mathrm{D}u_\varepsilon(v_\varepsilon(y))^{-1}\right]_{ij} \\ &= \frac{1}{\det(\mathrm{D}u_\varepsilon(v_\varepsilon(y)))} \cdot R_{i;j}\left(\left(\frac{\partial u_\varepsilon^{(r)}}{\partial x_s}(v_\varepsilon(y))\right)_{r,s}\right), \end{aligned} \tag{3.15}$$

where $R_{i;j}$ is a polynomial of degree $n-1$ in n^2 variables and $[\mathrm{D}v_\varepsilon(y)]_{ij}$ denotes the (i,j)-th entry of the Jacobian of v_ε at y. Using this equality, assumption (3.12), the inclusions $\operatorname{im} v_\varepsilon \subseteq W_\varepsilon \subseteq K$ and our assumption that $(u_\varepsilon)_\varepsilon$ is moderate, we obtain

$$\begin{aligned} \sup_{y \in V_\varepsilon} \left| \frac{\partial v_\varepsilon^{(i)}(y)}{\partial y_j} \right| &= \sup_{y \in V_\varepsilon} \left| \frac{1}{\det(\mathrm{D}u_\varepsilon(v_\varepsilon(y)))} \cdot R_{i;j}\left(\left(\frac{\partial u_\varepsilon^{(r)}}{\partial x_s}(v_\varepsilon(y))\right)_{r,s}\right) \right| \\ &\le \sup_{x \in W_\varepsilon} \underbrace{\frac{1}{|\det(\mathrm{D}u_\varepsilon(x))|}}_{\le \frac{1}{C_1}\varepsilon^{-N_1}} \cdot \sup_{x \in K} \underbrace{\left| R_{i;j}\left(\left(\frac{\partial u_\varepsilon^{(r)}}{\partial x_s}(x)\right)_{r,s}\right) \right|}_{\le C \cdot \varepsilon^{-N}} \\ &\le \frac{C}{C_1} \cdot \varepsilon^{-(N_1+N)} \end{aligned}$$

for some constant C, some fixed $N \in \mathbb{N}$ and ε sufficiently small. By the chain resp. the quotient rules, we find

$$
\begin{aligned}
&\frac{\partial^2 v_\varepsilon^{(i)}(y)}{\partial y_k \partial y_j} = \\
&= \frac{\partial}{\partial y_k}\left(\frac{1}{\det(\mathrm{D}u_\varepsilon(v_\varepsilon(y)))} \cdot R_{i;j}\left(\left(\frac{\partial u_\varepsilon^{(r)}}{\partial x_s}(v_\varepsilon(y))\right)_{r,s}\right)\right) \\
&= \frac{1}{\det(\mathrm{D}u_\varepsilon(v_\varepsilon(y)))^2} \cdot S_{i;j,k}\left(\left(\frac{\partial u_\varepsilon^{(r)}}{\partial x_s}(v_\varepsilon(y)), \frac{\partial^2 u_\varepsilon^{(r)}}{\partial x_s \partial x_t}(v_\varepsilon(y)) \cdot \frac{\partial v_\varepsilon^{(t)}}{\partial y_k}(y)\right)_{r,s,t}\right),
\end{aligned}
$$

where $S_{i;j,k}$ again is some polynomial. Estimating in a similar fashion to above, we see that also the second partial derivatives of $v_\varepsilon^{(i)}$ do not grow faster than some inverse power of ε. By induction, we also obtain the desired estimates for higher partial derivatives of $v_\varepsilon^{(i)}$, thus concluding the proof of the first claim of the proposition.

(2): The idea to extend the v_ε is to use two-member partitions of unity $(p_\varepsilon, 1-p_\varepsilon)$ where the plateau functions p_ε serve to retain the values of v_ε on K_ε. Since $\operatorname{dist}(K_\varepsilon, V_\varepsilon^c) \geq C_2\varepsilon^{N_2}$, it follows from Proposition 3.30 that there exists a net $(p_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ of plateau functions such that $p_\varepsilon|_{K_\varepsilon} = 1$ and $\operatorname{supp} p_\varepsilon \subseteq V_\varepsilon$ for all ε. Let $\tilde{v}_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$
\tilde{v}_\varepsilon(x) := \begin{cases} p_\varepsilon(x)\, v_\varepsilon(x) + (1 - p_\varepsilon(x))\, \tilde{x}_\varepsilon, & x \in V_\varepsilon \\ \tilde{x}_\varepsilon, & \text{otherwise} \end{cases}.
$$

Since for every ε the function p_ε is in $\mathcal{D}(V_\varepsilon)$, the functions $\tilde{v}_\varepsilon$ are smooth. By construction, $\tilde{v}_\varepsilon(x) = \tilde{x}_\varepsilon$ for all $x \in \mathbb{R}^n \backslash V_\varepsilon$ and, as $p_\varepsilon|_{K_\varepsilon} = 1$ for all $\varepsilon \in (0, \varepsilon_0]$, then $\tilde{v}_\varepsilon|_{K_\varepsilon} = v_\varepsilon|_{K_\varepsilon}$ also holds. To prove the moderateness of $(\tilde{v}_\varepsilon)_\varepsilon$ we have to show that for given $K \subset\subset \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ there exists some $N \in \mathbb{N}$ with $\sup_{y \in K} |\partial^\alpha \tilde{v}_\varepsilon(y)| = O(\varepsilon^{-N})$ as $\varepsilon \to 0$. As before, it suffices to consider $\partial^\alpha \tilde{v}_\varepsilon^{(i)}$, where $\tilde{v}_\varepsilon^{(i)}$ denotes the i-th component of $\tilde{v}_\varepsilon$. Let $K \subset\subset \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$. To obtain the $\mathcal{E}_M$-estimates, fix ε and split K into the sets $K \backslash V_\varepsilon$ and $K \cap V_\varepsilon$. Since $K \backslash V_\varepsilon$ is a proper subset of the open set $\mathbb{R}^n \backslash \operatorname{supp} p_\varepsilon$ and $\tilde{v}_\varepsilon$ restricted to this open set has the constant value $\tilde{x}_\varepsilon$, it follows that for all $y \in K \backslash V_\varepsilon$ all derivatives of $\tilde{v}_\varepsilon$ vanish. On $K \cap V_\varepsilon$ we write $\tilde{v}_\varepsilon^{(i)}(x)$ as $p_\varepsilon(x)\, v_\varepsilon(x)^{(i)} + (1 - p_\varepsilon(x))\, \tilde{x}_\varepsilon^{(i)}$. By the Leibniz rule, we obtain

$$
\begin{aligned}
\partial^\alpha \tilde{v}_\varepsilon^{(i)}(x) &= \partial^\alpha \left(p_\varepsilon(x)\, v_\varepsilon^{(i)}(x) + (1 - p_\varepsilon(x))\, \tilde{x}_\varepsilon^{(i)}\right) \\
&= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta p_\varepsilon(x)\, \partial^{\alpha-\beta} v_\varepsilon^{(i)}(x) - \partial^\alpha p_\varepsilon(x)\, \tilde{x}_\varepsilon^{(i)}
\end{aligned}
$$

for $x \in K \cap V_\varepsilon$. By the moderateness of $(p_\varepsilon)_\varepsilon$, the boundedness of $(\tilde{x}_\varepsilon)_\varepsilon$ and inequality (3.13) (established in the first part of this proof), it follows that

$\partial^\alpha \tilde{v}_\varepsilon^{(i)}$ is bounded on $K \cap V_\varepsilon$ by some inverse power of ε. Since all derivatives of $\tilde{v}_\varepsilon^{(i)}$ are constant (where all occurring values are contained in a compact set) resp. zero on $K \setminus V_\varepsilon$, corresponding estimates also hold for all $x \in K$. Finally, we show that $(\tilde{v}_\varepsilon)_\varepsilon$ is even uniformly bounded. By our assumption, $\operatorname{im} v_\varepsilon \subseteq W_\varepsilon \subseteq K$. Since $\tilde{v}_\varepsilon(x) = p_\varepsilon(x)\, v_\varepsilon(x) + (1 - p_\varepsilon(x))\, \tilde{x}_\varepsilon$ for all $x \in \mathbb{R}^n$, the image point $\tilde{v}_\varepsilon(x)$ is either in K or at least lies on the line connecting $v_\varepsilon(x)$ (which is in K) and $\tilde{x}_\varepsilon$ (which is in L). Hence, $\operatorname{im} \tilde{v}_\varepsilon$ is contained in the convex hull of $K \cup L$. The c-boundedness of $(v_\varepsilon)_\varepsilon$ into any open subset of $\mathbb{R}^n$ containing the convex hull of $K \cup L$ is a direct consequence. $\square$

If in the above proposition the W_ε are equal to some open W $(\subseteq K)$ for all ε and the compact sets K_ε are the images of a fixed compact subset of W under u_ε, condition (3.14) in the second part is automatically satisfied.

To prove this we first need a lemma. In what follows we will denote by $\overline{xy}$ the line segment $\{\lambda x + (1-\lambda) y \mid 0 \leq \lambda \leq 1\}$ where x and y are elements of some affine space.

3.33. Lemma: *Let $A \subseteq \mathbb{R}^n$ and $x \in A^\circ$. Let $y \in \partial A$ such that $|x - y| = \operatorname{dist}(x, \partial A)$. Then the half-open line segment $S = \overline{xy} \setminus \{y\}$ is a subset of A°.*

Proof: Obviously, S is connected and splits into $S \cap A^\circ$ and $S \cap \operatorname{ext}(A)$ since $S \cap \partial A = \emptyset$ by assumption (every $z \in S$ satisfies $|x - z| < |x - y|$). From $x \in S \cap A^\circ$, it follows that $S \cap \operatorname{ext}(A)$ is empty which establishes $S \subseteq A^\circ$. $\square$

3.34. Proposition: *Let U be an open subset of $\mathbb{R}^n$, W a (non-empty) open subset of U with $\overline{W} \subset\subset U$ and $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$. For all $\varepsilon \in (0, \varepsilon_0]$ let u_ε be injective on W with inverse $v_\varepsilon : u_\varepsilon(W) \to W$. Let $[(\tilde{x}_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}_c^n$ with $\tilde{x}_\varepsilon \in K' \subset\subset \mathbb{R}^n$ for all $\varepsilon \leq \varepsilon_0$, let K be a compact subset of W and $K_\varepsilon := u_\varepsilon(K)$. If*

$$\inf_{x \in W} |\det(\mathrm{D}u_\varepsilon(x))| \geq C_1 \varepsilon^{N_1}$$

for some $C_1 > 0$, $N_1 \in \mathbb{N}_0$ and for all $\varepsilon \leq \varepsilon_0$, then all v_ε are smooth and there exist $(K_\varepsilon, \tilde{x}_\varepsilon)$-extensions $\tilde{v}_\varepsilon$ of v_ε such that $(\tilde{v}_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. Furthermore, the net $(\tilde{v}_\varepsilon)_\varepsilon$ is uniformly bounded. In particular, $(\tilde{v}_\varepsilon)_\varepsilon$ is c-bounded into any open subset of $\mathbb{R}^n$ that contains the convex hull of $\overline{W} \cup K'$.

Proof: Set $V_\varepsilon := u_\varepsilon(W)$. All we have to do is to show that

$$\operatorname{dist}(K_\varepsilon, V_\varepsilon^c) \geq C \varepsilon^N$$

for some $C > 0$, a natural number N and sufficiently small ε. Applying Proposition 3.31 (2) then yields the desired result.

By Theorem 3.18, V_ε is open in $\mathbb{R}^n$ and u_ε maps W homeomorphically to V_ε. Choose $y_{1\varepsilon} \in \partial K_\varepsilon$ and $y_{2\varepsilon} \in \partial V_\varepsilon$ such that $\operatorname{dist}(K_\varepsilon, V_\varepsilon^c) =$

$\mathrm{dist}(\partial K_\varepsilon, \partial V_\varepsilon) = |y_{1\varepsilon} - y_{2\varepsilon}|$. Set $\eta := \mathrm{dist}(K, W^c) > 0$ and let $L := K + \overline{B_{\frac{\eta}{2}}(0)}$. Then L is a compact subset of W and $L_\varepsilon := u_\varepsilon(L)$ is a compact subset of V_ε. Set $\delta_\varepsilon := \mathrm{dist}(L_\varepsilon, V_\varepsilon^c) > 0$. Since, by construction, $K_\varepsilon \subseteq L_\varepsilon^\circ$, we have

$$\delta_\varepsilon \leq \mathrm{dist}(K_\varepsilon, V_\varepsilon^c) = |y_{1\varepsilon} - y_{2\varepsilon}|.$$

Choose some $\tilde{y}_{2\varepsilon}$ on the open line segment between $y_{1\varepsilon}$ and $y_{2\varepsilon}$ with

$$|\tilde{y}_{2\varepsilon} - y_{2\varepsilon}| < \delta_\varepsilon.$$

Since $y_{2\varepsilon} \in \partial V_\varepsilon$ and $\mathrm{dist}(L_\varepsilon, V_\varepsilon^c) = \delta_\varepsilon$, it follows that $\tilde{y}_{2\varepsilon} \notin L_\varepsilon$. By Lemma 3.33, $\overline{y_{1\varepsilon}y_{2\varepsilon}} \setminus \{y_{2\varepsilon}\}$ is a subset of V_ε and, hence, $\tilde{y}_{2\varepsilon} \in V_\varepsilon \backslash L_\varepsilon$. Let $x_{1\varepsilon} \in K$ and $\tilde{x}_{2\varepsilon} \in W \backslash L$ such that $u_\varepsilon(x_{1\varepsilon}) = y_{1\varepsilon}$ resp. $u_\varepsilon(\tilde{x}_{2\varepsilon}) = \tilde{y}_{2\varepsilon}$. Then, since $\mathrm{dist}(K, L^c) = \mathrm{dist}\left(K, (K + B_{\frac{\eta}{2}}(0))^c\right) = \frac{\eta}{2}$ and $\tilde{x}_{2\varepsilon} \in W \backslash L \subseteq L^c$, we have

$$|\tilde{x}_{2\varepsilon} - x_{1\varepsilon}| \geq \mathrm{dist}(\tilde{x}_{2\varepsilon}, K) \geq \frac{\eta}{2}.$$

Therefore,

$$\mathrm{dist}(K_\varepsilon, V_\varepsilon^c) = |y_{1\varepsilon} - y_{2\varepsilon}| \geq |y_{1\varepsilon} - \tilde{y}_{2\varepsilon}| = |u_\varepsilon(x_{1\varepsilon}) - u_\varepsilon(\tilde{x}_{2\varepsilon})|.$$

By the Mean Value Theorem (note that $\overline{y_{1\varepsilon}\tilde{y}_{2\varepsilon}} \subseteq V_\varepsilon$ by Lemma 3.33), we obtain

$$\begin{aligned} \frac{\eta}{2} &\leq |x_{1\varepsilon} - \tilde{x}_{2\varepsilon}| \\ &= |v_\varepsilon(u_\varepsilon(x_{1\varepsilon})) - v_\varepsilon(u_\varepsilon(\tilde{x}_{2\varepsilon}))| \\ &\leq \sup_{y \in V_\varepsilon} \|\mathrm{D}v_\varepsilon(y)\| \cdot |u_\varepsilon(x_{1\varepsilon}) - u_\varepsilon(\tilde{x}_{2\varepsilon})|. \end{aligned} \tag{3.16}$$

By Proposition 3.31, there exist $N \in \mathbb{N}$ and $C' > 0$, both independent of ε, and some $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$\sup_{y \in V_\varepsilon} \|\mathrm{D}v_\varepsilon(y)\| \leq C' \varepsilon^{-N}$$

for all $\varepsilon \leq \varepsilon_1$. Together with (3.16) this entails

$$|u_\varepsilon(x_{1\varepsilon}) - u_\varepsilon(\tilde{x}_{2\varepsilon})| \geq C \varepsilon^N$$

for $C_2 := \frac{\eta}{2C'}$ and $\varepsilon \leq \varepsilon_1$ and we are done. □

Now it is easy to prove

3.35. Theorem: *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}[U, \mathbb{R}^n]$. If u is ca-injective and $\det \circ \mathrm{D}u$ is strictly non-zero, then u is left invertible on any open subset W of U with $\overline{W} \subset\subset U$.*

Proof: Let W and W' be two open subsets of U with $\overline{W} \subset\subset W' \subseteq \overline{W'} \subset\subset U$. By the ca-injectivity of $u = [(u_\varepsilon)_\varepsilon]$, there exists some $\varepsilon_0 \in (0,1]$ such that $u_\varepsilon|_{\overline{W'}}$ is injective for all $\varepsilon \leq \varepsilon_0$. Let $v_\varepsilon : u_\varepsilon(W') \to W'$ be the inverse of $u_\varepsilon|_{W'}$. Now apply Proposition 3.34 to U, W', $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon$, $0 \in \{0\}$, $\overline{W}$ and $K_\varepsilon := u_\varepsilon(\overline{W})$. By the c-boundedness of u, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $u_\varepsilon(\overline{W}) \subseteq K$ for sufficiently small ε. We obtain that u is left invertible on W by $[W, \mathbb{R}^n, \tilde{v} := [(\tilde{v}_\varepsilon)_\varepsilon], B_l]$ where $\tilde{v}_\varepsilon$ is a smooth $(K_\varepsilon, 0)$-extension of v_ε and B_l can be any open subset of $\mathbb{R}^n$ that contains K. $\square$

Note that to construct the left inverse in Theorem 3.35 we used only *one* representative that is ca-injective. However, by the discussion following Corollary 3.13, we know that for left invertible generalised functions *all* representatives have this property. Hence, Theorem 3.35 immediately yields

3.36. Corollary: *Let U be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}[U, \mathbb{R}^n]$ and $\det \circ \mathrm{D}u$ strictly non-zero. If one representative of u is ca-injective, then all representatives have this property.*

At this point the question arises if we may prove a theorem with respect to ca-surjectivity and right invertibility corresponding to Theorem 3.35, i.e. a partial converse to Proposition 3.23. A quick glance at the results from which Theorem 3.35 was derived shows that matters turn out to be more complex as to such a “dual” statement: Given ca-injectivity of $(u_\varepsilon)_\varepsilon$ we have set-theoretic inverses $(v_\varepsilon)_\varepsilon$ on suitable open sets. These can be lifted to the level of moderate c-bounded nets by Proposition 3.34, yielding a left inverse for $[(u_\varepsilon)_\varepsilon]$. Dually, given ca-surjectivity of $(u_\varepsilon)_\varepsilon$, we fail when trying to imitate this argument since we do not even obtain continuous right inverses, in general.

However, we can show that local invertibility follows from the combination of ca-injectivity and ca-surjectivity and the assumption that $\det \circ \mathrm{D}u$ is strictly non-zero.

3.37. Theorem: *Let U and B be open subsets of $\mathbb{R}^n$ and $u \in \mathcal{G}[U, \mathbb{R}^n]$. If u is ca-injective and ca-surjective onto B and if $\det \circ \mathrm{D}u$ is strictly non-zero, then u is locally invertible on U.*

More precisely, for every $z \in U$ and every open subset B_r of B with $\overline{B_r} \subset\subset B$ there exist an open neighbourhood A of z with $\overline{A} \subset\subset U$, an open relatively compact subset B_l of $\mathbb{R}^n$ containing $\overline{B_r}$, and some $v \in \mathcal{G}(\mathbb{R}^n)^n$ such that u is invertible on A with inversion data $[A, \mathbb{R}^n, v, B_l, B_r]$. The set A can be chosen to contain any given $M \subset\subset U$. Furthermore, there exist representatives $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v such that $v_\varepsilon \circ u_\varepsilon|_A = \mathrm{id}_A$ and $u_\varepsilon \circ v_\varepsilon|_{B_r} = \mathrm{id}_{B_r}$ for sufficiently small ε.

Proof: Let $z \in U$, $(u_\varepsilon)_\varepsilon$ a representative of u and B_r an open subset of B with $\overline{B_r} \subset\subset B$. Let $\delta > 0$ such that $\overline{(B_r)_\delta} \subset\subset B$ for $(B_r)_\delta := B_r + B_\delta(0)$. By the ca-surjectivity of u, there exists a compact subset K of U with $\overline{(B_r)_\delta} \subseteq u_\varepsilon(K)$ for ε sufficiently small. Choose a compact subset L of U with $K \cup \{z\} \cup M \subset\subset L^\circ$ for some given $M \subset\subset U$. Set $A := L^\circ$. Then $\overline{B_r} \subseteq u_\varepsilon(A)$ for small ε. Let $\eta > 0$ such that the closure of $A_\eta := A + B_\eta(0)$ is a compact subset of U. From the ca-injectivity of u, it follows that u_ε is invertible (as a function) on A_η by, say, $w_\varepsilon : u_\varepsilon(A_\eta) \to A_\eta$ for ε small enough. Proposition 3.34 now yields the existence of smooth $(u_\varepsilon(\overline{A}), y)$-extensions v_ε of w_ε (for $y \in A_\eta$ fixed arbitrarily) such that $(v_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)^n$. Thus, $v_\varepsilon \circ u_\varepsilon|_A = \mathrm{id}_A$ and $u_\varepsilon \circ v_\varepsilon|_{B_r} = \mathrm{id}_{B_r}$. Since $B_r \subseteq u_\varepsilon(K) \subseteq u_\varepsilon(A)$, we have $v_\varepsilon(B_r) = w_\varepsilon(B_r) \subseteq K \subset\subset L^\circ = A$. Hence, $v_\varepsilon|_{B_r}$ is c-bounded into A. By the c-boundedness of u, we can find a compact subset K' of $\mathbb{R}^n$ such that $u_\varepsilon(\overline{A}) \subseteq K'$ for ε small. Finally, let B_l be an open relatively compact subset of $\mathbb{R}^n$ containing K'. Then $\overline{B_r} \subseteq u_\varepsilon(A) \subseteq B_l$ and, thus, u is invertible on A with inversion data $[A, \mathbb{R}^n, v, B_l, B_r]$. □

3.38. Remark:

(1) By the preceding theorem, we do not obtain an inverse of u on arbitrarily small open subsets of U (as was the case in Theorem 3.35). On the contrary, the size of the neighbourhood A of $z \in U$ depends on B_r. This does not constitute a deficiency of our proof, rather it originates from the necessity of proving the c-boundedness of $v|_{B_r}$ into A. As was discussed earlier, A cannot be forced smaller, in general, by shrinking B_r (cf. Example 3.6).

(2) In the proof of Theorem 3.37 we construct, given some representative of u, a net of smooth (classically) inverse functions v_ε. This means we can find smooth inverse functions to *any* given representative of u. However, the sets A and B_l depend on the chosen representative.

Finally, we demonstrate to what extent for an invertible u with inverse v there exist representatives $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v such that the compositions $v_\varepsilon \circ u_\varepsilon$ and $u_\varepsilon \circ v_\varepsilon$ classically are the identity (on suitable sets).

3.39. Theorem: *Let U be an open subset of $\mathbb{R}^n$, A an open subset of U and $u \in \mathcal{G}(U)^n$ invertible on A with inversion data $[A, V, v, B_l, B_r]$. For every representative $(u_\varepsilon)_\varepsilon$ of u and for every open subset W of B_r with $\overline{W} \subset\subset B_r$ the following hold: There exist an open subset A' of A with $\overline{A'} \subset\subset A$ and a moderate net of functions $(w_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)^n$ such that $w_\varepsilon \circ u_\varepsilon|_{A'} = \mathrm{id}_{A'}$ and $u_\varepsilon \circ w_\varepsilon|_W = \mathrm{id}_W$ for sufficiently small ε. Moreover, u is invertible on A' by*

$[A', \mathbb{R}^n, w := [(w_\varepsilon)_\varepsilon], B_l, W]$ and $w|_W = v|_W$ in $\mathcal{G}(W)^n$. The set A' can be chosen to contain any given $M \subset\subset A$.

Proof: Let $(u_\varepsilon)_\varepsilon$ be a representative of u, W an open subset of B_r with $\overline{W} \subset\subset B_r$, M a compact subset of A and $z \in M$. Let $\delta > 0$ such that $\overline{W_\delta} \subset\subset B_r$ where $W_\delta := W + B_\delta(0)$. By Propositions 3.16 and 3.23, we know that $(u_\varepsilon)_\varepsilon$ is ca-injective on A and ca-surjective on A onto B_r. Furthermore, Proposition 3.25 says that $\det \circ \mathrm{D}u$ is strictly non-zero on A. Then it follows from Theorem 3.37 (applied to A, B_r and W in place of U, B and B_r) that there exist an open neighbourhood A' of z in A with $M \subseteq A' \subseteq \overline{A'} \subset\subset A$ and some $w \in \mathcal{G}(\mathbb{R}^n)^n$ such that u is invertible on A' with inversion data $[A', \mathbb{R}^n, w, B_l, W]$. Furthermore, by Remark 3.38 (2), there exists a representative $(w_\varepsilon)_\varepsilon$ of w such that $w_\varepsilon \circ u_\varepsilon|_{A'} = \mathrm{id}_{A'}$ and $u_\varepsilon \circ w_\varepsilon|_W = \mathrm{id}_W$ for ε sufficiently small. The equality $w|_W = v|_W$ in $\mathcal{G}(W)^n$ follows from Proposition 3.5 (3). $\square$

3.4 Generalised inverse function theorems

The classical Inverse Function Theorem says that, solely from the invertibility of the derivative at a point x_0 of a given function f, we may deduce that on a suitable neighbourhood of x_0 the function itself is C^1-invertible. Conversely, by the chain rule, if f is C^1-invertible on some open set W, then its derivative is invertible at every $x \in W$. In analogy to the latter statement we proved in Section 3.2 that for every generalised function $u \in \mathcal{G}(U)^n$ invertible on A the determinant of the derivative is strictly non-zero at all points of A. Contrary to the classical case, however, this latter property at only *one* point is not sufficient to imply invertibility of u on some neighbourhood. Certainly, it provides ε-wise smooth inverses of a representative, but it says nothing about the sizes of the neighbourhoods on which those inverses are defined. In the following series of examples, we consider generalised functions defined on open subsets of $\mathbb{R}$ and examine their derivative at 0 and their (non-)invertibility behaviour on certain neighbourhoods of 0.

3.40. Example: Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ be defined by $u_\varepsilon(x) := \varepsilon \sin x$ (Figure 3.4). The derivative at 0 is $\mathrm{D}u_\varepsilon(0) = \varepsilon$, i.e. $\det \circ \mathrm{D}u(0)$ is strictly non-zero. Nevertheless, u is not invertible on any neighbourhood of 0 since it is not ca-surjective on $(-\alpha, \alpha)$ onto any open subset of $\mathbb{R}$.

Even if we demand that $\mathrm{D}u_\varepsilon(x_0)$ grows as $\frac{1}{\varepsilon}$, or at least is bounded away from 0, the situation does not get better.

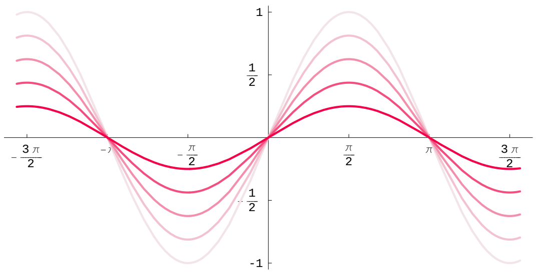

Figure 3.4: $u_\varepsilon(x) = \varepsilon \sin x$

3.41. Example: Consider $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ given by $u_\varepsilon(x) := \varepsilon \sin \frac{x}{\varepsilon}$ (Figure 3.5). The derivative at 0 is $\mathrm{D}u_\varepsilon(0) = 1$ for

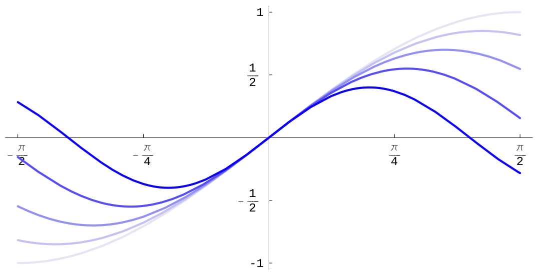

Figure 3.5: $u_\varepsilon(x) = \varepsilon \sin \frac{x}{\varepsilon}$

all ε. Again, u is not invertible on any neighbourhood of 0 since it is not ca-surjective on $(-\alpha, \alpha)$ onto any open subset of $\mathbb{R}$.

3.42. Example: Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ be given by $u_\varepsilon(x) := \varepsilon \sin \frac{x}{\varepsilon^2}$ (Figure 3.6). This time the derivative at 0

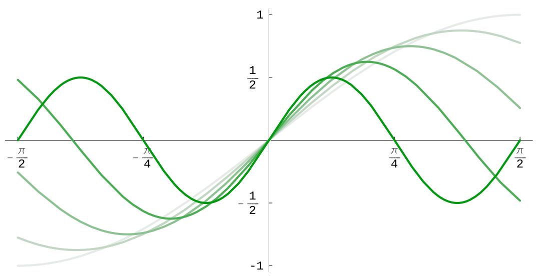

Figure 3.6: $u_\varepsilon(x) = \varepsilon \sin \frac{x}{\varepsilon^2}$

is $\mathrm{D}u_\varepsilon(0) = \frac{1}{\varepsilon}$, i.e. growing as $\varepsilon \to 0$. But still u is not invertible on any neighbourhood of 0, for the same reasons as before.

Thus it becomes apparent that the invertibility of the derivative at one point is not enough to ensure invertibility of u on any neighbourhood of that point. To stabilise the sizes of the sets on which the functions u_ε and their inverses are defined, it seems inevitable that we must impose conditions on u and/or its derivative even on some neighbourhood of x_0.

In Section 1.1, we presented a proof of the classical Inverse Function Theorem that keeps track of the minimum sizes of the neighbourhoods on which the function is invertible. In what follows we will make good use of those lower bounds in the proof of a generalised inverse function theorem. To begin with, we pin down an estimate for the determinant of a square matrix by its operator norm.

3.43. Proposition: *Let A be a square matrix with entries in $\mathbb{R}$. Then*

$$|\det(A)| \leq C \cdot \|A\|^n$$

holds, where $C > 0$ is some constant depending on the norms employed in $\mathbb{R}^n$.

Proof: By Hadamard's Inequality (see e.g. [Fis02], page 298),

$$|\det(A)| \leq \prod_{i=1}^{n} \|a_i\|_2 \leq \left(\sup_i \|a_i\|_2\right)^n, \tag{3.17}$$

where a_i denotes the i-th row of A. Since $N : A \mapsto \sup_i \|a_i\|_2$ defines a norm on the finite dimensional vector space $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$, there exists a constant $C' > 0$ such that $N(A) \leq C'\|A\|$ for all $A \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$. Together with (3.17) this yields the desired inequality. □

The quickest way to obtain an inverse function theorem for generalised functions u (with representative $(u_\varepsilon)_\varepsilon$) consists in assuming that the estimates of Theorem 1.3 hold uniformly in ε for all u_ε. Then Proposition 3.34 takes care of the common domain and the moderateness of the inverses of the u_ε. Recall that $\tilde{x} \approx y$ ($\tilde{x} \in \tilde{\mathbb{R}}^n$, $y \in \mathbb{R}^n$) signifies that y is the shadow of $\tilde{x}$, i.e. that for one (hence any) representative $(\tilde{x}_\varepsilon)_\varepsilon$ of x the net $(\tilde{x}_\varepsilon)_\varepsilon$ converges to y as $\varepsilon \to 0$ (see Definition 2.36).

3.44. Theorem: *Let U be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}[U, \mathbb{R}^n]$ and $x_0 \in U$. Let $\varepsilon_1 \in (0, 1]$, $y_0 \in \mathbb{R}^n$, $a, b > 0$ and $r > 0$ satisfying the following conditions:*

(i) $u(x_0) \approx y_0$.

(ii) $ab < 1$.

(iii) $\overline{B_r(x_0)} \subseteq U$.

If there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that for all $\varepsilon \leq \varepsilon_1$

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \leq a$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \leq b$ *for all* $x \in \overline{B_r(x_0)}$,

then u *is invertible on* $B_{\alpha r}(x_0)$ *with inversion data*

$$\left[B_{\alpha r}(x_0), \mathbb{R}^n, v, B, B_{\beta\frac{(1-ab)}{a}\gamma r}(y_0)\right],$$

where α *and* β *are arbitrary in* $(0,1)$, γ *is arbitrary in* $(0,\alpha)$ *and* $B \subseteq \mathbb{R}^n$ *is an arbitrary open set containing* $\overline{\bigcup_{\varepsilon\leq\varepsilon_2} u_\varepsilon(B_{\alpha r}(x_0))}$ *for some suitable* $\varepsilon_2 \leq \varepsilon_1$.

Furthermore, $v(y_0) \approx x_0$. *Also, there exists a representative* $(v_\varepsilon)_\varepsilon$ *of* v *such that*

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r}(x_0))} = u_\varepsilon|_{B_{\alpha r}(x_0)}{}^{-1}$$

for all $\varepsilon \leq \varepsilon_2$.

Proof: Since the demonstration of this theorem would be but a slimmed-down version of the proof of the next theorem, we omit it and refer to what follows. □

The preceding theorem, however, is not capable of handling situations such as jumps (cf. Example 3.3), which we consider as crucial due to their appearance in applications (see Chapter 4). This shortcoming stems from the assumption of uniform boundedness (with respect to ε) of the norms of both $\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)$ and $\mathrm{D}u_\varepsilon(x_0)^{-1}$: Typically, the former is violated by (representatives of) jump functions and the latter by their inverses.

So we present a result much more flexible than Theorem 3.44 (but including it): Essentially, we replace a by $a_\varepsilon\varepsilon^N$ and b by $b_\varepsilon\varepsilon^{-N}$.

3.45. Theorem: *Let* U *be an open subset of* $\mathbb{R}^n$, $u \in \mathcal{G}[U, \mathbb{R}^n]$ *and* $x_0 \in U$. *Let* $y_0 \in \mathbb{R}^n$, $\varepsilon_1 \in (0,1]$, $a_\varepsilon, b_\varepsilon > 0$ $(\varepsilon \leq \varepsilon_1)$, $N \in \mathbb{N}_0$, $d > 0$ *and* $r > 0$ *satisfying the following conditions:*

(i) $u(x_0) \approx y_0$.

(ii) $s := \sup\{a_\varepsilon \,|\, 0 < \varepsilon \leq \varepsilon_1\}$ *is finite.*

(iii) $a_\varepsilon b_\varepsilon + d\varepsilon^N \leq 1$ *for all* $\varepsilon \leq \varepsilon_1$.

(iv) $\overline{B_r(x_0)} \subseteq U$.

If there exists a representative $(u_\varepsilon)_\varepsilon$ *of* u *such that for all* $\varepsilon \leq \varepsilon_1$

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \le a_\varepsilon \varepsilon^N$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \le b_\varepsilon \varepsilon^{-N}$ *for all* $x \in \overline{B_r(x_0)}$,

then u *is invertible on* $B_{\alpha r}(x_0)$ *with inversion data*

$$\left[B_{\alpha r}(x_0), \mathbb{R}^n, v, B, B_{\beta \frac{d}{s}\gamma r}(y_0)\right],$$

where α *and* β *are arbitrary in* $(0,1)$, γ *is arbitrary in* $(0,\alpha)$ *and* $B \subseteq \mathbb{R}^n$ *is an arbitrary open set containing* $\overline{\bigcup_{\varepsilon \le \varepsilon_2} u_\varepsilon(B_{\alpha r}(x_0))}$ *for some suitable* $\varepsilon_2 \le \varepsilon_1$.

Furthermore, $v(y_0) \approx x_0$. *Also, there exists a representative* $(v_\varepsilon)_\varepsilon$ *of* v *such that*

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r}(x_0))} = {u_\varepsilon|_{B_{\alpha r}(x_0)}}^{-1}$$

for all $\varepsilon \le \varepsilon_2$.

Proof: We assume w.l.o.g. $x_0 = 0$ (otherwise, replace U by $U - x_0$ and $u_\varepsilon(x)$ by $u_\varepsilon(x + x_0)$) and $y_0 = 0$ (otherwise consider $u_\varepsilon(x) - y_0$); therefore, we have $u_\varepsilon(0) \approx 0$.

Let $\varepsilon \le \varepsilon_1$. Substituting a by $a_\varepsilon \varepsilon^N$ and b by $b_\varepsilon \varepsilon^{-N}$ in the Inverse Function Theorem 1.3 shows that (by Remark 1.4) u_ε is smoothly invertible on $B_r(0)$. Let $w_\varepsilon : V_\varepsilon \to B_r(0)$ denote the smooth inverse of $u_\varepsilon|_{B_r(0)}$, where $V_\varepsilon := u_\varepsilon(B_r(0))$ is open in $\mathbb{R}^n$. By (iii),

$$\frac{a_\varepsilon \varepsilon^N}{1 - a_\varepsilon b_\varepsilon} \le \frac{a_\varepsilon \varepsilon^N}{d\,\varepsilon^N} \le \frac{s}{d}$$

holds. Therefore, $\frac{a_\varepsilon \varepsilon^N}{1 - a_\varepsilon b_\varepsilon}$ being the value corresponding to c in Theorem 1.3, we obtain

$$|\mathrm{D}u_\varepsilon(x)^{-1}| \le \frac{s}{d}$$

for all $x \in \overline{B_r(0)}$. From Proposition 3.43, it follows that

$$|\det(\mathrm{D}u_\varepsilon(x))| = \left|\frac{1}{\det(\mathrm{D}u_\varepsilon(x)^{-1})}\right| \ge \frac{d^n}{Cs^n} \tag{3.18}$$

for some constant $C > 0$ and for all $x \in \overline{B_r(0)}$. Now let $\alpha \in (0,1)$ and $K_\varepsilon := u_\varepsilon(\overline{B_{\alpha r}(0)})$. From (3.18), it immediately follows by Proposition 3.34 that there exist $(K_\varepsilon, 0)$-extensions v_ε of w_ε such that $(v_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. In particular, $v_\varepsilon \circ u_\varepsilon|_{B_{\alpha r}(0)} = \mathrm{id}_{B_{\alpha r}(0)}$. Now let $\beta \in (0,1)$ and $\gamma \in (0,\alpha)$. Since $u_\varepsilon(0)$ converges to 0 for $\varepsilon \to 0$, there exists some $\varepsilon_2 \le \varepsilon_1$ such that

$$|u_\varepsilon(0)| \le (1-\beta)\frac{d}{s}\gamma\, r$$

for all $\varepsilon \le \varepsilon_2$. Thus, by Proposition 1.5, $B_{\beta \frac{d}{s}\gamma r}(0) \subseteq u_\varepsilon(B_{\gamma r}(0))$ for all $\varepsilon \le \varepsilon_2$. From now on, we always let $\varepsilon \le \varepsilon_2$. Since $u_\varepsilon(B_{\gamma r}(0)) \subseteq K_\varepsilon$, we

have $u_\varepsilon \circ v_\varepsilon|_{B_{\beta\frac{d}{s}\gamma r}(0)} = \mathrm{id}_{B_{\beta\frac{d}{s}\gamma r}(0)}$. Moreover, $(v_\varepsilon|_{B_{\beta\frac{d}{s}\gamma r}(0)})_\varepsilon$ is c-bounded into $B_{\alpha r}(0)$ since $v_\varepsilon(B_{\beta\frac{d}{s}\gamma r}(0)) \subseteq \overline{B_{\gamma r}(0)} \subseteq B_{\alpha r}(0)$. Furthermore, $(u_\varepsilon|_{B_{\alpha r}(0)})_\varepsilon$ is c-bounded into any open set $B \subseteq \mathbb{R}^n$ that contains $\overline{\bigcup_{\varepsilon\leq\varepsilon_2} u_\varepsilon(B_{\alpha r}(0))}$ since u is c-bounded into $\mathbb{R}^n$.
par Finally, applying Theorem 1.3 (1) and due to the fact that $v_\varepsilon|_{B_{\beta\frac{d}{s}\gamma r}(0)} = w_\varepsilon|_{B_{\beta\frac{d}{s}\gamma r}(0)}$, we get

$$|v_\varepsilon(0)| = |v_\varepsilon(0) - v_\varepsilon(u_\varepsilon(0))| \leq \frac{s}{d} \cdot |0 - u_\varepsilon(0)|.$$

Since $u_\varepsilon(0) \to 0$, this also shows that $v_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. $\square$

3.46. Remark: Condition 3.45 (1) appears in the first place to make sure that for every ε the inverse $\mathrm{D}u_\varepsilon(x_0)^{-1}$ exists and, therefore, to give meaning to 3.45 (2). However, it turns out that (2) actually implies a much stronger condition on $(\det(\mathrm{D}u_\varepsilon(x_0)))_\varepsilon$ than (1) does: By Proposition 3.43,

$$|\det(\mathrm{D}u_\varepsilon(x_0))| = \left|\frac{1}{\det(\mathrm{D}u_\varepsilon(x_0)^{-1})}\right| \geq \frac{1}{Ca_\varepsilon^n}\,\varepsilon^{-nN} \geq \frac{1}{Cs^n}\,\varepsilon^{-nN} \qquad (3.19)$$

for some constant $C > 0$ (cp. the preceding proof). This means that if $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\|$ is bounded by some positive power of ε, then the determinant of $\mathrm{D}u_\varepsilon(x_0)$ can be estimated from below by a negative power of ε, as ε tends to 0. In particular, $\det(\mathrm{D}u_\varepsilon(x_0))$ is strictly non-zero, a property that we already know to be satisfied by any invertible generalised function (cf. Proposition 3.25).
On the other hand, for $n \geq 2$ one cannot conclude from $\det(\mathrm{D}u_\varepsilon(x_0)) \geq g(\varepsilon)$ with $g(\varepsilon) \to \infty$ as $\varepsilon \to 0$, that condition (2) is satisfied: Consider $[(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^2)^2$ defined by $u_\varepsilon(x,y) := (x, g(\varepsilon)\,y)$. Then, for any $(x_0,y_0) \in \mathbb{R}^2$, $\mathrm{D}u_\varepsilon(x_0,y_0) = \begin{pmatrix} 1 & 0 \\ 0 & g(\varepsilon) \end{pmatrix}$ and $\det(\mathrm{D}u_\varepsilon(x_0,y_0)) = g(\varepsilon) \to \infty$ for $\varepsilon \to 0$. However,

$$\|\mathrm{D}u_\varepsilon(x_0,y_0)^{-1}\| = \left\|\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{g(\varepsilon)} \end{pmatrix}\right\| = \max\left(1, \frac{1}{g(\varepsilon)}\right) \to 1,$$

i.e. eventually $\|\mathrm{D}u_\varepsilon(x_0,y_0)^{-1}\|$ is not decreasing with ε.

Including 3.45 (iii) guarantees that Banach's Fixed Point Theorem can be applied (implicitly via Theorem 1.3).

The convergence of $(u_\varepsilon(x_0))_\varepsilon$ to some y_0 in Theorem 3.45 ensures that the images of the open ball $B_r(x_0)$ under the u_ε are not scattered wildly all over $\mathbb{R}^n$ but stay centred around y_0. One may suspect that this condition is stronger than necessary. As a matter of fact, the theorem still holds true if $u_\varepsilon(x_0)$ stays close enough to y_0 in the following sense: In the proof, convergence of $(u_\varepsilon(x_0))_\varepsilon$ is needed in one place only, namely to obtain ε_2 such that

$$|u_\varepsilon(x_0) - y_0| \le (1-\beta)\frac{d}{s}\,\gamma\, r \tag{3.20}$$

holds for all $\varepsilon \le \varepsilon_2$. Hence, u is invertible even if the convergence condition is weakened to (3.20).

The next proposition shows that the conditions of Theorem 3.45, in fact, are independent of the choice of the representative.

3.47. Proposition: *If one representative of $u \in \mathcal{G}(U)^n$ satisfies the conditions of Theorem 3.45, then every representative does.*

More precisely, if $(u_\varepsilon)_\varepsilon$ satisfies the conditions of the theorem with x_0, y_0, ε_1, a_ε, b_ε, s, N, d and r, then another given representative $(\bar{u}_\varepsilon)_\varepsilon$ of u satisfies them with x_0, y_0, N, r and suitable values for $\bar{\varepsilon}_1$, $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{s}$ and $\bar{d}$ which can be chosen to satisfy $\bar{\varepsilon}_1 \le \varepsilon_1$, $\bar{a}_\varepsilon \ge a_\varepsilon$ and $\bar{b}_\varepsilon \ge b_\varepsilon$ (for all $\varepsilon \le \bar{\varepsilon}_1$), $\bar{s} \ge s$ and $\bar{d} \le d$. Furthermore, we may suppose $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ for $\varepsilon \to 0$ and $\bar{s} \searrow s$ and $\bar{d} \nearrow d$ for $\bar{\varepsilon}_1 \to 0$.

Proof: Since the difference of $(u_\varepsilon)_\varepsilon$ and $(\bar{u}_\varepsilon)_\varepsilon$ is negligible, $\bar{u}_\varepsilon(x_0)$ converges to y_0 for $\varepsilon \to 0$. By Remark 3.46, $\det(\mathrm{D}u_\varepsilon(x_0))$ is strictly non-zero and, therefore, also $\det(\mathrm{D}\bar{u}_\varepsilon(x_0))$ has this property.

First, we show that w.l.o.g. we may assume that $(b_\varepsilon)_\varepsilon$ is moderate. On the one hand, we have

$$\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \le b_\varepsilon \varepsilon^{-N}. \tag{3.21}$$

On the other hand, by the moderateness of $\mathrm{D}u_\varepsilon$, there exist $C > 0$, $N_1 \in \mathbb{N}$ and $\varepsilon_2 \le \varepsilon_1$ such that

$$\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \le C\varepsilon^{-N_1} \tag{3.22}$$

for all $\varepsilon \le \varepsilon_2$. Inequalities (3.21) and (3.22) yield

$$\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \le b'_\varepsilon \varepsilon^{-N}$$

for $b'_\varepsilon := \min(b_\varepsilon, C\varepsilon^{N-N_1}) > 0$, a moderate net of positive numbers that still satisfies all other conditions of the theorem.

In the following, let $D > 0$ and $M \in \mathbb{N}$ such that $b_\varepsilon \le D\,\varepsilon^{-M}$ for all $\varepsilon \le \varepsilon_1$. By Proposition 2.31, the difference $(\mathrm{D}u_\varepsilon(x_0)^{-1} - \mathrm{D}\bar{u}_\varepsilon(x_0)^{-1})_\varepsilon$ is negligible. Therefore, we can choose $\varepsilon_2 \le \varepsilon_1$ such that

$$\|\mathrm{D}\bar{u}_\varepsilon(x_0)^{-1}\| \le \|\mathrm{D}u_\varepsilon(x_0)^{-1}\| + A\,\varepsilon^{2N+M+1} \le (a_\varepsilon + A\,\varepsilon^{N+M+1})\varepsilon^N \tag{3.23}$$

for some $A > 0$ and

$$\|\mathrm{D}\bar{u}_\varepsilon(x_0)-\mathrm{D}\bar{u}_\varepsilon(x)\| \le \|\mathrm{D}u_\varepsilon(x_0)-\mathrm{D}u_\varepsilon(x)\|+B\,\varepsilon \le (b_\varepsilon+B\,\varepsilon^{N+1})\varepsilon^{-N} \quad (3.24)$$

for all $x \in \overline{B_r(x_0)}$ for some $B > 0$. We set

$$\bar{a}_\varepsilon := a_\varepsilon + A\,\varepsilon^{N+M+1} \qquad \text{and} \qquad \bar{b}_\varepsilon := b_\varepsilon + B\,\varepsilon^{N+1}$$

for $\varepsilon \le \varepsilon_2$. Hence, $\bar{a}_\varepsilon \ge a_\varepsilon$ and $\bar{b}_\varepsilon \ge b_\varepsilon$ for all $\varepsilon \le \varepsilon_2$. Since $s = \sup\{a_\varepsilon \,|\, 0 < \varepsilon \le \varepsilon_1\}$ is finite and $A\,\varepsilon^{N+M+1}$ converges monotonously to zero, it follows that $\bar{s} := \sup\{\bar{a}_\varepsilon \,|\, 0 < \varepsilon \le \varepsilon_1\}$ is finite and greater or equal to s.

Finally, we check condition 3.45 (iii) for $\bar{a}_\varepsilon$ and $\bar{b}_\varepsilon$:

$$\begin{aligned}\bar{a}_\varepsilon\bar{b}_\varepsilon &= (a_\varepsilon + A\,\varepsilon^{N+M+1})(b_\varepsilon + B\,\varepsilon^{N+1})\\ &= \underbrace{a_\varepsilon b_\varepsilon}_{\le 1-d\varepsilon^N} + \underbrace{a_\varepsilon}_{\le s}\cdot B\,\varepsilon^{N+1} + \underbrace{b_\varepsilon}_{\le D\,\varepsilon^{-M}}\cdot A\,\varepsilon^{N+M+1} + A\,\varepsilon^{N+M+1}\cdot B\,\varepsilon^{N+1}\\ &\le 1 - \Big(d - \varepsilon\left(sB + AD + AB\,\varepsilon^{N+M+1}\right)\Big)\varepsilon^N.\end{aligned}$$

Now let $\bar{\varepsilon}_1 \le \varepsilon_2$ such that the expression in the brackets becomes positive for all $\varepsilon \le \bar{\varepsilon}_1$. Set $\bar{d} := d - \bar{\varepsilon}_1\left(sB + AD + AB\,\bar{\varepsilon}_1^{N+M+1}\right)$, then

$$\bar{a}_\varepsilon\bar{b}_\varepsilon \le 1 - \bar{d}\varepsilon^N$$

holds for all $\varepsilon \le \bar{\varepsilon}_1$. The convergences $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ for $\varepsilon \to 0$ and $\bar{s} \searrow s$ and $\bar{d} \nearrow d$ for $\bar{\varepsilon}_1 \to 0$ follow from the definitions of $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{s}$ and $\bar{d}$. ☐

3.48. Remark: Theorem 3.44 being a special case of Theorem 3.45, it is clear that the statement of Proposition 3.47 applies analogously to the situation of Theorem 3.44.

The following example shows that the inversion issue of the jump function of Example 3.3 is settled affirmatively by Theorem 3.45.

3.49. Example: Let $u \in \mathcal{G}((-\alpha,\alpha))$ (for $\alpha > 0$) be the generalised function modelling a jump with $u_\varepsilon(x) = x + \arctan\frac{x}{\varepsilon}$ as a representative. We already found in Example 3.3 that u is invertible on an open neighbourhood of 0. Indeed, $(u_\varepsilon)_\varepsilon$ satisfies all conditions of Theorem 3.45 with $x_0 = 0$, $y_0 = 0$, $\varepsilon_1 = 1$, $a_\varepsilon = \frac{1}{\varepsilon+1}$ (then $s = 1$), $b_\varepsilon = 1$, $N = 1$, $0 < d \le \frac{1}{2}$ and $0 < r < \alpha$.

The next example emphasises the role of 3.45 (iii): If this condition is violated, we cannot expect u to be invertible.

3.50. Example: Recall u from Example 3.9: A representative was given by $u_\varepsilon : (-\alpha, \alpha) \to \mathbb{R}$, $u_\varepsilon(x) = \sin \frac{x}{\varepsilon}$. Let $x_0 = 0$. Then $y_0 = 0$. No matter how small we choose ε_1 or r, we always end up with $N = 1$, $a_\varepsilon = 1$ and $b_\varepsilon = 2$. Since the product of a_ε and b_ε is already greater than 1, no $d > 0$ can be found consistent with condition 3.45 (iii). That is not surprising since we have already noted that (u_ε) is not ca-injective on any neighbourhood of 0 and, thus, u cannot be left invertible.

Despite the lack of left invertibility there is still hope for u from Example 3.50 to be right invertible since $(u_\varepsilon)_\varepsilon$ at least is ca-surjective onto $(-1, 1)$. Therefore, a theorem yielding right invertibility of generalised functions similar to u from Example 3.50, assuming properties of u similar to those of Theorem 3.45, might be desirable.

3.51. Theorem: *Let U be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}(U)^n$ and $x_0 \in U$. Let $y_0 \in \mathbb{R}^n$, $\varepsilon_1 \in (0, 1]$, $a_\varepsilon, b_\varepsilon > 0$ $(\varepsilon \le \varepsilon_1)$, $d > 0$ and $r > 0$ satisfying*

(i) $u(x_0) \approx y_0$,

(ii) $a_\varepsilon(b_\varepsilon + d) \le 1$ *for all* $\varepsilon \le \varepsilon_1$,

(iii) $\overline{B_r(x_0)} \subseteq U$,

and $N \in \mathbb{N}$. *If there exists a representative* $(u_\varepsilon)_\varepsilon$ *of* u *such that for all* $\varepsilon \le \varepsilon_1$

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \le a_\varepsilon \varepsilon^N$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \le b_\varepsilon \varepsilon^{-N}$ *for all* $x \in \overline{B_{r\varepsilon^N}(x_0)}$,

then u is right invertible on $B_{\alpha r \varepsilon_2^N}(x_0)$ with right inversion data

$$\left[B_{\alpha r \varepsilon_2^N}(x_0), \mathbb{R}^n, v, B_{\beta d \gamma r}(y_0)\right],$$

where α and β are arbitrary in $(0, 1)$, γ is arbitrary in $(0, \alpha)$ and for some suitable $\varepsilon_2 \le \varepsilon_1$.

Furthermore, $v(y_0) \approx x_0$. Also, there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r \varepsilon^N}(x_0))} = u_\varepsilon|_{B_{\alpha r \varepsilon^N}(x_0)}{}^{-1}$$

for all $\varepsilon \le \varepsilon_2$.

Proof: The main difference to Theorem 3.45 is the fact that the size of the ball where u_ε is injective is shrinking with ε. Consequently, no left inverse can be found without further conditions (cf. Examples 3.50 and 3.52). To prove the theorem just use (ii) instead of 3.45 (iii) to obtain an estimate for $\frac{a_\varepsilon \varepsilon^N}{1-a_\varepsilon b_\varepsilon}$ and replace $\frac{s}{d}$ by $\frac{\varepsilon^N}{d}$ and r by $r\varepsilon^N$ in the proof of Theorem 3.45, while omitting the part concerning the left inverse. □

Note that we do not require u to be c-bounded into $\mathbb{R}^n$. This is due to the fact that the c-boundedness of u is only necessary when composing with a *left* inverse, whereas the aim of the theorem is to produce a right inverse. Moreover, Condition 3.51 (ii) has a shape different from its equivalent in Theorem 3.45, corresponding to the difference in the estimates due to the replacement of r by $r\varepsilon^N$. Note that 3.51 (3) is weaker than 3.45 (3) and that 3.51 (ii) implies 3.45 (ii). The actual shape of 3.51 (ii) seems to be incomparable to the corresponding 3.45 (iii); it reflects the necessity of the proof to employ 3.51 (3). Finally, the convergence condition can again be exchanged for

$$|u_\varepsilon(x_0) - y_0| \le (1-\beta)\, d\, \gamma\, r$$

for all $\varepsilon \le \varepsilon_1$.

3.52. Example: Checking $u_\varepsilon(x) := \sin\frac{x}{\varepsilon}$ for the conditions of Theorem 3.51, we easily see that $(u_\varepsilon)_\varepsilon$ satisfies all the requirements with respect to $x_0 = 0$, $y_0 = 0$, $\varepsilon_1 = 1$, $0 < r < \frac{\pi}{2}$, $a_\varepsilon = 1$, $b_\varepsilon = 1 - \cos r$, $0 < d < 1 - \cos r$ and $N = 1$. Therefore, $u = [(u_\varepsilon)_\varepsilon]$ is right invertible on a suitable neighbourhood of 0.

Again, the conditions in Theorem 3.51 hold true independently of the choice of the represenative.

3.53. Proposition: *If one representative of $u \in \mathcal{G}(U)^n$ satisfies all conditions of Theorem 3.51, then every representative does.*

More precisely, if $(u_\varepsilon)_\varepsilon$ satisfies the conditions of the theorem with x_0, y_0, ε_1, a_ε, b_ε, N, d and r, then another given representative $(\bar{u}_\varepsilon)_\varepsilon$ of u satisfies them with x_0, y_0, N, r and suitable values for $\bar{\varepsilon}_1$, $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$ and $\bar{d}$ which can be chosen to satisfy $\bar{\varepsilon}_1 \le \varepsilon_1$, $\bar{a}_\varepsilon \ge a_\varepsilon$ and $\bar{b}_\varepsilon \ge b_\varepsilon$ (for all $\varepsilon \le \bar{\varepsilon}_1$) and $\bar{d} \le d$. Furthermore, we may suppose $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ for $\varepsilon \to 0$ and $\bar{d} \nearrow d$ for $\bar{\varepsilon}_1 \to 0$.

Proof: The proof closely resembles the proof of Proposition 3.47.

The difference $(u_\varepsilon)_\varepsilon - (\bar{u}_\varepsilon)_\varepsilon$ being negligible, $\bar{u}_\varepsilon(x_0)$ converges to y_0 as $\varepsilon \to 0$. Moreover, $\det(\bar{u}_\varepsilon(x_0))$ is strictly non-zero since $\det(u_\varepsilon(x_0))$ has this property. By Proposition 2.31, $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\|$ is strictly non-zero and, hence, so is $(a_\varepsilon)_\varepsilon$. In the following, let $D_1 > 0$ and $M_1 \in \mathbb{N}$ such that $a_\varepsilon \ge D_1 \varepsilon^{M_1}$ for all $\varepsilon \le \varepsilon_1$. Again, we may assume w.l.o.g. that $(b_\varepsilon)_\varepsilon$ is moderate. So let $D_2 > 0$ and $M_2 \in \mathbb{N}$ such that $b_\varepsilon \le D_2\, \varepsilon^{-M_2}$ for all $\varepsilon \le \varepsilon_1$.

As in the proof of Proposition 3.47, we can choose $\varepsilon_2 \leq \varepsilon_1$ such that

$$\|\mathrm{D}\bar{u}_\varepsilon(x_0)^{-1}\| \leq \|\mathrm{D}u_\varepsilon(x_0)^{-1}\| + A\,\varepsilon^{N+M_1+M_2+1} \leq (a_\varepsilon + A\,\varepsilon^{M_1+M_2+1})\varepsilon^N$$

for some $A > 0$ and

$$\|\mathrm{D}\bar{u}_\varepsilon(x_0) - \mathrm{D}\bar{u}_\varepsilon(x)\| \leq \|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| + B\,\varepsilon^{-N+1} \leq (b_\varepsilon + B\,\varepsilon)\varepsilon^{-N}$$

for all $x \in \overline{B_{r\varepsilon^N}(x_0)}$ for some $B > 0$. The index ε_2 and the constants A and B are chosen according to the negligibility of $(\mathrm{D}u_\varepsilon(x_0)^{-1} - \mathrm{D}\bar{u}_\varepsilon(x_0)^{-1})_\varepsilon$ and $(\mathrm{D}u_\varepsilon - \mathrm{D}\bar{u}_\varepsilon)_\varepsilon$ over the compact set $\overline{B_r(x_0)}$. We set

$$\bar{a}_\varepsilon := a_\varepsilon + A\,\varepsilon^{M_1+M_2+1} \qquad \text{and} \qquad \bar{b}_\varepsilon := b_\varepsilon + B\,\varepsilon$$

for $\varepsilon \leq \varepsilon_2$. Hence, $\bar{a}_\varepsilon \geq a_\varepsilon$ and $\bar{b}_\varepsilon \geq b_\varepsilon$ for all $\varepsilon \leq \varepsilon_2$. Consequently, also $\bar{a}_\varepsilon \geq D_1\varepsilon^{M_1}$ holds. It only remains to check condition 3.51 (ii) for $\bar{a}_\varepsilon$ and $\bar{b}_\varepsilon$:

$$\begin{aligned}
\bar{a}_\varepsilon\bar{b}_\varepsilon &= (a_\varepsilon + A\,\varepsilon^{M_1+M_2+1})(b_\varepsilon + B\,\varepsilon)\\
&= \underbrace{\underbrace{a_\varepsilon b_\varepsilon}_{\leq 1-a_\varepsilon d}}_{=1-\bar{a}_\varepsilon d + A\varepsilon^{M_1+M_2+1}d} + a_\varepsilon \cdot B\,\varepsilon + b_\varepsilon \cdot A\,\varepsilon^{M_1+M_2+1} + A\,\varepsilon^{M_1+M_2+1}\cdot B\,\varepsilon\\
&\leq 1 - \bar{a}_\varepsilon d + \bar{a}_\varepsilon \cdot \Bigg(\underbrace{\frac{1}{\bar{a}_\varepsilon}}_{\leq \frac{1}{D_1}\varepsilon^{-M_1}} A\varepsilon^{M_1+M_2+1}d + B\,\varepsilon\\
&\qquad + \underbrace{\frac{1}{\bar{a}_\varepsilon}}_{\leq \frac{1}{D_1}\varepsilon^{-M_1}} \underbrace{b_\varepsilon}_{\leq D_2\,\varepsilon^{-M_2}} \cdot A\,\varepsilon^{M_1+M_2+1} + \underbrace{\frac{1}{\bar{a}_\varepsilon}}_{\leq \frac{1}{D_1}\varepsilon^{-M_1}} A\,\varepsilon^{M_1+M_2+1}\cdot B\,\varepsilon \Bigg)\\
&\leq 1 - \bar{a}_\varepsilon \cdot \left(d - \varepsilon \left(\frac{Ad}{D_1}\varepsilon^{M_2} + B + \frac{AD_2}{D_1} + \frac{AB}{D_1}\,\varepsilon^{M_2+1} \right)\right).
\end{aligned}$$

Now let $\bar{\varepsilon}_1 \leq \varepsilon_2$ such that the expression in the brackets becomes positive for all $\varepsilon \leq \bar{\varepsilon}_1$. Set $\bar{d} := d - \bar{\varepsilon}_1 \left(B + \frac{AD_2}{D_1} + \frac{AB}{D_1}\,\bar{\varepsilon}_1^{M_2+1}\right)$, then

$$\bar{a}_\varepsilon\bar{b}_\varepsilon \leq 1 - \bar{a}_\varepsilon\bar{d}$$

holds for all $\varepsilon \leq \bar{\varepsilon}_1$. The convergences $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ for $\varepsilon \to 0$ and $\bar{d} \nearrow d$ for $\bar{\varepsilon}_1 \to 0$ follow from the definitions of $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$ and $\bar{d}$. $\square$

Now that we were successful in proving a "right inverse function theorem" the question arises if also a modification with respect to "only left invertible" is possible. Typically, the generalised functions being only left invertible are ca-injective on a fixed set but the interior of the intersection of the images of this set under u_ε is empty. In addition, we know that the inverse of any right invertible function is left invertible (cf. Proposition 3.5 (1)). So let us take a look at

3.54. Example: Consider $v \in \mathcal{G}((-1,1))$ that has $v_\varepsilon(x) := \varepsilon \arcsin x$ as a representative (Figure 3.7). This v is a right inverse to the function u we

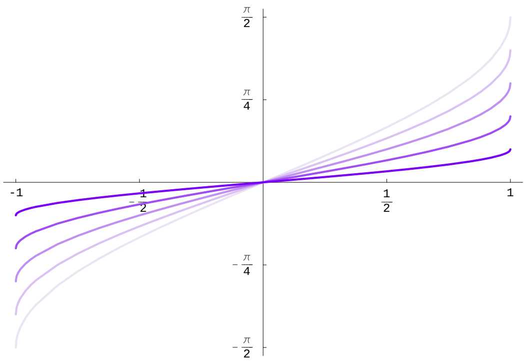

Figure 3.7: $v_\varepsilon(x) = \varepsilon \arcsin x$

studied in Examples 3.50 and 3.52. Since $\mathrm{D}v_\varepsilon(0)$ is the reciprocal value of $\mathrm{D}u_\varepsilon(0)$, it is not surprising to discover that $(v_\varepsilon)_\varepsilon$ satisfies estimates similar to 3.51 (2) and (3) with the sign of N reversed.

Indeed, reversing the sign of N in Theorem 3.45 (2) and (3) leads to sufficient conditions for left invertibility.

3.55. Theorem: *Let U be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}[U, \mathbb{R}^n]$ and $x_0 \in U$. Let $\varepsilon_1 \in (0,1]$, $a_\varepsilon, b_\varepsilon > 0$ ($\varepsilon \leq \varepsilon_1$), $N \in \mathbb{N}_0$, $d > 0$ and $r > 0$ satisfying the following conditions:*

(i) $s := \sup\{a_\varepsilon \,|\, 0 < \varepsilon \leq \varepsilon_1\}$ *is finite.*

(ii) $a_\varepsilon b_\varepsilon + d\varepsilon^N \leq 1$ *for all* $\varepsilon \leq \varepsilon_1$.

(iii) $\overline{B_r(x_0)} \subseteq U$.

If there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that for all $\varepsilon \leq \varepsilon_1$

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \leq a_\varepsilon \varepsilon^{-N}$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \leq b_\varepsilon \varepsilon^N$ *for all* $x \in \overline{B_r(x_0)}$,

then u is left invertible on $B_{\alpha r}(x_0)$ with left inversion data

$$[B_{\alpha r}(x_0), \mathbb{R}^n, v, B]$$

where α is arbitrary in $(0,1)$ and $B \subseteq \mathbb{R}^n$ is an arbitrary open set containing $\overline{\bigcup_{\varepsilon \leq \varepsilon_1} u_\varepsilon(B_{\alpha r}(x_0))}$.

Furthermore, there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r}(x_0))} = u_\varepsilon|_{B_{\alpha r}(x_0)}{}^{-1}$$

for all $\varepsilon \leq \varepsilon_2$.

Proof: To prove the theorem just use (ii) as in Theorem 3.45 to obtain an estimate for $\frac{a_\varepsilon \varepsilon^{-N}}{1-a_\varepsilon b_\varepsilon}$ and replace $a_\varepsilon \varepsilon^N$ by $a_\varepsilon \varepsilon^{-N}$, $b_\varepsilon \varepsilon^{-N}$ by $b_\varepsilon \varepsilon^N$ and d by ε^{2N} in the proof of Theorem 3.45, while omitting the part introducing the constant β and the part concerning the convergence of $v_\varepsilon(0)$ to 0. $\square$

The preceding theorem lacks the convergence condition on $(u_\varepsilon(x_0))_\varepsilon$ corresponding to 3.45 (i) since for the construction of a left inverse we do not care if the intersection of the images under u_ε still contains a non-empty open set.

3.56. Example: Let v be the generalised function from Example 3.54. Then $(v_\varepsilon)_\varepsilon$ satisfies the conditions of Theorem 3.55 with respect to $x_0 = 0$, $v_1 =$, $0 < r < \frac{\sqrt{3}}{2}$, $a_\varepsilon = 1$, $b_\varepsilon = \frac{1}{\sqrt{1-r^2}} - 1 < 1$, $N = 1$ and $0 < d \leq 2 - \frac{1}{\sqrt{1-r^2}}$.

Once more we show independence of the choice of the representative in Theorem 3.55.

3.57. Proposition: *If one representative of $u \in \mathcal{G}(U)^n$ satisfies all conditions of Theorem 3.45, then every representative does.*

More precisely, if $(u_\varepsilon)_\varepsilon$ satisfies the conditions of the theorem with x_0, ε_1, a_ε, b_ε, s, N, d and r, then another given representative $(\bar{u}_\varepsilon)_\varepsilon$ of u satisfies them with x_0, N, r and suitable values for $\bar{\varepsilon}_1$, $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{s}$ and $\bar{d}$ which can be chosen to satisfy $\bar{\varepsilon}_1 \leq \varepsilon_1$, $\bar{a}_\varepsilon \geq a_\varepsilon$ and $\bar{b}_\varepsilon \geq b_\varepsilon$ (for all $\varepsilon \leq \bar{\varepsilon}_1$), $\bar{s} \geq s$ and $\bar{d} \leq d$. Furthermore, we may suppose $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ for $\varepsilon \to 0$ and $\bar{s} \searrow s$ and $\bar{d} \nearrow d$ for $\bar{\varepsilon}_1 \to 0$.

Proof: The proof is nearly the same as the one of Proposition 3.47. $\bar{a}_\varepsilon$ and $\bar{b}_\varepsilon$, $\bar{s}$, $\bar{\varepsilon}_1$ and $\bar{d}$ are defined as before. Thus, they have the claimed properties. There are only two differences to the proof of Proposition 3.47: Again, we show that w.l.o.g. $(b_\varepsilon)_\varepsilon$ is moderate, only that this time $b'_\varepsilon := \min\left(b_\varepsilon, C\varepsilon^{-(N+N_1)}\right)$. And then, in the equivalents of the estimates (3.23) and (3.24), we find some $\varepsilon_2 \leq \varepsilon_1$ such that the negligible part in the respective estimate is less than $A\varepsilon^{M+1}$ resp. $B\varepsilon^{2N+1}$. $\square$

In classical inversion theory there are several theorems concerning the *global* injectivity of a given function (cf. [Par83]). One of them is from Gale and Nikaido ([GN65]).

3.58. Theorem (Gale-Nikaido): *Let U be an open subset of $\mathbb{R}^n$ and Ω a closed rectangular region of $\mathbb{R}^n$ with $\Omega \subseteq U$. Let $f : \Omega \to \mathbb{R}^n$ be a differentiable mapping. If every principal minor of $\mathrm{D}f(x)$ is positive for all $x \in \Omega$, then f is injective on Ω.*

The region Ω is of the form $\{(x_1, \ldots, x_n) \,|\, a_i \leq x_i \leq b_i, i = 1, \ldots, n\}$ where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$, i.e. Ω need not to be bounded. For the notion of differentiability on Ω see [GN65], page 84 or [Par83], page 17.

We use Theorem 3.58 to prove yet another "left inverse function theorem" that will be used in Chapter 4.

3.59. Theorem: *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}[U, \mathbb{R}^n]$. If $\det \circ \mathrm{D}u$ is strictly non-zero and if there exist a represenative $(u_\varepsilon)_\varepsilon$ of u and some $\varepsilon_0 \in (0, 1]$ such that every principal minor of $\mathrm{D}u_\varepsilon(x)$ is positive for all $x \in U$ and $\varepsilon \leq \varepsilon_0$, then u is left invertible on any open rectangular subset R of U with $\overline{R} \subset\subset U$.*

Proof: Let R be an open rectangular subset of U with $\overline{R} \subset\subset U$. Let $\delta > 0$ such that $\overline{R_\delta} \subseteq U$ where $R_\delta := R + \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, |x_i| < \delta, i = 1, \ldots, n\}$. Then $\overline{R_\delta}$ is a closed rectangular region and, by Theorem 3.58, every u_ε is injective for all $\varepsilon \leq \varepsilon_0$. We define $w_\varepsilon : u_\varepsilon(R_\delta) \to R_\delta$ by $w_\varepsilon := {u_\varepsilon|_{R_\delta}}^{-1}$. From Proposition 3.34, it follows that there exist $(u_\varepsilon(\overline{R}), 0)$-extensions v_ε of w_ε such that $(v_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. Since u is c-bounded into $\mathbb{R}^n$, there exists a compact subset K of $\mathbb{R}^n$ such that $u_\varepsilon(\overline{R}) \subseteq K$ for ε sufficiently small. Hence, u is left invertible on R with left inversion data $[R, \mathbb{R}^n, v := [(v_\varepsilon)_\varepsilon], B_l]$, where B_l is an arbitrary open subset of $\mathbb{R}^n$ containing K. $\square$

Finally, we take a look at the relation between the classical Inverse Function Theorem 1.3 and the generalised Inverse Function Theorem 3.45. On the C^∞ level we saw in Remark 3.2 (3) that if a smooth function $f : U \to V$ (with U and V open subsets of $\mathbb{R}^n$) is classically C^∞-invertible on a neighbourhood W of some point $x_0 \in U$ with smooth inverse g, then, obviously, $\sigma(f) = \iota(f)$ is strictly invertible on W with inversion data $[W, f(W), \sigma(g), f(W)]$. But what is the situation if f is not C^∞, i.e. if we cannot use the trivial embedding σ? In the following, we will show that our notion of invertibility and the generalised Inverse Function Theorem 3.45 are consistent with the classical Inverse Function Theorem 1.3, the latter taken for the special case $X = Y = \mathbb{R}^n$ and f a C^1-function. First, we need the following

3.60. Proposition: *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $f \in \mathrm{C}(U, V)$ and $f_\varepsilon \in \mathrm{C}(U, \mathbb{R}^m)$ for $\varepsilon \in (0, \varepsilon_0]$. Assume that for all compact subsets K of U there exists some compact subset L of V such* *that $f_\varepsilon(K) \subseteq L$ for sufficiently small ε. Furthermore, suppose that $(f_\varepsilon)_\varepsilon$ converges to f uniformly on compact subsets of U as $\varepsilon \to 0$. If g is a continuous function on V, then $(g \circ f_\varepsilon|_K)_\varepsilon$ converges uniformly to $g \circ f|_K$ for all compact sets K in U.*

Proof: Let $K \subset\subset U$ and let $L \subset\subset V$ such that $f(K), f_\varepsilon(K) \subseteq L$ for all $\varepsilon \leq \varepsilon_1$ for some $\varepsilon_1 \leq \varepsilon_0$. Let $\eta > 0$. Since g is continuous and L is compact, g is uniformly continuous on L. Now choose $\delta > 0$ such that for all $y_1, y_2 \in L$ with $|y_1 - y_2| < \delta$

$$|g(y_1) - g(y_2)| < \eta$$

holds. Choose $\varepsilon_2 \leq \varepsilon_1$ such that

$$|f_\varepsilon(x) - f(x)| < \delta$$

for all $\varepsilon \leq \varepsilon_2$. Since $f_\varepsilon(x), f(x) \in L$ for all $x \in K$ and $\varepsilon \leq \varepsilon_2$, it follows that

$$\sup_{x \in K} |g(f_\varepsilon(x)) - g(f(x))| < \eta$$

for all $\varepsilon \leq \varepsilon_2$. □

3.61. Theorem: *Let U be an open subset of $\mathbb{R}^n$, x_0 in U and f in $\mathrm{C}^1(U, \mathbb{R}^n)$ with $\det(\mathrm{D}f(x_0)) \neq 0$. Then the following hold:*

(1) $\iota(f) \in \mathcal{G}(U)^n$ satisfies the condition of Theorem 3.44 around x_0 and, therefore, is invertible on some neighbourhood of x_0.

(2) Assume that $g \in \mathrm{C}^1(V, W)$ is the inverse of $f|_W$ around $x_0 \in W$ given by the Inverse Function Theorem 1.3 and $v \in \mathcal{G}(\mathbb{R}^n)^n$ is the inverse of $\iota(f)$ obtained by Theorem 3.44 with inversion data $[B_s, \mathbb{R}^n, v, B_l, B_r]$. Then for every representative $(v_\varepsilon)_\varepsilon$ of v, $(v_\varepsilon)_\varepsilon$ and $(\mathrm{D}v_\varepsilon)_\varepsilon$ converge to g and $\mathrm{D}g$, respectively, uniformly on compact subsets of $B_r \cap V$.

Proof: (1): Let A be an open neighbourhood of x_0 with $\overline{A} \subset\subset U$. Since all the conditions of Theorem 3.44 have to be satisfied only on an arbitrarily small open ball with centre x_0, it suffices to show that $\iota(f)$ has a representative $(u_\varepsilon)_\varepsilon$ such that the $u_\varepsilon|_A$ satisfy the conditions assumed in Theorem 3.44. We will even prove a (formally) stronger statement, namely that every representative $(v_\varepsilon)_\varepsilon$ of $\iota(f)|_A$ satisfies the conditions of Theorem 3.44. This, in turn, will be established once we have shown that there exists at least one representative $(f_\varepsilon)_\varepsilon$ of $\iota(f)|_A$ satisfying the relevant conditions, due to Remark 3.48.

Let ψ be an element of $\mathcal{D}(U)$ with $\psi \equiv 1$ in some neighbourhood of $\overline{A}$. Then ψf has compact support and can be embedded into $\mathcal{G}(U)$ by ι_0. Using the formula of Chapter 2, we obtain that $\big(((\psi f) * \rho_\varepsilon)|_U\big)_\varepsilon$ is a representative of $\iota_0(\psi f)$, and thus of $\iota(\psi f)$ (Proposition 2.11). Restriction to A yields that $(f_\varepsilon)_\varepsilon := \big(((\psi f) * \rho_\varepsilon)|_A\big)_\varepsilon$ is a representative of $\iota(\psi f)|_A$. Since $\hat{\iota}$ is a sheaf morphism (Proposition 2.13), we have

$$\iota_U(\psi f)|_A = \iota_A(\psi f|_A) = \iota_A(f|_A) = \iota_U(f)|_A,$$

establishing $(f_\varepsilon)_\varepsilon$ as a representative of $\iota_U(f)|_A$. In the following, we will denote $\iota(f) = \iota_U(f)$ simply by ιf.
By Proposition 2.39, $\mathrm{D}^j f_\varepsilon$ converges uniformly to $\mathrm{D}^j f|_A$ on compact subsets of A as $\varepsilon \to 0$ for $j = 0, 1$. Hence, $(\iota f)|_A$ is c-bounded into $\mathbb{R}^n$ and $(\iota f)|_A$ satisfies condition 3.44 (i), i.e. $(\iota f)|_A(x_0) \approx f(x_0)$. Furthermore, we have

$$\mathrm{D}f_\varepsilon(x_0) \to \mathrm{D}f(x_0) \quad \text{for} \quad \varepsilon \to 0. \tag{3.25}$$

Since the determinant function det is continuous, this yields

$$\det(\mathrm{D}f_\varepsilon(x_0)) \to \det(\mathrm{D}f(x_0)) \neq 0 \quad \text{for} \quad \varepsilon \to 0.$$

Hence, $\det(\mathrm{D}f_\varepsilon(x_0))$ is non-zero and, thus, satisfies 3.44 (1) for ε sufficiently small, say $\varepsilon \leq \varepsilon_1$.
Now let $C := \|\mathrm{D}f(x_0)^{-1}\|$. By (3.25), and since inversion on $\mathrm{GL}_n(\mathbb{R})$, i.e. $\varphi \mapsto \varphi^{-1}$, is continuous, we obtain $\mathrm{D}f_\varepsilon(x_0)^{-1} \to \mathrm{D}f(x_0)^{-1}$, and thus $\|\mathrm{D}f_\varepsilon(x_0)^{-1}\| \to \|\mathrm{D}f(x_0)^{-1}\|$ for $\varepsilon \to 0$. Therefore, for fixed $a > C$ there exists $\varepsilon_2 \leq \varepsilon_1$ such that $\|\mathrm{D}f_\varepsilon(x_0)^{-1}\| \leq a$, showing that 3.44 (2) is satisfied.
Note that, by $\mathrm{D}(\iota f) = \iota(\mathrm{D}f)$ (Theorem 2.9) and by $\iota(\mathrm{D}f)|_A = \iota(\psi \cdot \mathrm{D}f)|_A$ (same line of argument as for $\iota(f)|_A$), we have

$$\mathrm{D}f_\varepsilon = ((\psi \cdot \mathrm{D}f) * \rho_\varepsilon)|_A + N_\varepsilon$$

for some $(N_\varepsilon)_\varepsilon \in \mathcal{N}(A)^{n^2}$. Setting $g := \psi \cdot \mathrm{D}f$ and substituting z for $\frac{y}{\varepsilon}$, we obtain

$$\begin{aligned}
&\|\mathrm{D}f_\varepsilon(x_0) - \mathrm{D}f_\varepsilon(x)\| \leq \\
&\qquad \leq \|g * \rho_\varepsilon\,(x_0) - g * \rho_\varepsilon\,(x)\| + \|M_\varepsilon(x)\| \\
&\qquad = \Big\| \int_{\mathbb{R}^n} \big(g(x_0 - y) - g(x - y)\big) \cdot \rho_\varepsilon(y)\; dy \Big\| + \|M_\varepsilon(x)\| \\
&\qquad = \Big\| \int_{\mathbb{R}^n} \big(g(x_0 - \varepsilon z) - g(x - \varepsilon z)\big) \cdot \rho(z)\; dz \Big\| + \|M_\varepsilon(x)\| \\
&\qquad \leq \|g(x_0 - \varepsilon z) - g(x - \varepsilon z)\| \cdot \int_{\mathbb{R}^n} |\rho(z)|\; dz + \|M_\varepsilon(x)\|
\end{aligned} \tag{3.26}$$

where $(M_\varepsilon)_\varepsilon \in \mathcal{N}(A)^{n^2}$. The function g is continuous and has compact support. Hence, g is uniformly continuous on $\mathbb{R}^n$. Let $\eta > 0$ with $a \cdot \eta \cdot (\int_{\mathbb{R}^n} |\rho(z)|\, dz + 1) < 1$ and set

$$b := \eta \cdot \left(\int\limits_{\mathbb{R}^n} |\rho(z)|\, dz + 1 \right)$$

(yielding $ab < 1$ as required by 3.44 (ii)). Choose $r > 0$ according to the uniform continuity of g such that $\overline{B_r(x_0)} \subseteq A$ (i.e. let r satisfy also 3.44 (iii)). Now, for every $x \in \overline{B_r(x_0)}$, we have $|(x_0 - \varepsilon z) - (x - \varepsilon z)| = |x_0 - x| \le r$ and, hence,

$$\|g(x_0 - \varepsilon z) - g(x - \varepsilon z)\| \le \eta. \tag{3.27}$$

Let $\varepsilon_3 \le \varepsilon_2$ such that

$$\sup_{x \in \overline{B_r(x_0)}} \|M_\varepsilon(x)\| \le \eta. \tag{3.28}$$

Estimating the last expression in (3.26) by (3.27) and (3.28), we finally obtain

$$\|\mathrm{D}f_\varepsilon(x_0) - \mathrm{D}f_\varepsilon(x)\| \le \eta \cdot \int\limits_{\mathbb{R}^n} |\rho(z)|\, dz + \eta = b$$

for all $\varepsilon \le \varepsilon_3$, showing that also 3.44 (3) is satisfied.

(2): Obviously, if the assertion holds for one representative of v, it is true for every representative. By Theorem 3.44, there exist representatives $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ of $\iota(f)$ resp. v such that $u_\varepsilon \circ v_\varepsilon|_{B_r} = \mathrm{id}_{B_r}$.
First, we establish that $(f \circ v_\varepsilon)_\varepsilon$ converges to the identity on $B_r \cap V$ uniformly on $B_r \cap V$: By the inclusion $v_\varepsilon(B_r \cap V) \subseteq \overline{B_s}$, we obtain

$$\begin{aligned} \sup_{x \in B_r \cap V} |f \circ v_\varepsilon(x) - x| &= \sup_{x \in B_r \cap V} |f(v_\varepsilon(x)) - u_\varepsilon(v_\varepsilon(x))| \\ &\le \sup_{y \in \overline{B_s}} |f(y) - u_\varepsilon(y)|. \end{aligned}$$

By Proposition 2.39, the right hand side converges to 0 for $\varepsilon \to 0$ and, hence, so does the left hand side.
Next, we show that for all $K \subset\subset B_r \cap V$ there exists some $L \subset\subset B_r \cap V$ such that $f \circ v_\varepsilon(K) \subseteq L$ for sufficiently small ε: Let $K \subset\subset B_r \cap V$ and $\delta > 0$ such that $K + \overline{B_\delta(0)} \subset\subset B_r \cap V$. By the uniform convergence of $(f \circ v_\varepsilon)_\varepsilon$ to $\mathrm{id}_{B_r \cap V}$, there exists some $\eta \in (0, 1]$ such that

$$|f \circ v_\varepsilon(x) - x| < \delta$$

for all $x \in K$ and $\varepsilon \le \eta$. Thus,

$$f \circ v_\varepsilon(K) \subseteq K + \overline{B_\delta(0)} \subset\subset B_r \cap V$$

for all $\varepsilon \leq \eta$.
Finally, we apply Proposition 3.60 to $\mathrm{id}_{B_r \cap V}$, $(f \circ v_\varepsilon)_\varepsilon$ and g to obtain that $(v_\varepsilon)_\varepsilon$ converges to g uniformly on compact subsets of $B_r \cap V$.

To prove the uniform convergence of the derivatives on compact sets, we first show that $\mathrm{D}f(v_\varepsilon(\,.\,)) \circ \mathrm{D}v_\varepsilon(\,.\,)$ converges to the identity I uniformly on $B_r \cap V$: By $v_\varepsilon(B_r \cap V) \subseteq \overline{B_s}$,

$$\begin{aligned}
\sup_{x \in B_r \cap V} &\|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - I\| \\
&= \sup_{x \in B_r \cap V} \|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - \mathrm{D}u_\varepsilon(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x)\| \\
&\leq \sup_{x \in B_r \cap V} \big\|\big(\mathrm{D}f(v_\varepsilon(x)) - \mathrm{D}u_\varepsilon(v_\varepsilon(x))\big) \circ \mathrm{D}v_\varepsilon(x)\big\| \\
&\leq \sup_{z \in \overline{B_s}} \|\mathrm{D}f(z) - \mathrm{D}u_\varepsilon(z)\| \cdot \|\mathrm{D}u_\varepsilon(z)^{-1}\|
\end{aligned}$$

holds. As shown in the proof of Theorem 3.44 (resp. Theorem 3.45), $(\mathrm{D}u_\varepsilon(\,.\,)^{-1})_\varepsilon$ is uniformly bounded on $\overline{B_s}$ with respect to ε. By Proposition 2.39, $(\mathrm{D}u_\varepsilon)_\varepsilon$ converges to $\mathrm{D}f$ uniformly on the compact set $\overline{B_s}$ for $\varepsilon \to 0$. Hence,

$$\sup_{x \in B_r \cap V} \|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - I\| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.29}$$

Next, we apply Proposition 3.60 to g, $(v_\varepsilon)_\varepsilon$ and $\mathrm{D}f$ to obtain that

$$\sup_{x \in L} \|\mathrm{D}f(g(x)) - \mathrm{D}f(v_\varepsilon(x))\| \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{3.30}$$

for all compact subsets L of $B_r \cap V$.
Finally, let $K \subset\subset B_r \cap V$ and $x \in K$. Then

$$\begin{aligned}
&\|\mathrm{D}v_\varepsilon(x) - \mathrm{D}g(x)\| = \\
&= \|\mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - \mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}g(x)\| \\
&\leq \|\mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - \mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(g(x)) \circ \mathrm{D}g(x)\| \\
&\quad + \|\mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(g(x)) \circ \mathrm{D}g(x) - \mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}g(x)\| \\
&\leq \|\mathrm{D}f(v_\varepsilon(x))^{-1}\| \cdot \Big(\|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - I\| \\
&\qquad\qquad + \|\mathrm{D}f(g(x)) - \mathrm{D}f(v_\varepsilon(x))\| \cdot \|\mathrm{D}g(x)\|\Big)
\end{aligned}$$

holds. $\mathrm{D}f(v_\varepsilon(\,.\,))^{-1}$ (by $v_\varepsilon(K) \subseteq \overline{B_s}$) and $\mathrm{D}g$ are bounded on K, independently of ε. By (3.29) and (3.30), the two expressions in the bracket converge to 0 uniformly on K as $\varepsilon \to 0$, thereby concluding the proof. □

Chapter 4

A "discontinuous coordinate transformation" in general relativity

In this chapter, we will apply the notions and some of the results of the inversion theory of generalised functions developed in the preceding chapter to a problem in general relativity. This builds upon results of M. Kunzinger and R. Steinbauer (cf. [Ste00], [Ste98], [KS99b] and [GKOS01]) which will be reviewed here to the extent needed. We shall begin with a short introduction to so-called impulsive pp-waves, whose description by two different spacetime metrics (one distributional and one continuous) gives rise to a "discontinuous coordinate transformation" (Section 4.1). Replacing the distributional form of the metric by a generalised one leads to generalised geodesic equations which we will study in Section 4.2. Using these generalised geodesics, we obtain a generalised coordinate transformation modelling the discontinuous one. In Section 4.3, we will show that this transformation is indeed locally invertible in the sense of Chapter 3.

4.1 Impulsive pp-waves

The class of *plane fronted gravitational waves with parallel rays* or, for short, *pp-waves* was first considered by Brinkmann ([Bri23]) already in 1923 and rediscovered subsequently by several authors, among them Rosen ([Ros37]), Robinson in 1956 (cf. [EK62], page 88), Hély ([Hél59]) and Peres ([Per59]).

The most common way to define pp-waves is as spacetimes admitting a *covariantly constant null vector field* k^a. It is possible to physically interpret such a field as the rays of gravitational (or other null) waves. Introducing a null coordinate u by the condition $\partial_a u = k_a$, the metric of a pp-wave may be written in the form

$$ds^2 = h(u,x,y)\,du^2 - du\,dv + dx^2 + dy^2, \tag{4.1}$$

where h, called the *wave profile*, is an arbitrary (smooth) function of its arguments. Often (4.1) is referred to as the *Brinkmann form* of the pp-wave metric. A pp-wave of the form $h(u,x,y) = \rho(u)f(x,y)$ is called a *sandwich pp-wave* ([BPR59]) if ρ is non-vanishing only in some finite region $u_0 \le u \le u_1$ of spacetime. The gravitational field then is confined to that region, with flat space in "front" of $(u \le u_0)$ resp. "behind" $(u \ge u_1)$ the wave. In [Pen68], Ch. 4, R. Penrose introduced *impulsive pp-waves* as an idealisation (*impulsive limit*) of sandwich waves of infinitely short duration (say $u_0, u_1 \to 0$) but still producing a nontrivial effect in the sense that ρ equals the Dirac-δ, i.e. the metric taking the form

$$ds^2 = f(x,y)\,\delta(u)\,du^2 - du\,dv + dx^2 + dy^2. \tag{4.2}$$

This spacetime is flat everywhere except for the null hyperplane $u = 0$ where a δ-like impulse is located. The corresponding geodesic equations are given by

$$\begin{aligned} u'' &= 0,\\ x^{i\,\prime\prime} &= \frac{1}{2}\,\partial_i f\,u'^2\,\delta,\\ v'' &= f\,\dot\delta\,u'^2 + 2\sum_{i=1}^{2}\partial_i f\,x^{i\,\prime}\,u'\,\delta, \end{aligned}$$

where $'$ denotes the derivative with respect to an affine parameter, $\dot\delta$ is the derivative of the δ-distribution and $(x^1,x^2) = (x,y)$. Since $u'' = 0$, we may use u as a new affine parameter (thereby excluding only trivial geodesics parallel to the shock hypersurface), leading to

$$\begin{aligned} \ddot x^i(u) &= \frac{1}{2}\,\partial_i f(x^1(u),x^2(u))\,\delta(u),\\ \ddot v(u) &= f(x^1(u),x^2(u))\,\dot\delta(u) + 2\sum_{i=1}^{2}\partial_i f(x^1(u),x^2(u))\,\dot x^i(u)\,\delta(u), \end{aligned} \tag{4.3}$$

where the upper dot $\dot{}$ denotes the derivative with respect to u and we have written out all the dependencies explicitly. Heuristically, we expect the geodesics to be broken, possibly refracted straight lines. However, taking a closer look at system (4.3), it turns out that the x^i components (as second antiderivatives of δ) have the shape of kink functions and, consequently, the right hand side of the equation for v involves the (in $\mathcal{D}'$) ill-defined product $H\delta$ ("Heaviside times delta"). Nevertheless, attempts have been made to solve this system not well-defined in $\mathcal{D}'$ (cf. [FPV88] and [Bal97]) by simply setting $H\delta = \frac{1}{2}\delta$ (indeed, $H\delta \approx \frac{1}{2}\delta$ holds), leading to the "solutions"

$$\begin{aligned} x^i(u) &= x_0^i + \dot{x}_0^i\,(1+u) + \frac{1}{2}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\,u_+, \\ v(u) &= v_0 + \dot{v}_0\,(1+u) + f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\,H(u) \\ &\qquad + \sum_{i=1}^{2} \partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\Big(\dot{x}_0^i + \frac{1}{4}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\Big)\,u_+, \end{aligned} \tag{4.4}$$

where u_+ denotes the kink function $u \mapsto H(u)\,u$. In [Ste98], [KS99b], [Ste00] resp. [GKOS01], M. Kunzinger and R. Steinbauer justified that somewhat ad-hoc approach by providing a mathematically rigorous method of treating equations such as (4.3) and arriving at the same answer: They regularised the given equations, solved them in a suitable Colombeau algebra and showed that the solution indeed is associated to (4.4) (see the following section for details).

In the literature impulsive pp-waves frequently have also been described by a different spacetime metric which is actually continuous (see [Pen72], [PV98] and, for the general case, [AB97]). The latter is derived from the *Rosen form* (cf. [Ros37]) and is given by

$$\begin{aligned} ds^2 &= -du\,dV + (1 + \frac{1}{2}\,\partial_{11} f\,u_+)^2\,dX^2 + (1 + \frac{1}{2}\,\partial_{22} f\,u_+)^2\,dY^2 \\ &\quad + \frac{1}{2}\,\partial_{12} f\,\Delta f\,u_+^2\,dX\,dY + 2u_+\,\partial_{12} f\,dX\,dY + \frac{1}{4}(\partial_{12} f)^2\,u_+^2\,(dX^2 + dY^2), \end{aligned} \tag{4.5}$$

where for simplicity we have suppressed the dependence of the function f on its arguments, i.e. $f(X,Y)$. This suggests to look for a change of coordinates transforming (4.2) into (4.5). Of course, such a transformation cannot even be continuous.

Comparing the metrics described by (4.2) and (4.5), it turns out that the coordinate lines in (4.5) are exactly given by the "distributional geodesics" of the metric (4.2) with vanishing initial speed in the x, y and v-directions, giving rise to the "discontinuous coordinate transformation"

$$\begin{aligned} u &= u, \\ x^i &= X^i + \frac{1}{2}\,\partial_i f(X^k)\,u_+, \\ v &= V + f(X^k)\,H(u) + \frac{1}{4}\sum_{i=1}^{2} \partial_i f(X^k)^2\,u_+, \end{aligned} \tag{4.6}$$

where we write (X^k) for $(X^1, X^2) = (X, Y)$ and again $(x^1, x^2) = (x, y)$. Hence, from a physical point of view, the two approaches to impulsive pp-waves are equivalent. However, besides changing the manifold structure, the transformation once more involves products of distributions ill-defined in the linear theory. M. Kunzinger and R. Steinbauer gave meaning to the term "physically equivalent" by interpreting the discontinuous transformation as the distributional shadow of a generalised transformation in $\mathcal{G}$: After replacing the distributional spacetime metric (4.2) by a generalised one, they applied a generalised change of coordinates modelling the distributional one. Then they calculated the distributional shadow of the transformed generalised metric to arrive precisely at the continuous form (4.5) (cf. [KS99a], [Ste00] and [GKOS01]).
However, they did not arrive at a complete result in terms of inversion of generalised functions as developed in the preceding chapter. In the following section we will retrace—resp. complement and improve where necessary—the construction of a generalised solution of the regularised geodesic equations corresponding to (4.3). In Section 4.3, we will then show that the transformation (4.6) can indeed be viewed as a generalised function and that this function is locally invertible on some open set containing the half space $(-\infty, 0] \times \mathbb{R}^3$ in the sense of Definition 3.28.

4.2 Description of the geodesics for impulsivse pp-waves in $\mathcal{G}$

In this section, we study the geodesic equations corresponding to the regularisation of the distributional metric (4.2), following the approach taken in [Ste98], [Ste00] and [GKOS01]. While our presentation includes a lemma and a theorem by M. Kunzinger and R. Steinbauer regarding the existence and uniqueness of the generalised geodesics, we will study the solutions of the geodesic equations in greater depth and, thus, provide a basis for the constructions and discussions of the following section.

To carry out the programme indicated at the end of the preceding section we first have to regularise the spacetime metric and, more importantly, the geodesic equations (4.3). We will not employ some given embedding ι of $\mathcal{D}'$ into $\mathcal{G}$; rather, we will use a fairly general approach for modelling delta, following [GKOS01].

4.1. Definition: *A strict delta net is a net* $(\delta_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ *satisfying*

(a) $\operatorname{supp}(\delta_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$,

(b) $\int \delta_\varepsilon(x)\,dx \to 1$ *for* $\varepsilon \to 0$,

(c) $\int |\delta_\varepsilon(x)|\,dx \leq C$ *for some* $C > 0$ *and small* ε.

A strict delta function is a generalised function $D = [(\delta_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$ *with* $(\delta_\varepsilon)_\varepsilon$ *a strict delta net.*

Note that condition (a) is chosen in order to avoid technicalities in the following calculations which, however, remain valid if (a) is replaced by

(a′) $\operatorname{supp}(\delta_\varepsilon) \to \{0\}$ for $\varepsilon \to 0$.

We will base our considerations on the generalised metric $\hat{g}$ on $\mathbb{R}^4$ given by

$$\hat{ds}^2 = f(x^1, x^2)\, D(u)\, du^2 - du\, dv + (dx^1)^2 + (dx^2)^2, \tag{4.7}$$

where D is a strict delta function. Therefore, the geodesic equations (4.3) appear in the following regularised form:

$$\begin{aligned} \ddot{x}^i(u) &= \frac{1}{2}\,\partial_i f(x^1(u), x^2(u))\, D(u), \\ \ddot{v}(u) &= f(x^1(u), x^2(u))\, \dot{D}(u) + 2\sum_{i=1}^{2} \partial_i f(x^1(u), x^2(u))\, \dot{x}^i(u)\, D(u). \end{aligned} \tag{4.8}$$

The solution of this system, on the level of representatives and for fixed ε, is obtained by means of the following lemma ([GKOS01], Lemma 5.3.1). The initial conditions are chosen in $u = -1$, i.e. "long before the shock".

4.2. Lemma: *Let* $g : \mathbb{R}^n \to \mathbb{R}^n$ *and* $h : \mathbb{R} \to \mathbb{R}^n$ *be smooth and* $(\delta_\varepsilon)_\varepsilon$ *a net of smooth functions satisfying conditions (a) and (c) as above. For any* $x_0, \dot{x}_0 \in \mathbb{R}^n$ *and any* $\varepsilon \in (0,1]$ *consider the system*

$$\begin{aligned} \ddot{x}_\varepsilon(t) &= g(x_\varepsilon(t))\delta_\varepsilon(t) + h(t) \\ x_\varepsilon(-1) &= x_0 \\ \dot{x}_\varepsilon(-1) &= \dot{x}_0. \end{aligned} \tag{4.9}$$

Let $b > 0$, $Q := \int_{-1}^{1}\int_{-1}^{s} |h(r)| dr\, ds$, $I := \{x \in \mathbb{R}^n \,|\, |x - x_0| \leq b + |\dot{x}_0| + Q\}$ *and*

$$\alpha := \min\left(\frac{b}{C\|g\|_{\infty,I} + |\dot{x}_0|}, \frac{1}{2LC}, 1\right),$$

with L *a Lipschitz constant for* g *on* I*. Then (4.9) has a smooth solution* x_ε *on* $J_\varepsilon := [-1, \alpha - \varepsilon]$ *which is unique in* $\{x_\varepsilon \in \mathrm{C}^2(J_\varepsilon, \mathbb{R}) \,|\, |x_\varepsilon(t) - x_0| \leq b + |\dot{x}_0| + Q\}$*. Furthermore, for* ε *sufficiently small (e.g.* $\varepsilon \leq \frac{\alpha}{2}$*)* x_ε *is globally defined and both* $(x_\varepsilon)_\varepsilon$ *and* $(\dot{x}_\varepsilon)_\varepsilon$ *are uniformly bounded on compact subsets of* $\mathbb{R}$.

For fixed initial values, the above lemma can be used to ensure the existence of a solution of the generalised geodesic equations (4.8) (as is done in [Ste98], [Ste00] resp. [GKOS01]). However, in view of inverting the generalised coordinate transformation induced by the generalised geodesics (with vanishing initial speed in the x^1, x^2 and v-directions), we need to know more about the dependence of the solution on the initial values. Therefore, before presenting the theorem of M. Kunzinger and R. Steinbauer, we study the solutions provided by Lemma 4.2 in more detail.

The sets I and J_ε and the constants α and L depend on the initial values x_0 and $\dot{x}_0$. Nevertheless, they can be chosen uniformly for $(x_0, \dot{x}_0)$ ranging over some compact set $K \subset\subset \mathbb{R}^{2n}$: For $\beta(K) := \sup_{z \in \mathrm{pr}_2(K)} |z|$, set $I(K) := \mathrm{pr}_1(K) + \overline{B_{b+\beta(K)+Q}(0)}$, $L(K) := \max_{z \in I(K)} \|\mathrm{D}g(z)\|$, $\alpha(K)$ as in Lemma 4.2 (replacing I, $|\dot{x}_0|$, L by $I(K)$, $\beta(K)$, $L(K)$, respectively) and, finally, $J_\varepsilon(K) := [-1, \alpha(K) - \varepsilon]$. Hence, for $\varepsilon \le \varepsilon(K) := \frac{\alpha(K)}{2}$ and $(x_0, \dot{x}_0) \in K$, the solutions $x_\varepsilon(x_0, \dot{x}_0)$ are globally defined.

By the Existence and Uniqueness Theorem for ODEs 1.7, x_ε also depends smoothly on the initial values, i.e. $x_\varepsilon \in \mathrm{C}^\infty(K^\circ \times \mathbb{R})$ for $K \subset\subset \mathbb{R}^{2n}$ and $\varepsilon \le \varepsilon(K)$.

Our next task is to combine the (maximal) solutions obtained by Lemma 4.2 (keeping in mind that their domains depend on the initial values and on ε) to a "solution" on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times (0,1]$. More precisely, we have to construct a net $(x_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)^{(0,1]}$ of smooth functions such that for every $K \subset\subset \mathbb{R}^{2n}$ there exists some $\varepsilon_K \in (0,1]$ such that $x_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is the global solution of (4.9) for all $(x_0, \dot{x}_0) \in K$ and $\varepsilon \le \varepsilon_K$.

4.3. Proposition: *There exists $(x_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)^{(0,1]}$ such that for every $K \subset\subset \mathbb{R}^{2n}$ there exists some $\varepsilon_K \in (0,1]$ such that $x_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is the global solution of (4.9) for all $(x_0, \dot{x}_0) \in K$ and $\varepsilon \le \varepsilon_K$.*

Proof: Let $(K_m)_m$ be an increasing sequence of compact subsets of $\mathbb{R}^{2n}$ satisfying $K_m \subset\subset K_{m+1}^\circ$ which exhausts $\mathbb{R}^{2n}$. Set $D_m := (\varepsilon(K_{m+1}), \varepsilon(K_m)] \times K_m$ and $D := \bigcup_{m=1}^\infty D_m$. Now, we may define a function $y : D \to \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^n)$, $(\varepsilon, x_0, \dot{x}_0) \mapsto y_\varepsilon(x_0, \dot{x}_0, \,.\,)$ such that $y_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is the global solution of (4.9). Let $\sigma_m \in \mathcal{D}(K_m^\circ)$ such that $0 \le \sigma_m \le 1$ and $\sigma_m|_{K_{m-1}} = 1$. For $\varepsilon \in (\varepsilon(K_{m+1}), \varepsilon(K_m)]$ we define

$$x_\varepsilon(x_0, \dot{x}_0, t) := \begin{cases} \sigma_m(x_0, \dot{x}_0) \cdot y_\varepsilon(x_0, \dot{x}_0, t), & (x_0, \dot{x}_0) \in K_m^\circ \\ 0, & (x_0, \dot{x}_0) \in \mathbb{R}^{2n} \setminus \operatorname{supp} \sigma_m \end{cases}.$$

Then $x_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $x_\varepsilon|_{K_{m-1} \times \mathbb{R}} = y_\varepsilon|_{K_{m-1} \times \mathbb{R}}$. Since for $\varepsilon \in (0, \varepsilon(K_m)]$ and $(x_0, \dot{x}_0) \in K_m$ the function $y_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is a global solution, $x_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is a global solution for $\varepsilon \in (0, \varepsilon(K_m)]$ and $(x_0, \dot{x}_0) \in K_{m-1}$.

Finally, for every $K \subset\subset \mathbb{R}^{2n}$ there exists some $m \in \mathbb{N}$ such that $K \subseteq K_m$. For $\varepsilon_K := \varepsilon(K_{m+1})$ the function $x_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is the global solution of (4.9) for all $(x_0, \dot{x}_0) \in K$ and $\varepsilon \leq \varepsilon_K$. $\square$

We will call a net as in Proposition 4.3 an *asymptotic solution* of the system of differential equations (4.9). Note that the asypmtotic solution $(x_\varepsilon)_\varepsilon$ is a net of functions depending on time *and* initial values.

Next, we show uniform boundedness of the asymptotic solution $(x_\varepsilon)_\varepsilon$ on compact sets, a crucial ingredient for our proof of moderateness of the generalised coordinate transformation in Section 4.3.

4.4. Proposition: *The asymptotic solution* $(x_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)^{(0,1]}$ *is uniformly bounded on compact subsets of* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Proof: Let $K \times L \times J \subset\subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\varepsilon \leq \varepsilon_{K\times L}$. Then, on $K \times L \times \mathbb{R}$, x_ε can be written as

$$x_\varepsilon(x_0, \dot{x}_0, t) = \\ \begin{cases} x_0 + \dot{x}_0(t+1) + \int_t^{-1}\int_s^{-1} h(r)dr\, ds, & t \in (-\infty, -1] \\ x_0 + \dot{x}_0(t+1) + \int_{-\varepsilon}^t \int_{-\varepsilon}^s g(x_\varepsilon(x_0, \dot{x}_0, r))\delta_\varepsilon(r)dr\, ds & \\ \qquad + \int_{-1}^t\int_{-1}^s h(r)dr\, ds, & t \in [-1, \varepsilon] \\ x_\varepsilon(x_0, \dot{x}_0, \varepsilon) + \dot{x}_\varepsilon(x_0, \dot{x}_0, \varepsilon)(t-\varepsilon) + \int_\varepsilon^t\int_\varepsilon^s h(r)dr\, ds, & t \in [\varepsilon, \infty) \end{cases}.$$

For $(x_0, \dot{x}_0, t) \in K \times L \times (J \cap (-\infty, -1])$ we have

$$\begin{aligned} |x_\varepsilon(x_0,&\dot{x}_0, t)| \leq \\ &\leq \sup_{x_0\in K}|x_0| + \sup_{\dot{x}_0 \in L}|\dot{x}_0| \cdot (\sup_{t\in J}|t| + 1) + \sup_{t \in J}\int_t^{-1}\int_s^{-1}|h(r)|\, dr\, ds \\ &< \infty. \end{aligned}$$

Now, let $(x_0, \dot{x}_0, t) \in K \times L \times (J \cap [-1, \varepsilon]) \subseteq K \times L \times J_\varepsilon(\{x_0, \dot{x}_0\})$. Then, immediately by Lemma 4.2, $|x_\varepsilon(x_0, \dot{x}_0, t)|$ is bounded by

$$|x_\varepsilon(x_0, \dot{x}_0, t)| \leq \sup_{x_0 \in K}|x_0| + \sup_{\dot{x}_0\in L}|\dot{x}_0| + b + Q.$$

Finally, let $(x_0, \dot{x}_0, t) \in K \times L \times (J \cap [\varepsilon, \infty))$. Observe that

$$\dot{x}_\varepsilon(x_0, \dot{x}_0, \varepsilon) = \dot{x}_0 + \int_{-\varepsilon}^{\varepsilon} g(x_\varepsilon(x_0, \dot{x}_0, s))\, \delta_\varepsilon(s)\, ds + \int_{-1}^{\varepsilon} h(s)\, ds.$$

Hence,

$$
\begin{aligned}
|x_\varepsilon(x_0,&\dot x_0,t)| \leq \\
&\leq |x_\varepsilon(x_0,\dot x_0,\varepsilon)| + |\dot x_\varepsilon(x_0,\dot x_0,\varepsilon)||t-\varepsilon| + \int_\varepsilon^\varepsilon \int_\varepsilon^s |h(r)|\,dr\,ds \\
&\leq (\sup_{x_0\in K}|x_0| + \sup_{\dot x_0\in L}|\dot x_0| + b + Q) \\
&\qquad + (\sup_{\dot x_0\in L}|\dot x_0| + \|g\|_{\infty,I(K\times L)}\cdot C + \|h\|_{1,[-1,1]})\cdot(\sup_{t\in J}|t|+1) \\
&\qquad + \sup_{t\in J}\int_\varepsilon^t \int_\varepsilon^s |h(r)|\,dr\,ds \\
&< \infty,
\end{aligned}
$$

which concludes the proof of the proposition. □

We return to the results of M. Kunzinger and R. Steinbauer and cite the theorem stating the existence and uniqueness of generalised geodesics ([GKOS01], Theorem 5.3.2).

4.5. Theorem: *Let $[(\delta_\varepsilon)_\varepsilon]$ be a strict delta function, $f \in \mathrm{C}^\infty(\mathbb{R}^2,\mathbb{R})$ and let $x_0^1, \dot x_0^1, x_0^2, \dot x_0^2, v_0, \dot v_0 \in \mathbb{R}$. Then the system of generalised differential equations given (on the level of representatives) by*

$$
\begin{aligned}
\ddot x_\varepsilon^i(u) &= \frac{1}{2}\partial_i f(x_\varepsilon^1(u), x_\varepsilon^2(u))\,\delta_\varepsilon(u) \\
\ddot v_\varepsilon(u) &= f(x_\varepsilon^1(u), x_\varepsilon^2(u))\,\dot\delta_\varepsilon(u) + 2\sum_{i=1}^2 \partial_i f(x_\varepsilon^1(u), x_\varepsilon^2(u))\,\dot x_\varepsilon^i(u)\,\delta_\varepsilon(u)
\end{aligned}
\tag{4.10}
$$

with initial conditions

$$
x_\varepsilon^i(-1) = x_0^i, \quad \dot x_\varepsilon^i(-1) = \dot x_0^i, \quad v_\varepsilon(-1) = v_0, \quad \dot v_\varepsilon(-1) = \dot v_0
$$

has a unique, c-bounded solution $\big([(x_\varepsilon^1)_\varepsilon], [(x_\varepsilon^2)_\varepsilon], [(v_\varepsilon)_\varepsilon]\big) \in \mathcal{G}(\mathbb{R})^3$. Hence, $\gamma : u \mapsto ([(x_\varepsilon^1)_\varepsilon], [(x_\varepsilon^2)_\varepsilon], [(v_\varepsilon)_\varepsilon], u)(u) \in \mathcal{G}[\mathbb{R},\mathbb{R}^4]$ is the unique solution to the geodesic equation for the generalised metric (4.7)*. Furthermore, $(x_\varepsilon^i, v_\varepsilon)$ solves* (4.10) *classically for ε sufficiently small.*

Note that the asymptotic solution constructed in Proposition 4.3 is a representative of the generalised solution of (4.10). Observe that the latter actually deserves the name “solution”, despite all the subtleties of the glueing process employed in Proposition 4.3: Due to the form of the ideal $\mathcal{N}$, it is sufficient for equations to hold in $\mathcal{G}$ if they are satisfied “only” for small ε on compact sets on the level of representatives.

Also note that the theorem claims the moderateness and c-boundedness of the solution for fixed initial values only, i.e. as (generalised) functions depending only on the real variable u. If we are to use the geodesics as coordinate lines of a generalised coordinate transformation as indicated in the preceding section, we still have to show the moderateness (and c-boundedness) of the solution depending also on the initial values. We will do this (for the special case $\dot{x}_0^i = 0$ and $\dot{v}_0 = 0$) in the next section.

Finally, M. Kunzinger and R. Steinbauer proved that the distributional shadow of the generalised geodesics obtained by Theorem 4.5 is indeed the "solution" (4.4) obtained by [FPV88] and [Bal97] ([GKOS01], Theorem 5.3.3):

4.6. Theorem: *The unique solution of the geodesic equation given by* (4.10) *satisfies the following association relations:*

$$\begin{aligned} x_\varepsilon^i(u) &\approx x_0^i + \dot{x}_0^i\,(1+u) + \frac{1}{2}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\,u_+ \\ v_\varepsilon(u) &\approx v_0 + \dot{v}_0\,(1+u) + f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\,H(u) \\ &\qquad + \sum_{i=1}^{2} \partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2) \Big(\dot{x}_0^i + \frac{1}{4}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\Big)\,u_+. \end{aligned}$$

The first line even holds in the sense of C^0*-association, i.e.*

$$x_\varepsilon^i \to x_0^i + \dot{x}_0^i\,(1+u) + \frac{1}{2}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\,u_+$$

as $\varepsilon \to 0$ *uniformly on compact subsets of* $\mathbb{R}$.

Again, note that the (generalised) functions in the above theorem depend only on u, whereas the initial conditions x_0^i, $\dot{x}_0^i$, v_0 and $\dot{v}_0$ are fixed.

4.3 Inversion of the generalised coordinate transformation

In this section, we will prove that the generalised coordinate transformation $[(t_\varepsilon)_\varepsilon]$ (to be defined below) is indeed locally invertible on some open set containing the half space $(-\infty, 0] \times \mathbb{R}^3$ in the sense of Definition 3.28. Part of the inversion problem was already solved by M. Kunzinger and R. Steinbauer in [KS99a], [Ste00] resp. [GKOS01]: They showed that on suitable subsets of $\mathbb{R}^4$ the t_ε are diffeomorphisms for ε sufficiently small (we will give a slightly modified proof suitable for our needs). However, neither did they give an accurate definition of $(t_\varepsilon)_\varepsilon$ as a net in $\mathrm{C}^\infty(\mathbb{R}^4, \mathbb{R}^4)^{(0,1]}$ (as can be derived from Proposition 4.3) nor did they explicitly prove the moderateness and c-boundedness of $(t_\varepsilon)_\varepsilon$ (as a net of functions with four real arguments). Furthermore, lacking a notion of invertibility of generalised functions as developed in Chapter 3, the question of a common domain for the inverses of the t_ε was not raised in [KS99a] resp. [GKOS01]. In [Ste00], the problem of the common domain was addressed but not satisfactorily solved.

Frequently in this section we will have to consider only the first three components of four-vectors resp. functions with four components. To ease notation, we introduce the following general convention: For an element $x = (x^1, \dots, x^n)$ of $\mathbb{R}^n$ ($n \geq 2$), set $\hat{x} := (x^1, \dots, x^{n-1})$ and for functions f from some set into $\mathbb{R}^n$, $f = (f^1, \dots, f^n)$, set $\hat{f} := (f^1, \dots, f^{n-1})$. In addition, we will often meet the situation where, for a function $f = (f^1, \dots, f^n)$ of $x = (x^1, \dots, x^n)$, only f^n depends on x^n. Here, we will not formally distinguish between $\hat{f}$ considered as a function of x (n variables) and of $\hat{x}$ ($n-1$ variables). The respective meaning will be clear from the context.

We start by defining a net $(t_\varepsilon)_\varepsilon$ of smooth functions modelling the "discontinuous coordinate transformation" (4.4). As discussed in Section 4.1, the coordinate transformation is given by the equation for the geodesics with vanishing initial speed in the x^1, x^2 and v-directions. Hence, we set

$$x^i_\varepsilon(-1) = x^i_0, \quad \dot{x}^i_\varepsilon(-1) = 0, \quad v_\varepsilon(-1) = v_0, \quad \dot{v}_\varepsilon(-1) = 0. \tag{4.11}$$

Let $(x^i_\varepsilon)_\varepsilon$ be the asymptotic solution of the first line of (4.10) with initial conditions (4.11) obtained by Proposition 4.3. Using x^i_ε in the second line of (4.10) yields an asymptotic solution for the entire system of differential equations. Thus, we may define the net of transformations $(t_\varepsilon)_\varepsilon$ by $t_\varepsilon := (u, x^1_\varepsilon, x^2_\varepsilon, v_\varepsilon) : \mathbb{R}^4 \to \mathbb{R}^4$,

$$t_\varepsilon : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} U \\ x^i_\varepsilon(X^k, U) \\ v_\varepsilon(X^k, V, U) \end{pmatrix},$$

where $(X^k) = (X^1, X^2)$ and x^i_ε and v_ε are given implicitly (with (X^1, X^2) in a compact subset of $\mathbb{R}^2$ and for sufficiently small ε) by

$$x^i_\varepsilon(X^k, U) = X^i + \frac{1}{2} \int\limits_{-\varepsilon}^{U} \int\limits_{-\varepsilon}^{s} \partial_i f(x^j_\varepsilon(X^k, r))\, \delta_\varepsilon(r)\, dr\, ds,$$

$$v_\varepsilon(X^k, V, U) = V + \int\limits_{-\varepsilon}^{U} f(x_\varepsilon^j(X^k, s))\, \delta_\varepsilon(s)\, ds$$
$$+ \int\limits_{-\varepsilon}^{U} \int\limits_{-\varepsilon}^{s} \sum_{i=1}^{2} \partial_i f(x_\varepsilon^j(X^k, r))\, \dot{x}_\varepsilon^i(X^k, r)\, \delta_\varepsilon(r)\, dr\, ds.$$

The "discontinuous coordinate transformation" will then be denoted by $t := (u, x^1, x^2, v) : \mathbb{R}^4 \to \mathbb{R}^4$. It is given by

$$t : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) = U \\ x^i(X^k, U) = X^i + \frac{1}{2}\partial_i f(X^k)\, U_+ \\ v(X^k, V, U) = V + f(X^k)\, H(U) + \frac{1}{4}\sum_{i=1}^{2} \partial_i f(X^k)^2\, U_+ \end{pmatrix}.$$

At this point, let us briefly outline the strategy of this section from a more technical point of view: In a first step we show the moderateness and c-boundedness of $(t_\varepsilon)_\varepsilon$, together with the boundedness of some of its derivatives, where the full dependency on all four real arguments is taken into account. A crucial feature for the invertibility of $T := [(t_\varepsilon)_\varepsilon]$ consists in the injectivity of t_ε and the property of $\det \circ \mathrm{D}t_\varepsilon$ being strictly non-zero on sufficiently large sets, for small ε. Essentially, this is achieved by the above mentioned result of M. Kunzinger and R. Steinbauer. We quote this as Proposition 4.8, at the same time correcting some minor flaws as to the shapes of the sets of injectivity and their dependence on the relevant parameters. The main difficulty in establishing the local invertibility of T consists in proving that there exist open sets P such that, for ε small, the intersection of the $t_\varepsilon(P)$ has non-empty interior (we even show that the sets P contain arbitrarily large (bounded) parts of the left half space $U \leq 0$). Technically, we accomplish this by a twofold application of Theorem 4.16, a slight variant of Theorem 3.20. To do so, we need two ingredients:

- Uniform convergence: Since t is discontinuous, we have to cut out the discontinuous term from the last component, thereby defining an auxiliary continuous transformation s. Constructing s_ε in an analogous manner out of t_ε, we establish, in several steps, the uniform convergence of s_ε to s.

- Injectivity: For t_ε resp. s_ε, this is provided by Proposition 4.8. As to s, injectivity (on some open superset of the half space $U \leq 0$) follows from Lemma 4.10.

Now Theorem 4.16 can be applied, first to s and s_{ε_0} and then to s_{ε_0} and t_ε (for some ε_0 and $\varepsilon \leq \varepsilon_0$), carrying over the property of having nontrivial interior independent of ε from $s(P)$, via $s_{\varepsilon_0}(P)$, to $t_\varepsilon(P)$.

Now, as announced, we start by showing that $T := [(t_\varepsilon)_\varepsilon]$ is a c-bounded generalised function in $\mathcal{G}[\mathbb{R}^4, \mathbb{R}^4]$.

4.7. Proposition: *$T = [(t_\varepsilon)_\varepsilon]$ is an element of $\mathcal{G}[\mathbb{R}^4, \mathbb{R}^4]$. Furthermore, $\big(\frac{\partial^a}{\partial(X^1)^a}\frac{\partial^b}{\partial(X^2)^b}\, x^i_\varepsilon\big)_\varepsilon$ and $\big(\frac{\partial^a}{\partial(X^1)^a}\frac{\partial^b}{\partial(X^2)^b}\frac{\partial}{\partial U}\, x^i_\varepsilon\big)_\varepsilon$ are c-bounded from $\mathbb{R}^3$ into $\mathbb{R}$ for $a, b \in \mathbb{N}_0$ and $i = 1, 2$.*

Proof: In this proof, $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial X^j}$ will be denoted by ∂_U resp. ∂_{X^j}. Moreover, by $\partial^\alpha_X x^i_\varepsilon$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ we will denote $\partial^{\alpha_1}_{X^1}\partial^{\alpha_2}_{X^2} x^i_\varepsilon$.

First, we show the moderateness of $(x^i_\varepsilon)_\varepsilon$. Let $K \times I \subset\subset \mathbb{R}^3$. By Proposition 4.4, $(x^i_\varepsilon)_\varepsilon$ is uniformly bounded on compact subsets of $\mathbb{R}^4$. The first partial derivative with respect to U on $K \times L$ can be estimated by

$$|\partial_U x^i_\varepsilon(X^k, U)| \le \frac{1}{2}\int_{-\varepsilon}^{U} \underbrace{|\partial_i f(x^j_\varepsilon(X^k, s))|}_{\text{bounded}}\, |\delta_\varepsilon(s)|\, ds \le \frac{1}{2} C_1 C,$$

where $C_1 > 0$. The $\mathcal{E}_M$-estimates of the higher partial derivatives of x^i_ε with respect to U follow inductively from

$$\partial^2_U x^i_\varepsilon(X^k, U) = \frac{1}{2}\partial_i f(x^j_\varepsilon(X^k, U))\, \delta_\varepsilon(U).$$

Next, we consider partial derivatives with respect to X^1 resp. X^2. For $|\alpha| = 1$, we have to find estimates for

$$\partial_{X^j} x^i_\varepsilon(X^k, U) = \delta^i_j + \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s}\sum_{m=1}^{2} \partial_m\partial_i f(x^l_\varepsilon(X^k, r))\, \partial_{X^j} x^m_\varepsilon(X^k, r)\, \delta_\varepsilon(r)\, dr ds. \tag{4.12}$$

For some compact set $L \subset\subset \mathbb{R}^2$, $u_0 \in [-1, \infty)$ and ε small, let

$$C_{L,u_0} := \sup_{\substack{(X^k,U)\in L\times[-1,u_0] \\ i,j\in\{1,2\}}} \big|\partial_i\partial_j f(x^l_\varepsilon(X^k, U))\big| < \infty$$

and

$$g_\varepsilon(L, u_0) := \sup_{\substack{(X^k,U)\in L\times[-1,u_0] \\ j\in\{1,2\}}} \sum_{i=1}^{2} \big|\partial_{X^j} x^i_\varepsilon(X^k, U)\big|.$$

By (4.12), we obtain

$$|g_\varepsilon(L, u_0)| \le 1 + C\, C_{L,u_0}\int_{-\varepsilon}^{u_0} |g_\varepsilon(L, s)|\, ds.$$

From Gronwall's Lemma, it follows that

$$|g_\varepsilon(L,u_0)| \leq e^{|C\,C_{L,u_0}\int_{-\varepsilon}^{u_0} 1\,ds|} \leq e^{C\,C_{L,u_0}(u_0+1)},$$

implying that for small ε also $\partial_{X^j} x_\varepsilon^i$ remains uniformly bounded on compact subsets of $\mathbb{R}^3$ with respect to ε (note that $\partial_{X^j} x_\varepsilon^i(X^k,U) = \delta_j^i$ for $U \leq -\varepsilon$). The higher order derivatives we obtain by induction: Let $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \geq 2$ and assume that $(\partial_X^\beta x_\varepsilon^i)_\varepsilon$ is c-bounded from $\mathbb{R}^3$ into $\mathbb{R}$ for $|\beta| \leq |\alpha|$. Since

$$\begin{aligned}
&\partial_{X^j}\partial_X^\alpha x_\varepsilon^i(X^k,U) = \\
&\quad = \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \delta_\varepsilon(r)\cdot P_{j,\alpha}\Big(\big(\partial^\beta\partial_i f(x_\varepsilon^l(X^k,r))\big)_{|\beta|\leq|\alpha|}, \\
&\qquad\qquad\qquad \big(\partial_X^\beta x_\varepsilon^m(X^k,r)\big)_{|\beta|\leq|\alpha|,m=1,2}\Big)\,dr\,ds \\
&\quad + \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \delta_\varepsilon(r)\sum_{(m,n)=(1,1)}^{(2,2)} \partial_n\partial_m\partial_i f(x_\varepsilon^l(X^k,r))\cdot\partial_{X^j} x_\varepsilon^n(X^k,r) \\
&\qquad\qquad\qquad \cdot\,\partial_X^\alpha x_\varepsilon^m(X^k,r)\,dr\,ds \\
&\quad + \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \delta_\varepsilon(r)\sum_{m=1}^{2}\partial_m\partial_i f(x_\varepsilon^l(X^k,r))\cdot\partial_{X^j}\partial_X^\alpha x_\varepsilon^m(X^k,r)\,dr\,ds,
\end{aligned}$$

where $P_{j,\alpha}$ is a polynomial, we obtain for $(X^k,U) \in K \times I$ and sufficiently small values of ε

$$|\partial_{X^j}\partial_X^\alpha x_\varepsilon^i(X^k,U)| \leq C_1 + \frac{1}{2}C_2\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} |\delta_\varepsilon(r)|\sum_{m=1}^{2}|\partial_{X^j}\partial_X^\alpha x_\varepsilon^m(X^k,r)|\,dr\,ds,$$

where $C_1, C_2 > 0$. Estimating in the same way as in the case $|\alpha| = 1$ yields that also $\big(\partial_{X^j}\partial_X^\alpha x_\varepsilon^i\big)_\varepsilon$ is c-bounded from $\mathbb{R}^3$ into $\mathbb{R}$.
Now, consider $\partial_X^\alpha\partial_U x_\varepsilon^i$ for $\alpha \in \mathbb{N}_0^2$. Observe that

$$\partial_X^\alpha\partial_U x_\varepsilon^i(X^k,U) = \frac{1}{2}\int_{-\varepsilon}^{U} \underbrace{\partial_X^\alpha\big(\partial_i f(x_\varepsilon^j(X^k,s))\big)}_{(*)}\,\delta_\varepsilon(s)\,ds. \tag{4.13}$$

By the chain rule, $(*)$ is a polynomial in $\partial^\beta\partial_i f(x_\varepsilon^j(X^k,s))$ and $\partial_X^\beta x_\varepsilon^i(X^k,s)$ for $|\beta| \leq |\alpha|$ and $i = 1,2$, which are all c-bounded from $\mathbb{R}^3$ into $\mathbb{R}$. Thus, condition (c) on $(\delta_\varepsilon)_\varepsilon$ yields the c-boundedness of $\big(\partial_X^\alpha\partial_U x_\varepsilon^i\big)_\varepsilon$.
Finally, the $\mathcal{E}_M$-estimates for $\partial_X^\alpha\partial_U^m x_\varepsilon^i$ for $\alpha \in \mathbb{N}_0^2$ and $m \geq 2$ follow inductively by differentiating equation (4.13).

The moderateness and c-boundedness of $(v_\varepsilon)_\varepsilon$ follow immediately from the moderateness of $(x^i_\varepsilon)_\varepsilon$ resp. the c-boundedness of $(x^i_\varepsilon)_\varepsilon$ and $(\partial_U x^i_\varepsilon)_\varepsilon$ and condition (c) on $(\delta_\varepsilon)_\varepsilon$. $\square$

In [KS99a], [Ste00] resp. [GKOS01] (Theorem 5.3.6), M. Kunzinger and R. Steinbauer claim that for sufficiently small ε the functions t_ε are diffeomorphisms on a suitable open subset Ω of $\mathbb{R}^4$ containing the shock hyperplane $U = 0$. To show this they employ Theorem 3.58 (Gale and Nikaido). However, a closer look at their proof reveals that the condition of Theorem 3.58 (requiring every principal minor of $\mathrm{D}t_\varepsilon(x)$ for $x \in \Omega$ to be positive) is established only on sets of the form $(-\infty, \eta\,] \times K \times \mathbb{R}$ for sufficiently small ε, say $\varepsilon \leq \varepsilon_0$, where K is a compact subset of $\mathbb{R}^2$ and η and ε_0 both depend on K. Furthermore, they use the uniform boundedness of $(x^i_\varepsilon)_\varepsilon$ on compact subsets of $\mathbb{R}^4$ (Proposition 4.4) which they prove only for compact subsets of $\mathbb{R}$ for fixed initial values x^i_0 and $\dot{x}^i_0$ (Lemma 4.2 resp. [GKOS01], Lemma 5.3.1).
For the convenience of the reader, we restate Theorem 5.3.6 of [GKOS01], claiming only that which is explicitly shown in [KS99a], [Ste00] resp. [GKOS01], and prove it in full detail to make all dependencies clear.

4.8. Proposition: *For every $K \subset\subset \mathbb{R}^2$ and $\delta > 0$ there exist $\eta > 0$ and $\varepsilon_0 \in (0,1]$ such that every principal minor of $\mathrm{D}t_\varepsilon(U, X^i, V)$ stays in $(1-\delta, 1+\delta)$ for all $(U, X^i, V) \in (-\infty, \eta\,] \times K \times \mathbb{R}$ and $\varepsilon \leq \varepsilon_0$. In particular,* $\det \circ \mathrm{D}T$ *is strictly non-zero on $(-\infty, \eta\,] \times K \times \mathbb{R}$ and every principal minor of $\mathrm{D}t_\varepsilon(U, X^i, V)$ is positive for $(U, X^i, V) \in (-\infty, \eta\,] \times K \times \mathbb{R}$ and $\varepsilon \leq \varepsilon_0$.*

Proof: Since

$$\mathrm{D}t_\varepsilon = \frac{\partial(u, x^1_\varepsilon, x^2_\varepsilon, v_\varepsilon)}{\partial(U, X^1, X^2, V)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{\partial x^1_\varepsilon}{\partial U} & \frac{\partial x^1_\varepsilon}{\partial X^1} & \frac{\partial x^1_\varepsilon}{\partial X^2} & 0 \\ \frac{\partial x^2_\varepsilon}{\partial U} & \frac{\partial x^2_\varepsilon}{\partial X^1} & \frac{\partial x^2_\varepsilon}{\partial X^2} & 0 \\ \frac{\partial v_\varepsilon}{\partial U} & \frac{\partial v_\varepsilon}{\partial X^1} & \frac{\partial v_\varepsilon}{\partial X^2} & 1 \end{vmatrix},$$

we have to find estimates for

$$\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k, U) = \delta^i_j + \frac{1}{2} \int\limits_{-\varepsilon}^{U} \int\limits_{-\varepsilon}^{s} \sum_{m=1}^{2} \partial_m \partial_i f(x^l_\varepsilon(X^k, r))\, \frac{\partial x^m_\varepsilon}{\partial X^j}(X^k, r)\, \delta_\varepsilon(r)\, dr ds. \tag{4.14}$$

By Proposition 4.7, $\left(\frac{\partial x^i_\varepsilon}{\partial X^j}\right)_\varepsilon$ is c-bounded from $\mathbb{R}^3$ into $\mathbb{R}$. In particular, there exists $C_1 > 0$ such that for small ε

$$\sup_{\substack{(X^k, U) \in K \times [-1,1] \\ i,j \in \{1,2\}}} \left| \frac{\partial x^i_\varepsilon}{\partial X^j}(X^k, U) \right| \leq C_1.$$

Hence, (4.14) yields

$$\left|\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k,U)-\delta^i_j\right|\leq C\,C_{K,1}\,C_1\,(U+\varepsilon)_+$$

for $(X^k,U)\in K\times(-\infty,1]$ and sufficiently small ε (note that $\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k,U)=\delta^i_j$ for $U\leq-\varepsilon$; $C_{K,1}$ has the same meaning as in the proof of Proposition 4.7). Thus,

$$\sup_{(X^k,U)\in K\times(-\infty,\eta]}\left|\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k,U)-\delta^i_j\right|$$

stays arbitrarily close to 0 for all $\varepsilon\leq\varepsilon_0$ if $\eta>0$ and $\varepsilon_0\in(0,1]$ are chosen accordingly. □

We will say a smooth net $(u_\varepsilon)_\varepsilon:(-a,b)\times\mathbb{R}^n\times\mathbb{R}\to(-a,b)\times\mathbb{R}^n\times\mathbb{R}$ (for $a,b\in\mathbb{R}^+\cup\{\infty\}$) has *property (E)* if for every compact subset K of $\mathbb{R}^n$ there exist $\alpha\in(0,b)$ and $\varepsilon_0\in(0,1]$ such that u_ε is injective on $(-a,\alpha]\times K\times\mathbb{R}$ for all $\varepsilon\leq\varepsilon_0$. The net $(u_\varepsilon)_\varepsilon$ has *property (E+)* if for every compact subset K of $\mathbb{R}^n$ there exist $\alpha\in(0,b)$ and $\varepsilon_0\in(0,1]$ such that u_ε is injective on $(-a,\alpha]\times K\times\mathbb{R}$ and $(\det\circ\mathrm{D}u_\varepsilon)_\varepsilon$ is strictly non-zero, uniformly on $(-a,\alpha]\times K\times\mathbb{R}$ for all $\varepsilon\leq\varepsilon_0$, i.e. an estimate as (3.8) holds for all $(U,X,V)\in(-a,\alpha]\times K\times\mathbb{R}$.

Combining the preceding proposition and Theorem 3.58 of Gale and Nikaido (as was the intention all along), it follows at once that $(t_\varepsilon)_\varepsilon$ has property (E+).

For T to be left invertible (on suitable subsets of $\mathbb{R}^4$) it suffices that $(t_\varepsilon)_\varepsilon$ possesses property (E+): By applying Theorem 3.59 to $(t_\varepsilon)_\varepsilon$, we immediately obtain

4.9. Corollary: *For every open relatively compact subset W of $\mathbb{R}^2$ there exists some $\alpha>0$ such that for all $\beta>0$ and for all bounded open intervals I the generalised function T is left invertible on $(-\beta,\alpha)\times W\times I$.*

As a further (rather plausible) ingredient for the proof of local invertibility of T we will need the fact that the first three components of the "discontinuous transformation" t modelled by T constitute an injective function on some open set containing the half space $(-\infty,0]\times\mathbb{R}^2$. This is established by the following lemma, setting $g=\frac{1}{2}\mathrm{D}f$. Two examples will then show that in the special case $f(X,Y)=X^2-Y^2$ considered by Penrose in [Pen68] such a neighbourhood is given by $(-\infty,1)\times\mathbb{R}^2$, whereas for general (smooth) f a rectangular set of injectivity, i.e. one of the form $(\alpha,\beta)\times\mathbb{R}^2$, does not necessarily exist.

4.10. Lemma: *Let*

$$\begin{aligned} F: \quad (-a,b)\times\mathbb{R}^n \quad &\to \quad (-a,b)\times\mathbb{R}^n \\ \begin{pmatrix} U \\ X \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} U \\ X + g(X)\,U_+ \end{pmatrix}. \end{aligned}$$

where $a, b \in \mathbb{R}^+ \cup \{\infty\}$ *and* $g \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$. *Then there exists an open set* W *containing* $(-a, 0] \times \mathbb{R}^n$ *such that* $F|_W$ *is injective.*

Proof: For $X \in \mathbb{R}^n$ let

$$h(X) := \sup_{z \in \overline{B_{|X|}(0)}} \|\mathrm{D}g(z)\|.$$

The function h is continuous, nonnegative and non-decreasing with $|X|$. Now set

$$W := \left\{ (U, X) \in (-a, b) \times \mathbb{R}^n \,\middle|\, -a < U < \min\left(b, \frac{1}{h(X)}\right) \right\}$$

(here we use the convention $\frac{1}{0} := \infty$). Let $(U_1, X_1), (U_2, X_2) \in W$ and $F(U_1, X_1) = F(U_2, X_2)$. Then $U_1 = U_2 =: U$ and $U < \frac{1}{h(X_i)}$ for $i = 1, 2$. For $U \le 0$, we immediately obtain $X_1 = X_2$. Now let $U > 0$ and assume $X_1 \ne X_2$ with $|X_1| \ge |X_2|$, w.l.o.g. From

$$X_1 + g(X_1)\,U = X_2 + g(X_2)\,U,$$

it follows, noting that $U \cdot h(X_1) < 1$, that

$$\begin{aligned} |X_1 - X_2| &= U \cdot |g(X_2) - g(X_1)| \\ &\le U \cdot \sup_{z \in \overline{B_{|X_1|}(0)}} \|\mathrm{D}g(z)\| \cdot |X_2 - X_1| \\ &< |X_2 - X_1|, \end{aligned}$$

concluding the proof by contradiction. □

In the following two examples, we consider F as in Lemma 4.10, where g is given by $\frac{1}{2}\,\mathrm{D}f$ for certain functions $f : \mathbb{R}^2 \to \mathbb{R}$. The map $F = \hat{t}$ then represents the first three components of t for the function f at hand.

4.11. Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(X, Y) := X^2 - Y^2$. This special case was considered by R. Penrose in [Pen68] (cp. also [GKOS01], components 1,2,4 of (5.45) on page 463). In this case, an easy computation shows that $\hat{t}$ is injective even on $(-\infty, 1) \times \mathbb{R}^2$. The value 1 is maximal since $\hat{t}(1, X, Y_1) = (1, 2X, 0) = \hat{t}(1, X, Y_2)$ for all $X, Y_1, Y_2 \in \mathbb{R}$.

4.12. Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(X,Y) := -\frac{1}{2}(X^4 + Y^4)$. For every $\eta > 0$ the function $\hat{t}$ is non-injective on $\{\eta\} \times \mathbb{R}^2$ since $(\eta, 0, 0) = \hat{t}(\eta, 0, 0) = \hat{t}(\eta, \frac{1}{\sqrt{\eta}}, \frac{1}{\sqrt{\eta}}) = \hat{t}(\eta, -\frac{1}{\sqrt{\eta}}, -\frac{1}{\sqrt{\eta}})$. Hence, on every set of the form $(-\alpha, \beta) \times \mathbb{R}^2$ $(\alpha, \beta > 0)$, $\hat{t}$ is non-injective. However, $\hat{t}$ is injective on $W = \{(U, X, Y) \,|\, U < \frac{1}{3}(X^2 + Y^2)^{-1}\}$.

So far, we ensured the invertibility of the functions t_ε and $\hat{t}$ on certain subsets of $\mathbb{R}^4$ resp. $\mathbb{R}^3$. To invert the generalised function T, however, we also have to consider the sets on which the inverses of the t_ε are defined. The next step will be to prove that the images of certain sets under the t_ε intersect with non-empty interior, the main idea being that if t_ε stays close enough to t, then also the image of some set W under t_ε stays close to $t(W)$. Therefore, we will need convergence of $(t_\varepsilon)_\varepsilon$ to t as $\varepsilon \to 0$ in the appropriate sense. Theorem 4.6 only tells us that $(x^i_\varepsilon(\,.\,, X^1, X^2, V))_\varepsilon$ converges to $x^i(\,.\,, X^1, X^2, V)$ $(\varepsilon \to 0)$ uniformly on compact subsets of $\mathbb{R}$ for fixed $(X^1, X^2, V) \in \mathbb{R}^3$. We will show that we even have uniform convergence of $(x^i_\varepsilon)_\varepsilon$ to x^i on arbitrary compact subsets of $\mathbb{R}^4$. Obviously, this is impossible for v_ε since v is discontinuous. However, dropping the part of v_ε converging (pointwise for $U \neq 0$) to the term involving the Heaviside function, we again can prove uniform convergence on arbitrary compact sets. To this end, we define

$$w(X^k, V, U) := V + \frac{1}{4} \sum_{i=1}^{2} \partial_i f(X^k)^2 \, U_+$$

and

$$w_\varepsilon(X^k, V, U) := V + \int_{-\varepsilon}^{U} \int_{-\varepsilon}^{s} \sum_{i=1}^{2} \partial_i f(x^j_\varepsilon(X^k, r)) \, \dot{x}^i_\varepsilon(X^k, r) \, \delta_\varepsilon(r) \, dr \, ds.$$

Furthermore, let

$$s := (u, x^1, x^2, w)$$

and

$$s_\varepsilon := (u, x^1_\varepsilon, x^2_\varepsilon, w_\varepsilon).$$

Obviously, $\hat{t} = \hat{s}$, implying that also $\hat{s}$ is injective on some open set containing the half space $(-\infty, 0] \times \mathbb{R}^2$. Moreover, since all principal minors of $\mathrm{D}t_\varepsilon$ are independent of the derivatives of v_ε, Proposition 4.8 also holds for $(s_\varepsilon)_\varepsilon$. Therefore, also $(s_\varepsilon)_\varepsilon$ has property (E+).

In a first step, we will show that $\dot{t}_\varepsilon \to \dot{t}$ uniformly on compact subsets of $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}^3$ for $\varepsilon \to 0$. Differentiating t with respect to U yields

$$\begin{aligned}
\dot{u}(U) &= 1,\\
\dot{x}^i(X^k, U) &= \frac{1}{2}\partial_i f(X^k)\, H(U),\\
\dot{v}(X^k, V, U) &= f(X^k)\,\delta(U) + \frac{1}{4}\sum_{i=1}^{2}\partial_i f(X^k)^2\, H(U),
\end{aligned}$$

and $\dot{t}_\varepsilon$ is given by

$$\begin{aligned}
\dot{u}(U) &= 1,\\
\dot{x}^i_\varepsilon(X^k, U) &= \frac{1}{2}\int\limits_{-\varepsilon}^{U} \partial_i f(x^j_\varepsilon(X^k, s))\,\delta_\varepsilon(s)\, ds,\\
\dot{v}_\varepsilon(X^k, V, U) &= f(x^j_\varepsilon(X^k, U))\,\delta_\varepsilon(U)\\
&\quad + \int\limits_{-\varepsilon}^{U}\sum_{i=1}^{2}\partial_i f(x^j_\varepsilon(X^k, s))\,\dot{x}^i_\varepsilon(X^k, s)\,\delta_\varepsilon(s)\, ds.
\end{aligned}$$

4.13. Lemma: *$\dot{t}_\varepsilon \to \dot{t}$ as $\varepsilon \to 0$, uniformly on compact subsets of $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}^3$.*

Proof: First, we show that

$$M^i_\varepsilon(K_1, K_2) := \sup_{(X^k,U)\in K_1\times K_2} |\partial_i f(x^j_\varepsilon(X^k, \varepsilon U)) - \partial_i f(X^k)| \to 0$$

for $\varepsilon \to 0$ ($i = 1, 2$), where $K_1 \times K_2 \subset\subset \mathbb{R}^2 \times \mathbb{R}$: By the boundedness properties of x^i_ε established in Proposition 4.7 and by condition (c), we have

$$\begin{aligned}
&\sup_{(X^k,U)\in K_1\times K_2} \left| x^i_\varepsilon(X^k, \varepsilon U) - X^k \right| \le\\
&\le \frac{1}{2}\sup_{(X^k,U)\in K_1\times K_2} \int\limits_{-\varepsilon}^{\varepsilon U}\int\limits_{-\varepsilon}^{s} \left|\partial_i f(x^j_\varepsilon(X^k, r))\right| \left|\delta_\varepsilon(r)\right| dr\, ds\\
&\le \frac{1}{2}\sup_{\substack{(X^k)\in K_1\\ r\in[-1,\sup_{U\in K_2}|U|]}} \left|\partial_i f(x^j_\varepsilon(X^k, r))\right| \cdot \sup_{U\in K_2}\left|\int\limits_{-\varepsilon}^{\varepsilon U} C\, ds\right|\\
&\le \varepsilon\cdot \underbrace{\frac{C}{2}\sup_{\substack{(X^k)\in K_1\\ r\in[-1,\sup_{U\in K_2}|U|]}} \left|\partial_i f(x^j_\varepsilon(X^k, r))\right| \cdot \sup_{U\in K_2}\left|U+1\right|}_{\text{bounded}},
\end{aligned}$$

which vanishes in the limit for $\varepsilon \to 0$. Therefore, $M^i_\varepsilon(K_1, K_2) \to 0$ as $\varepsilon \to 0$.

Also, note that $\sup_{U\in L} |\int_{-\varepsilon}^U \delta_\varepsilon(s)\,ds - H(U)| \to 0$ as $\varepsilon \to 0$ for any compact set $L \subset\subset \mathbb{R}\backslash\{0\}$, by conditions (a) and (b).

Now, let $L \times K \times M \subset\subset (\mathbb{R}\backslash\{0\}) \times \mathbb{R}^2 \times \mathbb{R}$. Since both $\dot{t}_\varepsilon$ and $\dot{t}$ are, in fact, independent of V, we only have to take estimates on $L \times K$. By the properties of the strict delta net and the above considerations,

$$
\begin{aligned}
&\sup_{(X^k,U)\in K\times L} \left|\dot{x}^i_\varepsilon(X^k,U) - \dot{x}^i(X^k,U)\right| = \\
&\quad = \sup_{(X^k,U)\in K\times L} \left|\frac{1}{2}\int\limits_{-\varepsilon}^{U} \partial_i f(x^j_\varepsilon(X^k,s))\delta_\varepsilon(s)\,ds - \frac{1}{2}\partial_i f(X^k)H(U)\right| \\
&\quad \le \frac{1}{2}\sup_{(X^k)\in K}\int\limits_{-\varepsilon}^{\varepsilon}\left|\partial_i f(x^j_\varepsilon(X^k,s)) - \partial_i f(X^k)\right|\left|\delta_\varepsilon(s)\right|ds \\
&\qquad\qquad + \frac{1}{2}\sup_{(X^k)\in K}\left|\partial_i f(X^k)\right|\cdot\sup_{U\in L}\left|\int\limits_{-\varepsilon}^{U}\delta_\varepsilon(s)\,ds - H(U)\right| \\
&\quad \le \frac{1}{2}\cdot\underbrace{\sup_{|s|\le 1}\left|\partial_i f(x^j_\varepsilon(X^k,\varepsilon s)) - \partial_i f(X^k)\right|}_{=M^i_\varepsilon(K,|s|\le 1)\to 0}\cdot\underbrace{\int\limits_{-\varepsilon}^{\varepsilon}\left|\delta_\varepsilon(s)\right|ds}_{\le C} \\
&\qquad\qquad + \frac{1}{2}\underbrace{\sup_{(X^k)\in K}\left|\partial_i f(X^k)\right|}_{\text{bounded}}\cdot\underbrace{\sup_{U\in L}\left|\int\limits_{-\varepsilon}^{U}\delta_\varepsilon(s)\,ds - H(U)\right|}_{\to 0} \\
&\quad \to 0
\end{aligned}
$$

for $\varepsilon \to 0$. Hence, the claim follows for $\dot{x}^i_\varepsilon$. Concerning $\dot{v}_\varepsilon$, we have

$$
\begin{aligned}
&\sup_{(U,X^k,V)\in L\times K\times M}\left|\dot{v}_\varepsilon(U,X^k,V) - \dot{v}(U,X^k,V)\right| \le \\
&\quad \le \sup_{(X^k,U)\in K\times L}\left|f(x^j_\varepsilon(X^k,U))\,\delta_\varepsilon(u) - 0\right| \\
&\quad + \sup_{(X^k,U)\in K\times L}\sum_{i=1}^{2}\left|\underbrace{\int\limits_{-\varepsilon}^{U}\partial_i f(x^j_\varepsilon(X^k,s))\,\dot{x}^i_\varepsilon(X^k,s)\,\delta_\varepsilon(s)\,ds - \frac{1}{4}\,\partial_i f(X^k)^2\,H(u)}_{(*)}\right|
\end{aligned}
$$

since $0 \notin L$ and $\delta(U) = 0$ for $U \neq 0$. The first term above vanishes in the limit $\varepsilon \to 0$ due to condition (a). Concerning the last term, we write out

$$
\begin{aligned}
(*) &= \int_{-\varepsilon}^{U} \partial_i f(x^j_\varepsilon(X^k,s))\,\delta_\varepsilon(s)\,\frac{1}{2}\int_{-\varepsilon}^{s} \partial_i f(x^j_\varepsilon(X^k,r))\,\delta_\varepsilon(r)\,dr\,ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{4}\,\partial_i f(X^j)^2\,H(u) \\
&= \frac{1}{2}\int_{-\varepsilon}^{U} \partial_i f(x^j_\varepsilon(X^k,s))\delta_\varepsilon(s)\int_{-\varepsilon}^{s}\Big(\partial_i f(x^j_\varepsilon(X^k,r)) - \partial_i f(X^k)\Big)\delta_\varepsilon(r)\,dr\,ds \\
&\quad + \frac{1}{2}\int_{-\varepsilon}^{U}\Big(\partial_i f(x^j_\varepsilon(X^k,s)) - \partial_i f(X^k)\Big)\,\delta_\varepsilon(s)\int_{-\varepsilon}^{s}\partial_i f(X^k)\,\delta_\varepsilon(r)\,dr\,ds \\
&\quad + \frac{1}{2}\,\partial_i f(X^k)^2\Big(\int_{-\varepsilon}^{U}\delta_\varepsilon(s)\int_{-\varepsilon}^{s}\delta_\varepsilon(r)\,dr\,ds - \frac{1}{2}\,H(u)\Big).
\end{aligned}
$$

Integrating $\int_{-\varepsilon}^{U}\delta_\varepsilon(s)\int_{-\varepsilon}^{s}\delta_\varepsilon(r)\,dr\,ds$ by parts gives $\frac{1}{2}\Big(\int_{-\varepsilon}^{U}\delta_\varepsilon(s)ds\Big)^2$. Now, due to the boundedness properties of x^i_ε and condition (c), we may estimate

$$
\begin{aligned}
&\sup_{(X^k,U)\in K\times L}|(*)| \le \\
&\le \frac{1}{2}\cdot\underbrace{M^i_\varepsilon(K,|r|\le 1)}_{\to 0}\cdot\underbrace{\sup_{\substack{(X^k)\in K\\ |r|\le 1}}\Big|\partial_i f(x^j_\varepsilon(X^k,r))\Big|}_{\text{bounded}}\cdot\underbrace{\int_{-\varepsilon}^{\varepsilon}|\delta_\varepsilon(s)|\int_{-\varepsilon}^{s}|\delta_\varepsilon(r)|\,dr\,ds}_{\le C^2} \\
&\quad + \frac{1}{2}\cdot\underbrace{M^i_\varepsilon(K,|s|\le 1)}_{\to 0}\cdot\underbrace{\sup_{(X^k)\in K}\Big|\partial_i f(X^k)\Big|}_{\text{bounded}}\cdot\underbrace{\int_{-\varepsilon}^{\varepsilon}|\delta_\varepsilon(s)|\int_{-\varepsilon}^{s}|\delta_\varepsilon(r)|\,dr\,ds}_{\le C^2} \\
&\quad + \frac{1}{2}\cdot\underbrace{\sup_{(X^k)\in K}\Big|\partial_i f(X^k)^2\Big|}_{\text{bounded}}\cdot\underbrace{\Big|\frac{1}{2}\Big(\int_{-\varepsilon}^{U}\delta_\varepsilon(s)ds\Big)^2 - \frac{1}{2}\,H(u)\Big|}_{\to 0} \\
&\to 0 \quad (\varepsilon\to 0),
\end{aligned}
$$

thereby concluding the proof of the lemma. □

An inspection of the proof of the above lemma shows that also $\dot{s}_\varepsilon \to \dot{s}$ for $\varepsilon \to 0$ uniformly on compact subsets of $(\mathbb{R}\backslash\{0\})\times\mathbb{R}^3$.

4.14. Lemma: *Let $f_\varepsilon, f \in \mathrm{C}(\mathbb{R}^n,\mathbb{R})$ (for $\varepsilon\in(0,1]$). Suppose that $\partial_n f_\varepsilon(x,t)$ and $\partial_n f(x,t)$ exist for all $(x,t)\in\mathbb{R}^{n-1}\times(\mathbb{R}\backslash\{0\})$ and that $\partial_n f_\varepsilon(x,\,.\,)$ and*

$\partial_n f(x, .)$ are piecewise continuous (with one-sided limits existing) for all $x \in \mathbb{R}^{n-1}$. Let $c \in \mathbb{R}$ with $c < 0$. If

(1) $f_\varepsilon \to f$ for $\varepsilon \to 0$ uniformly on $K \times \{c\}$ for all $K \subset\subset \mathbb{R}^{n-1}$,

(2) $\partial_n f_\varepsilon \to \partial_n f$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^{n-1} \times (\mathbb{R}\backslash\{0\})$, and

(3) $\|\partial_n f_\varepsilon - \partial_n f\|_{\infty,K\times([-d,d]\backslash\{0\})}$ is uniformly bounded for any compact set $K \subset\subset \mathbb{R}^{n-1}$ and some $d > 0$,

then $f_\varepsilon \to f$ for $\varepsilon \to 0$ uniformly on arbitrary compact subsets of $\mathbb{R}^n$.

Proof: It suffices to show the uniform convergence on compact sets of the form $L = K \times [-a, a] \subset\subset \mathbb{R}^{n-1} \times \mathbb{R}$ with $a > c$.

Let $\eta > 0$. Fix some $0 < b < \min(a, d, |c|)$ such that

$$\|\partial_n f_\varepsilon - \partial_n f\|_{\infty,K\times([-b,b]\backslash\{0\})} < \frac{\eta}{6b}.$$

Now, choose $\varepsilon_0 \in (0, 1]$ such that

$$\sup_{x\in L} |f_\varepsilon(x, c) - f(x, c)| < \frac{\eta}{3} \quad \text{and} \quad \|\partial_n f_\varepsilon - \partial_n f\|_{\infty,Q} < \frac{\eta}{6(a-b)}$$

for all $\varepsilon \le \varepsilon_0$, where $Q := K \times ([-a, -b] \cup [b, a])$. Then, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\sup_{\substack{(x,t)\in L\\ t\ge b}} & |f_\varepsilon(x, t) - f(x, t)| \le \\
&\le \sup_{x\in K} |f_\varepsilon(x, c) - f(x, c)| + \sup_{(x,t)\in L} \int_c^t |\partial_n f_\varepsilon(x, s) - \partial_n f(x, s)|\, ds \\
&< \frac{\eta}{3} + \sup_{x\in K} \int_c^{-b} |\partial_n f_\varepsilon(x, s) - \partial_n f(x, s)|\, ds \\
&\qquad\qquad\qquad + \sup_{x\in K} \int_{-b}^{b} |\partial_n f_\varepsilon(x, s) - \partial_n f(x, s)|\, ds \\
&\qquad\qquad\qquad + \sup_{x\in K} \int_{b}^{a} |\partial_n f_\varepsilon(x, s) - \partial_n f(x, s)|\, ds \\
&\le \frac{\eta}{3} + 2(a-b) \cdot \|\partial_n f_\varepsilon - \partial_n f\|_{\infty,Q} + 2b \cdot \|\partial_n f_\varepsilon - \partial_n f\|_{\infty,K\times([-b,b]\backslash\{0\})} \\
&< \frac{\eta}{3} + 2(a-b) \cdot \frac{\eta}{6(a-b)} + 2b \cdot \frac{\eta}{6b} \\
&= \eta.
\end{aligned}$$

For $-a \le t \le b$, the estimate is similar (even easier), involving less terms. Hence, the claim follows. □

Now we are ready to prove

4.15. Lemma: *$s_\varepsilon \to s$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^4$.*

Proof: We show that for each component function of s_ε the conditions of Lemma 4.14 are satisfied with respect to s. The symbol ∂_n in Lemma 4.14, if applied to x^i_ε resp. w_ε, is understood to denote the derivatives of x^i_ε resp. w_ε with respect to U.

$\dot{x}^i_\varepsilon$ resp. $\dot{w}_\varepsilon$ are smooth on $\mathbb{R}^3$ resp. $\mathbb{R}^4$, x^i and w are smooth on $\mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ resp. $\mathbb{R}^3 \times (\mathbb{R}\backslash\{0\})$. $\dot{x}^i(X^1, X^2, \,.\,)$ and $\dot{w}(X^1, X^2, V, \,.\,)$ are piecewise continuous for all $(X^1, X^2) \in \mathbb{R}^2$ resp. $(X^1, X^2, V) \in \mathbb{R}^3$. For $U = -1$ the integral terms of $x^i_\varepsilon(\,.\,, \,.\,, U)$ and $w_\varepsilon(\,.\,, \,.\,, \,.\,, U)$ vanish and $x^i_\varepsilon = x^i$ and $w_\varepsilon = w$. Hence, condition (1) is satisfied. By Lemma 4.13, $\dot{x}^i_\varepsilon \to \dot{x}^i$ and $\dot{w}_\varepsilon \to \dot{w}$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ resp. $\mathbb{R}^3 \times (\mathbb{R}\backslash\{0\})$, i.e. they satisfy condition (2). Finally, by Theorem 4.5, $\dot{x}^i_\varepsilon$ is uniformly bounded on compact sets and, therefore, this is also true for $\dot{w}_\varepsilon$. Since both $\dot{x}^i$ and $\dot{w}$ are bounded on any bounded subset of $\mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ resp. $\mathbb{R}^3 \times (\mathbb{R}\backslash\{0\})$, also condition (3) is satisfied and the claim follows. □

Recall that for vectors $x \in \mathbb{R}^n$ resp. $\mathbb{R}^n$-valued functions f, the notation $\hat{x}$ resp. $\hat{f}$ indicates that the last component is to be dropped.

In the sequel, we will often have to make use of cylinders rather than balls. Therefore, for $x = (\hat{x}, x^n) \in \mathbb{R}^n$, let $B^Z_{\delta,\eta}(x)$ denote the cylinder $B_\delta(\hat{x}) \times (x^n - \eta, x^n + \eta)$.

We will employ a slightly modified form of Theorem 3.20 where the balls $B_\delta(0)$ are replaced by cylinders $B^Z_{\delta,\eta}(0)$. We leave it to the reader to adapt the proof of 3.20 to the case of cylinders.

4.16. Theorem: *Let U be an open subset of $\mathbb{R}^n$, $f, g \in \mathrm{C}(U, \mathbb{R}^n)$ both injective and W a connected open subset of $\mathbb{R}^n$ with $\overline{W} \subset\subset f(U)$. Choose $y \in W$ and $\delta, \eta > 0$ with $y + B^Z_{\delta,\eta}(0) \subseteq W$ such that the closure of $W_{\delta,\eta} := W + B^Z_{\delta,\eta}(0)$ is still a subset of $f(U)$. If, for $A := f^{-1}(\overline{W_{\delta,\eta}})$ and $f = (\hat{f}, f^n)$ resp. $g = (\hat{g}, g^n)$, both*

$$\|\hat{g} - \hat{f}\|_{\infty,A} < \delta \quad \text{and} \quad \|g^n - f^n\|_{\infty,A} < \eta$$

hold, then

$$\overline{W} \subseteq g(A)^\circ.$$

Now we are ready to prove that the domains of suitable inverses of the t_ε intersect with non-empty interior. The following theorem yields the desired result for an entire class of c-bounded nets (also denoted by $(t_\varepsilon)_\varepsilon$) of smooth functions of which our particular $(t_\varepsilon)_\varepsilon$ at hand is only a special case.

4.17. Theorem: *Let $a, b \in \mathbb{R}^+ \cup \{\infty\}$. Let the functions t_ε, s_ε (for every $\varepsilon \in (0,1]$) and s satisfy the following assumptions:*

(1) $t_\varepsilon : (-a,b) \times \mathbb{R}^n \times \mathbb{R} \to (-a,b) \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{pmatrix} U \\ X \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & := U \\ x_\varepsilon(U,X) & \\ v_\varepsilon(U,X,V) & := V + g_\varepsilon(U,X) + h_\varepsilon(U,X) \end{pmatrix},$$

where $x_\varepsilon \in \mathrm{C}^\infty((-a,b) \times \mathbb{R}^n, \mathbb{R}^n)$ and $g_\varepsilon, h_\varepsilon \in \mathrm{C}^\infty((-a,b) \times \mathbb{R}^n, \mathbb{R})$. Assume that $(t_\varepsilon)_\varepsilon$ has property (E), i.e. that for every compact subset K of $\mathbb{R}^n$ there exist $\alpha \in (0,b)$ and $\varepsilon' \in (0,1]$ such that t_ε is injective on $(-a,\alpha] \times K \times \mathbb{R}$ for all $\varepsilon \le \varepsilon'$. Furthermore, suppose that $(h_\varepsilon)_\varepsilon$ is uniformly bounded on compact subsets of $(-a,b) \times \mathbb{R}^n$.

(2) $s_\varepsilon : (-a,b) \times \mathbb{R}^n \times \mathbb{R} \to (-a,b) \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{pmatrix} U \\ X \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & = U \\ x_\varepsilon(U,X) & \\ w_\varepsilon(U,X,V) & := V + g_\varepsilon(U,X) \end{pmatrix}.$$

By (1), s_ε is smooth. Suppose that also $(s_\varepsilon)_\varepsilon$ has property (E).

(3) $s : (-a,b) \times \mathbb{R}^n \times \mathbb{R} \to (-a,b) \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{pmatrix} U \\ X \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & = U \\ x(U,X) & \\ w(U,X,V) & := V + g(U,X) \end{pmatrix},$$

where $x \in \mathrm{C}((-a,b) \times \mathbb{R}^n, \mathbb{R}^n)$ and $g, h \in \mathrm{C}((-a,b) \times \mathbb{R}^n, \mathbb{R})$. Assume that for $\hat{s} := (u,x) : (-a,b) \times \mathbb{R}^n \to (-a,b) \times \mathbb{R}^n$, there exists some open set W containing $(-a,0] \times \mathbb{R}^n$ such that $\hat{s}|_W$ is injective.

Finally, suppose $s_\varepsilon \to s$ for $\varepsilon \to 0$ uniformly on compact sets.

Then the following holds: For every p on the hyperplane $U = 0$ there exist open neighbourhoods P of p with $P \subseteq W \times \mathbb{R}$ and Q of $q := s(q)$ with $Q \subseteq s(W \times \mathbb{R})$, and some $\varepsilon_0 \in (0,1]$ such that

$$\overline{Q} \subseteq t_\varepsilon(P)$$

for all $\varepsilon \le \varepsilon_0$.

Proof: Since, by assumption, $\hat{s}$ is injective on W, it follows from Theorem 3.18 that $\hat{s}(W)$ is open in $\mathbb{R}^{n+1}$ and $\hat{s}|_W : W \to \hat{s}(W)$ is a homeomorphism. Note that with $\hat{s}|_W$, also $s|_{W\times\mathbb{R}}$ is a homeomorphism and that $s(W \times \mathbb{R})$ equals the open set $\hat{s}(W) \times \mathbb{R}$. We will simply write $\hat{s}$ and s in place of $\hat{s}|_W$ resp. $s|_{W\times\mathbb{R}}$. Analogously to $\hat{s}$, we define $\hat{t}_\varepsilon := \hat{s}_\varepsilon := (u, x_\varepsilon)$. Then $\hat{t}_\varepsilon = \hat{s}_\varepsilon \to \hat{s}$ uniformly on compact sets for $\varepsilon \to 0$.

Let $p = (0, x_p, v_p)$ be a point of the hyperplane $U = 0$, $q := s(p) = (0, x_q, v_q)$, $\hat{p} = (0, x_p)$ and $\hat{q} = \hat{s}(\hat{p}) = (0, x_q)$. Let $R \subseteq \mathbb{R}^n$ be an open bounded cuboid (or, more generally, a bounded open set satisfying $\overline{R}^\circ = R$) containing x_p. Since $(-a, 0] \times \mathbb{R}^n \subseteq W$ and W is open, there exist $\alpha \in (0, \min(a, b))$ and $\lambda > 0$ such that

$$(-a, \alpha] \times \overline{R_\lambda} \subseteq W,$$

where $R_\lambda := R + B_\lambda(0)$. Then s is injective on $(-a, \alpha] \times \overline{R_\lambda} \times \mathbb{R}$. By property (E), we can assume w.l.o.g. (making α smaller if necessary) that there exists $\varepsilon_1 \in (0, 1]$ such that also $(t_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ are injective on $(-a, \alpha] \times \overline{R_\lambda} \times \mathbb{R}$ for all $\varepsilon \leq \varepsilon_1$. Defining

$$G := (-a, \alpha) \times R_\lambda \times \mathbb{R},$$

we have, in particular, that s, t_ε and s_ε (for $\varepsilon \leq \varepsilon_1$) are injective on G.

Fix $\gamma \in (0, \alpha)$ and $\beta \in [\gamma, a)$. Since $\hat{s}(W)$ is open and $\hat{s}^{-1}$ is continuous, there exists some $\delta > 0$ with $\hat{s}^{-1}(\overline{B_{3\delta}(\hat{q})}) \subseteq (-\beta, \gamma) \times R$, i.e. $\overline{B_{3\delta}(\hat{q})} \subseteq \hat{s}((-\beta, \gamma) \times R)$. Let $\mu \in (\beta, a)$. Choose $\eta \geq \delta$ and $\varepsilon_2 \leq \varepsilon_1$ such that

$$\|v_\varepsilon - w_\varepsilon\|_{\infty,[-\mu,\alpha]\times\overline{R_\lambda}\times\mathbb{R}} = \|h_\varepsilon\|_{\infty,[-\mu,\alpha]\times\overline{R_\lambda}} < \eta \tag{4.15}$$

for all $\varepsilon \leq \varepsilon_2$. Since $s(W \times \mathbb{R}) = \hat{s}(W) \times \mathbb{R}$, it follows that $\overline{B^Z_{3\delta,2\eta+\delta}(q)} = \overline{B_{3\delta}(\hat{q})} \times [v_q - (2\eta + \delta), v_q + (2\eta + \delta)]$ is a compact subset of $s(W \times \mathbb{R})$. Now let I be a bounded open interval in $\mathbb{R}$ such that

$$s^{-1}\left(\overline{B^Z_{3\delta,2\eta+\delta}(q)}\right) \subseteq (-\beta, \gamma) \times R \times I =: P$$

which is possible since only the last component of s is dependent on V and this dependence is a linear one. Applying s to both sides of this inclusion yields

$$\overline{B^Z_{3\delta,2\eta+\delta}(q)} \subseteq s(P). \tag{4.16}$$

Observe that $p \in P \subseteq \overline{P} \subset\subset G$ and $q \in s(P)$. Choose $\varepsilon_0 \leq \varepsilon_2$ such that

$$\|\hat{s}_\varepsilon - \hat{s}\|_{\infty,\overline{P}} < \frac{\delta}{2} \tag{4.17}$$

and

$$\|w_\varepsilon - w\|_{\infty,\overline{P}} < \frac{\delta}{2} \tag{4.18}$$

for all $\varepsilon \leq \varepsilon_0$ (note that in (4.17) we consider $\hat{s}_\varepsilon$ and $\hat{s}$ as functions with the same arguments as s_ε resp. s, yet being independent of the last (real) argument). The set $s(P)$ is open since $s|_{W\times\mathbb{R}}$ is a homeomorphism, and bounded because $s(\overline{P})$ is compact. Consequently,

$$Q_0' := s(P)\backslash(\partial s(P) + \overline{B^Z_{\delta,\delta}(0)})$$

is open and bounded. By (4.16) and by definition of Q_0',

$$\overline{B^Z_{2\delta,2\eta}(q)} \subseteq Q_0' \tag{4.19}$$

holds. Now let Q' be the connected component of Q_0' containing q, hence also containing the (connected) set $\overline{B^Z_{2\delta,2\eta}(q)}$. Obviously, Q' is open, bounded and connected.

Now the plan is to apply Theorem 4.16 with G, s, s_{ε_0}, Q', q, δ and δ in place of U, f, g, W, y, δ and η. For this, we have to verify the respective list of assumptions:

- s and s_{ε_0} are continuous and injective on the open set G. This is satisfied due to our construction.
- Q' is open and connected (see above). $\overline{Q'}$ is a compact subset of the (open) set $s(G)$: Noting that $s(\overline{P})$ is compact, this follows from

$$\overline{s(P)} \subseteq \overline{s(\overline{P})} = s(\overline{P}) \subset\subset s(G).$$

- $B^Z_{\delta,\delta}(q) \subseteq Q'$, due to $\delta \leq \eta$ and (4.19).
- $\overline{Q' + B^Z_{\delta,\delta}(0)} \subseteq s(G)$: We even show $\overline{Q_0' + B^Z_{\delta,\delta}(0)} \subseteq s(G)$. For this, it suffices to prove $Q_0' + B^Z_{\delta,\delta}(0) \subseteq s(P)$, implying $\overline{Q_0' + B^Z_{\delta,\delta}(0)} \subseteq \overline{s(P)} \subseteq s(G)$ (for the last inclusion see above). By way of contradiction, we assume that $z = (\hat{z},\tau) \in Q_0'$, $y = (\hat{y},\sigma) \in B^Z_{\delta,\delta}(0)$, yet $z + y = (\hat{z} + \hat{y}, \tau + \sigma) \notin s(P)$. Since $z \in s(P)$, there exists a point $z + \nu y$ on the line segment connecting z and $z + y$, with $0 < \nu \leq 1$ due to $s(P)$ being open. From $z = (z + \nu y) - \nu y$, it follows that $z \in \partial s(P) + B^Z_{\delta,\delta}(0)$, contradicting $z \in Q_0'$.
- For $M' := s^{-1}(\overline{Q' + B^Z_{\delta,\delta}(0)})$, (4.17) and (4.18) yield

$$\|\hat{s}_\varepsilon - \hat{s}\|_{\infty,M'} < \delta \quad \text{and} \quad \|w_\varepsilon - w\|_{\infty,M'} < \delta,$$

where we have taken into account that

$$M' = s^{-1}(\overline{Q' + B^Z_{\delta,\delta}(0)}) \subseteq s^{-1}(\overline{s(P)}) \subseteq s^{-1}(s(\overline{P})) = \overline{P}.$$

Having thus checked that all assumptions are satisfied, we obtain from Theorem 4.16

$$\overline{Q'} \subseteq s_{\varepsilon_0}(M')^\circ \subseteq s_{\varepsilon_0}(\overline{P})^\circ. \tag{4.20}$$

Now we set out to apply Theorem 4.16 once more to derive an analogous statement with respect to t_ε. Similarly to above, set

$$Q_0 := Q' \backslash (\partial Q' + \overline{B^Z_{\delta,\eta}(0)}). \tag{4.21}$$

Again, Q_0 is open and bounded. By (4.19) and the definitions of Q' and Q_0,

$$\overline{B^Z_{\delta,\eta}(q)} \subseteq Q_0$$

holds. With Q denoting the connected component of Q_0 containing q, we even have

$$\overline{B^Z_{\delta,\eta}(q)} \subseteq Q. \tag{4.22}$$

As before, we check the list of assumptions in Theorem 4.16, this time with respect to G, s_{ε_0}, t_ε (for fixed $\varepsilon \leq \varepsilon_0$), Q, q, δ and η in place of U, f, g, W, y, δ and η:

- s_{ε_0} and t_ε are continuous and injective on the open set G due to our construction.

- Q is open and connected. By (4.20) and the definition of Q, we have

  $$\overline{Q} \subseteq \overline{Q'} \subseteq s_{\varepsilon_0}(\overline{P}) \subseteq s_{\varepsilon_0}(G),$$

  showing that $\overline{Q}$ is a compact subset of $s_{\varepsilon_0}(G)$.

- $B^Z_{\delta,\eta}(q) \subseteq Q$, due to (4.22).

- $\overline{Q + B^Z_{\delta,\eta}(0)} \subseteq s_{\varepsilon_0}(G)$: Again, it suffices to show $Q_0 + B^Z_{\delta,\eta}(0) \subseteq Q'$. This, in turn, is derived by an analogous line of argument as in the checklist for the first application of Theorem 4.16.

- Set $M := s_{\varepsilon_0}^{-1}(\overline{Q + B^Z_{\delta,\eta}(0)})$. By $\overline{Q + B^Z_{\delta,\eta}(0)} \subseteq \overline{Q'} \subseteq s_{\varepsilon_0}(\overline{P})$ (see (4.20)), we have $M \subseteq \overline{P} \subseteq [-\mu, \alpha] \times \overline{R_\lambda} \times \mathbb{R}$. Therefore, (4.15) immediately yields

  $$\|v_\varepsilon - w_\varepsilon\|_{\infty,M} < \eta.$$

  By $\hat{t}_\varepsilon = \hat{s}_\varepsilon$ and (4.17), we obtain

  $$\|\hat{t}_\varepsilon - \hat{s}_{\varepsilon_0}\|_{\infty,M} \leq \|\hat{s}_\varepsilon - \hat{s}\|_{\infty,\overline{P}} + \|\hat{s} - \hat{s}_{\varepsilon_0}\|_{\infty,\overline{P}} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Now, Theorem 4.16 yields

$$\overline{Q} \subseteq t_\varepsilon(M)^\circ \subseteq t_\varepsilon(\overline{P}),$$

and hence, t_ε being a homeomorphism on G and $\overline{R}^\circ = R$,

$$\overline{Q} \subseteq t_\varepsilon(P)$$

for all $\varepsilon \leq \varepsilon_0$. □

4.18. Remark: As to sizes and shapes of P resp. Q, an inspection of the preceding proof reveals the following:

(1) P can be chosen as having the form $(-\beta, \gamma) \times R \times I$ where $-\beta < 0$ is arbitrarily close to $-a$, R and I are arbitrarily large, yet bounded open sets (I being of a certain minimum size, depending on $\|h_\varepsilon\|_\infty$ on compact sets for small ε) and γ has to be sufficiently small, depending (via α) on R and the injectivity behaviour of s, $(t_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ for $U > 0$. $G = (-a, \alpha) \times R_\lambda \times \mathbb{R}$ is an open superset of $\overline{P}$, serving as common domain of injectivity for s, s_{ε_0} and t_ε when applying Theorem 4.16.

(2) Q results from $s(P)$ by twofold application of the operation "remove the outermost strip of width δ (resp. η for the last coordinate in step 2) and keep only the connected component containing q". The maximum size of δ, in turn, essentially depends on $s(P)$ around $U = 0$ and has to satisfy $\delta \leq \frac{\gamma}{3}$. However, δ can be chosen arbitrarily small.

(3) In the case of $x(U, X) = X + f_0(X)\, U_+$ (occurring in our study of pp-waves), the maximal size of δ, for small γ, is about $\frac{\gamma}{3}$ since $\hat{s}((-\beta, \gamma) \times R)$ approaches $(-\beta, \gamma) \times R$ for $U \to 0+$ (cp. Examples 4.11 and 4.12).

Finally, we show that if the nets of smooth functions in Theorem 4.17 are representatives of generalised functions T and S which additionally satisfy property (E+), then T is invertible around any point on the shock hyperplane.

4.19. Theorem: *Let $(t_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ and s be as in Theorem 4.17. If, in addition, $(t_\varepsilon)_\varepsilon$ has property (E+) and*

$$T := [(t_\varepsilon)_\varepsilon] \in \mathcal{G}[(-a, b) \times \mathbb{R}^n \times \mathbb{R}, (-a, b) \times \mathbb{R}^n \times \mathbb{R}]$$

and

$$S := [(s_\varepsilon)_\varepsilon] \in \mathcal{G}[(-a, b) \times \mathbb{R}^n \times \mathbb{R}, (-a, b) \times \mathbb{R}^n \times \mathbb{R}],$$

then, for every p on the hyperplane $U = 0$, there exists an open neighbourhood A of p in $(-a, b) \times \mathbb{R}^n \times \mathbb{R}$ such that T is invertible on A with inversion data $[A, \mathbb{R}^{n+2}, T^\diamond, B, Q]$ where $T^\diamond \in \mathcal{G}[\mathbb{R}^{n+2}, D]$ and B, Q and D are suitable bounded open subsets of $(-a, b) \times \mathbb{R}^n \times \mathbb{R}$ with $Q \subseteq B$ and $A \subseteq D$.

Proof: Let α, R_λ, $G = (-a, \alpha) \times R_\lambda \times \mathbb{R}$, P, Q and ε_0 be as in the proof of Theorem 4.17. Recall that then, among other things, the following holds (for all $\varepsilon \leq \varepsilon_0$):

- $p \in P \subseteq \overline{P} \subset\subset G$.
- t_ε is injective on G.
- $\overline{Q} \subseteq t_\varepsilon(P)$. (4.23)

Assume that α was chosen according to property (E+), i.e. we have in addition:

- There exist $\varepsilon' \leq \varepsilon_0$, $C' > 0$ and $N' \in \mathbb{N}$ such that

$$\inf_{(U,X,V)\in G} |\det(\mathrm{D}t_\varepsilon(U, X, V))| \geq C'\varepsilon^{N'}$$

for all $\varepsilon \leq \varepsilon'$.

Let A and D_1 be open subsets of G such that

$$\overline{P} \subset\subset A \subseteq \overline{A} \subset\subset D_1 \subseteq \overline{D_1} \subset\subset G.$$

Then $p \in A$ and $K_\varepsilon := t_\varepsilon(\overline{A})$ is compact for all $\varepsilon \leq \varepsilon_0$. By property (E+) and $D_1 \subseteq G$, we have

$$\inf_{(U,X,V)\in D_1} |\det(\mathrm{D}t_\varepsilon(U, X, V))| \geq C'\varepsilon^{N'}$$

for all $\varepsilon \leq \varepsilon'$. Hence, Proposition 3.34 applied to $(-a, b) \times \mathbb{R}^n \times \mathbb{R}$, D_1, $(t_\varepsilon)_\varepsilon$, $(t_\varepsilon|_{D_1}{}^{-1})_\varepsilon$, p, $\{p\}$, $\overline{A}$ and K_ε in place of U, W, $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon$, $[(\tilde{x}_\varepsilon)_\varepsilon]$, K', K and K_ε yields the existence of (K_ε, p)-extensions $t_\varepsilon^\diamond$ of $t_\varepsilon|_{D_1}{}^{-1}$ such that $(t_\varepsilon^\diamond)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^{n+2})^{n+2}$. The net $(t_\varepsilon^\diamond)_\varepsilon$ is c-bounded into any (bounded) open subset D of $\mathbb{R}^{n+2}$ that contains the convex hull of $\overline{D_1} \cup \{p\} = \overline{D_1}$. Set $T^\diamond := [(t_\varepsilon^\diamond)_\varepsilon] \in \mathcal{G}[\mathbb{R}^{n+2}, D]$. On the one hand, by (4.23), we have

$$Q \subseteq t_\varepsilon(P) \subseteq t_\varepsilon(A) \subseteq K_\varepsilon$$

and, therefore, $t_\varepsilon^\diamond(Q) = t_\varepsilon|_{D_1}{}^{-1}(Q) \subseteq P \subseteq \overline{P} \subset\subset A$, implying that $(t_\varepsilon^\diamond|_Q)_\varepsilon$ is c-bounded into A. Moreover,

$$t_\varepsilon \circ t_\varepsilon^\diamond|_Q = t_\varepsilon \circ t_\varepsilon^{-1}|_Q = \mathrm{id}_Q,$$

establishing $[A, \mathbb{R}^{n+2}, T^\diamond, Q]$ as a right inverse of T on A. On the other hand, since $t_\varepsilon(A) \subseteq K_\varepsilon$, we have

$$t_\varepsilon^\diamond \circ t_\varepsilon|_A = t_\varepsilon^\diamond|_{K_\varepsilon} \circ t_\varepsilon|_A = t_\varepsilon^{-1}|_{K_\varepsilon} \circ t_\varepsilon|_A = \mathrm{id}_A\,.$$

By the c-boundedness of $(t_\varepsilon)_\varepsilon$, there exists some $K' \subset\subset (-a,b) \times \mathbb{R}^n \times \mathbb{R}$ with $t_\varepsilon(\overline{A}) \subseteq K'$ for sufficiently small ε. Hence, $(t_\varepsilon|_A)_\varepsilon$ is c-bounded into any (bounded) open set B containing K'. It follows that $[A, \mathbb{R}^{n+2}, T^\diamond, B]$ is a left inverse of T on A. Combining these results, we obtain that T is invertible on A with inversion data $[A, \mathbb{R}^{n+2}, T^\diamond, B, Q]$. $\square$

4.20. Remark: Again we comment on sizes and shapes of the sets involved in the proof of the preceding theorem.

(1) Concerning R_λ resp. R, α, G, P and Q see Remark 4.18.

(2) Both A and D_1 are bounded open sets with their (compact) closures nested in between $\overline{P}$ and G, where D_1 and $\overline{A}$ play the roles of W resp. K in Proposition 3.34.

(3) B and D are introduced as supersets of $t_\varepsilon(\overline{A})$ resp. the convex hull of $\overline{D_1}$ and serve as target sets for the c-boundedness.

Finally, we apply Theorem 4.19 to the special case of $T = [(t_\varepsilon)_\varepsilon]$ and t as occurring in our study of pp-waves. Thus, we assume that $n = 2$, $a = b = \infty$ and $(t_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ and s are of the form

$$t_\varepsilon : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} U \\ x^i_\varepsilon(X^k, U) \\ v_\varepsilon(X^k, V, U) \end{pmatrix},$$

$$s_\varepsilon : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} U \\ x^i_\varepsilon(X^k, U) \\ w_\varepsilon(X^k, V, U) \end{pmatrix},$$

$$s : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) = U \\ x^i(X^k, U) = X^i + \frac{1}{2}\partial_i f(X^k)\, U_+ \\ v(X^k, V, U) = V + \frac{1}{4}\sum_{i=1}^{2} \partial_i f(X^k)^2\, U_+. \end{pmatrix},$$

where

$$x^i_\varepsilon(X^k, U) = X^i + \frac{1}{2} \int\limits_{-\varepsilon}^{U} \int\limits_{-\varepsilon}^{s} \partial_i f(x^j_\varepsilon(X^k, r))\, \delta_\varepsilon(r)\, dr\, ds,$$

$$\begin{aligned} v_\varepsilon(X^k,V,U) &= V + \int\limits_{-\varepsilon}^{U} f(x_\varepsilon^j(X^k,s))\,\delta_\varepsilon(s)\,ds \\ &\quad + \int\limits_{-\varepsilon}^{U}\int\limits_{-\varepsilon}^{s}\sum_{i=1}^{2}\partial_i f(x_\varepsilon^j(X^k,r))\,\dot{x}_\varepsilon^i(X^k,r)\,\delta_\varepsilon(r)\,dr\,ds, \\ w_\varepsilon(X^k,V,U) &= V + \int\limits_{-\varepsilon}^{U}\int\limits_{-\varepsilon}^{s}\sum_{i=1}^{2}\partial_i f(x_\varepsilon^j(X^k,r))\,\dot{x}_\varepsilon^i(X^k,r)\,\delta_\varepsilon(r)\,dr\,ds. \end{aligned}$$

Having collected the necessary tools, we can now establish the main result of this section concerning the invertibility of the generalised coordinate transformation T.

4.21. Theorem: *The generalised coordinate transformation $T = [(t_\varepsilon)_\varepsilon]$ is locally invertible (in the sense of Definition 3.28) on some open set Ω containing the half space $(-\infty,0]\times\mathbb{R}^3$.*

Proof: By Proposition 4.8, $(t_\varepsilon)_\varepsilon$ as well as $(s_\varepsilon)_\varepsilon$ possess property (E+). Moreover, $\hat{s}$ is injective on some open set W containing $(-\infty,0]\times\mathbb{R}^2$ by Lemma 4.10. Then, by Theorem 4.19, for every p on the hyperplane $U=0$ there exists an open neighbourhood $A(p)\subseteq\mathbb{R}^4$ such that T is invertible on $A(p)$. Recall that each $A(p)$ contains some set $P=(-\beta,\gamma)\times R\times I$ as discussed in Remark 4.18. In particular, all of $\beta>0$, R and I (both bounded) can be chosen arbitrarily large. Forming the union Ω of a family of $A(p)$ with the corresponding sets P covering the left half space, we obtain that the generalised function T is locally invertible on Ω, constituting an open set containing $(-\infty,0]\times\mathbb{R}^3$. $\square$

Chapter 5

Differential equations in generalised functions

Since J. F. Colombeau introduced his method of embedding $\mathcal{D}'$ into a differential algebra whose product coincides with the pointwise product of smooth functions, different types of differential equations in generalised functions have been studied. There are those describing the geodesics of impulsive gravitational waves (pp-waves) (see Chapter 4 or cf. [Ste98], [Ste99], and, for solutions of the geodesic equations in the full Colombeau algebra, [KS99b]). In [Lig96], J. Ligęza considers linear differential equations, while in [Lig97] and [Lig98] he finds periodic solutions of linear ODEs of first and second order. M. Oberguggenberger and R. Hermann presented several results regarding the (global) solvability of differential equations given by tempered generalised functions (cf. [HO99] and [GKOS01]). In [KOSV04], generalised flows and (globally defined) singular ODEs on differentiable manifolds are studied. However, there exists no local theory of differential equations over the special Colombeau algebra so far. The aim of this chapter is to lay the foundations to such an approach. We will present generalised versions of the Existence and Uniqueness Theorem for ODEs 1.7 (Section 5.1) and Frobenius' Theorem 1.8 (Section 5.2).

5.1 Ordinary differential equations in generalised functions

Let I be an open interval in $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, $F \in \mathcal{G}(I \times U)^n$, $t_0 \in I$ and $\tilde{x}_0 \in \tilde{U}_c$. We are interested in finding solutions u in $\mathcal{G}(J)^n$ of the initial value problem

$$u'(t) = F(t, u(t)), \quad u(t_0) = \tilde{x}_0, \tag{5.1}$$

where $J \subseteq I$ is an interval in $\mathbb{R}$ with $t_0 \in J$. Note that in order to be able to compose F with u the generalised function u has to be c-bounded. Therefore, the requirement for $\tilde{x}_0$ to be compactly supported does not constitute a restriction on possible initial value problems, but stems from the fact that *any* point value of u is a compactly supported generalised point.
We will give sufficient conditions to guarantee a (unique) solution to (5.1). For the proof we need a result of Weissinger. The proof of the following theorem (for Banach spaces) can be found in [Heu89] (page 138f, 12.1).

5.1. Theorem (Weissinger's Fixed Point Theorem): *Let A be a closed subset of a metric space (M,d), $\sum_{k=1}^{\infty}\alpha_k$ a convergent series of positive numbers and $T : A \to A$ a map satisfying*

$$d(T^k u, T^k v) \leq \alpha_k \cdot d(u,v)$$

for all $u, v \in A$ and $k \in \mathbb{N}$. Then T possesses a unique fixed point $u \in A$. This fixed point is the limit of the iterative sequence $(T^k u_0)_{k\in\mathbb{N}}$, where u_0 is an arbitrary initial value in A. Furthermore, the error estimate

$$d(u_k, u) \leq \left(\sum_{i=k}^{\infty}\alpha_i\right) \cdot d(u_0, u_1)$$

holds.

5.2. Theorem: *Let I be an open interval in $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, $\tilde{t}_0$ a near-standard point in $\tilde{I}_c$ with $\tilde{t}_0 \approx t_0 \in I$, $\tilde{x}_0 = [(\tilde{x}_{0\varepsilon})_\varepsilon] \in \tilde{U}_c$ and $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(I \times U)^n$. Let $\varepsilon_0 \in (0,1]$ and L a compact subset of U such that $\tilde{x}_{0\varepsilon} \in L$ for all $\varepsilon \leq \varepsilon_0$. Let $\alpha, \beta > 0$ such that*

$$Q := \overline{B_\alpha(t_0)} \times \overline{L_\beta} \subseteq I \times U,$$

where $L_\beta := L + B_\beta(0)$. If there exists some $a > 0$ such that

$$\sup_{(t,x)\in Q} |F_\varepsilon(t,x)| \leq a \tag{5.2}$$

for all $\varepsilon \leq \varepsilon_0$, then for fixed $h \in \left(0, \min\left(\alpha, \frac{\beta}{a}\right)\right)$ there exists $u \in \mathcal{G}[J, L_\beta]$ that is a solution of the initial value problem

$$u'(t) = F(t, u(t)), \quad u(\tilde{t}_0) = \tilde{x}_0, \tag{5.3}$$

where $J := [t_0 - h, t_0 + h]$.
Furthermore, there exist representatives $(u_\varepsilon)_\varepsilon$, $(\tilde{t}_{0\varepsilon})_\varepsilon$, $(\tilde{x}_{0\varepsilon})_\varepsilon$ of u, $\tilde{t}_0$, $\tilde{x}_0$, respectively, such that

$$u_\varepsilon'(t) = F_\varepsilon(t, u_\varepsilon(t)), \quad u_\varepsilon(t_0) = \tilde{x}_{0\varepsilon},$$

holds for all $t \in J$ and ε sufficiently small.

The solution is unique in $\mathcal{G}[J, L_\beta]$ if, in addition,

$$\sup_{(t,x)\in Q} |\partial_2 F_\varepsilon(t,x)| = O(|\log\varepsilon|) \tag{5.4}$$

holds.

Proof: We consider the differential equation on the level of representatives. For each ε we follow the proof of the classical Existence and Uniqueness Theorem for ODEs as can be found in [Heu89] (page 139ff, 12.2). However, in order to obtain a net of solutions defined on a common interval, we have to keep track of the constants depending on ε. Thus, we give the proof in full detail.

Existence: Let $(\tilde{t}_{0\varepsilon})_\varepsilon$ be a representative of $\tilde{t}_0$. Set $c := \min\left(\alpha, \frac{\beta}{a}\right)$ and choose some $h \in (0,c)$. Let $\varepsilon_1 \le \varepsilon_0$ such that $|\tilde{t}_{0\varepsilon} - t_0| < \frac{1}{2}(c-h)$ for all $\varepsilon \le \varepsilon_1$. Now fix some $\varepsilon \le \varepsilon_1$. Observe that for $t \in J$

$$|\tilde{t}_{0\varepsilon} - t| \le |\tilde{t}_{0\varepsilon} - t_0| + |t_0 - t| \le \left(\frac{1}{2}(c-h) + h\right) = \frac{1}{2}(c+h) < c \tag{5.5}$$

holds. The function u_ε is a solution of the initial value problem

$$u_\varepsilon'(t) = F_\varepsilon(t, u_\varepsilon(t)), \quad u_\varepsilon(\tilde{t}_{0\varepsilon}) = \tilde{x}_{0\varepsilon}, \tag{5.6}$$

if and only if it solves

$$u_\varepsilon(t) = \tilde{x}_{0\varepsilon} + \int_{\tilde{t}_{0\varepsilon}}^{t} F_\varepsilon(s, u_\varepsilon(s))ds. \tag{5.7}$$

The idea of the proof is to find a fixed point (by Weissinger's Fixed Point Theorem 5.1) of the integral operator defined by the right hand side of (5.7). To this end, we set

$$A := \{f \in \mathrm{C}(J, \mathbb{R}^n) \mid \operatorname{im} f \subseteq \overline{L_\beta}\}.$$

A is non-empty and a closed subset of the Banach space $\mathrm{C}(J,\mathbb{R}^n)$. We define $T_\varepsilon : A \to \mathrm{C}(J, \mathbb{R}^n)$ by

$$(T_\varepsilon f)(t) := \tilde{x}_{0\varepsilon} + \int_{\tilde{t}_{0\varepsilon}}^{t} F_\varepsilon(s, f(s))ds.$$

T_ε maps A into A since, by (5.5),

$$\left|\int_{\tilde{t}_{0\varepsilon}}^{t} F_\varepsilon(s,f(s))\,ds\right| \le \left|\int_{\tilde{t}_{0\varepsilon}}^{t} a\,ds\right| \le |\tilde{t}_{0\varepsilon} - t|\cdot a \le \frac{1}{2}(c+h)\cdot a < c\cdot a \le \beta \tag{5.8}$$

and, hence,

$$(T_\varepsilon f)(t) = \tilde{x}_{0\varepsilon} + \int\limits_{\tilde{t}_{0\varepsilon}}^{t} F_\varepsilon(s, f(s))ds \in L + B_\beta(0) \subseteq \overline{L_\beta}$$

for all $t \in J$. By Lemma 3.11 and Remark 3.12, there exists a constant $C_K > 0$ such that for all $(t, x), (t, y) \in \overline{B_\alpha(t_0)} \times \overline{L_\beta}$ the estimate

$$\begin{aligned}|F_\varepsilon(t,x) - F_\varepsilon(t,y)| &\leq C_K \sup_{z \in K}(|F_\varepsilon(t,z)| + |\partial_2 F_\varepsilon(t,z)|) \cdot |x-y| \\ &\leq \underbrace{C_K \sup_{(t,z)\in\overline{B_\alpha(t_0)}\times K} (|F_\varepsilon(t,z)| + |\partial_2 F_\varepsilon(t,z)|)}_{C_\varepsilon :=} \cdot |x-y|\end{aligned}$$

holds, where C_K only depends on K, a compact subset of U with $\overline{L_\beta} \subseteq K^\circ$. By induction, we prove that

$$\big|(T_\varepsilon^k f)(t) - (T_\varepsilon^k g)(t)\big| \leq \frac{|t - \tilde{t}_{0\varepsilon}|^k}{k!}\, C_\varepsilon^k\, \|f - g\|_\infty \tag{5.9}$$

holds for all $t \in J$: For $k = 0$ the inequality is trivially satisfied. Now let us assume that (5.9) holds for some k. Then

$$\begin{aligned}\big|(T_\varepsilon^{k+1} f)(t) - (T_\varepsilon^{k+1} g)(t)\big| &= \big|\big(T_\varepsilon(T_\varepsilon^k f)\big)(t) - \big(T_\varepsilon(T_\varepsilon^k g)\big)(t)\big| \\ &\leq \Bigg|\int\limits_{\tilde{t}_{0\varepsilon}}^{t} \big|F_\varepsilon\big(s, (T_\varepsilon^k f)(s)\big) - F_\varepsilon\big(s, (T_\varepsilon^k g)(s)\big)\big| ds\Bigg| \\ &\leq C_\varepsilon \Bigg|\int\limits_{\tilde{t}_{0\varepsilon}}^{t} \frac{|s - \tilde{t}_{0\varepsilon}|^k}{k!} \cdot C_\varepsilon^k \cdot \|f-g\|_\infty ds\Bigg| \\ &\leq \frac{|t - \tilde{t}_{0\varepsilon}|^{k+1}}{(k+1)!}\, C_\varepsilon^{k+1}\, \|f - g\|_\infty,\end{aligned}$$

and, thus, (5.9) holds for all $k \in \mathbb{N}$. From (5.9), it follows immediately that

$$\|T_\varepsilon^k f - T_\varepsilon^k g\|_\infty \leq \frac{(c\, C_\varepsilon)^k}{k!} \|f - g\|_\infty.$$

Since $\sum_{k=0}^{\infty} \frac{(c\,C_\varepsilon)^k}{k!} = e^{c\,C_\varepsilon} < \infty$, it follows from Weissinger's Fixed Point Theorem 5.1 that for every $\varepsilon \leq \varepsilon_0$ there exists a unique element $u_\varepsilon \in A$ which satisfies $T_\varepsilon u_\varepsilon = u_\varepsilon$, and which is therefore a solution of (5.6).

We still have to show the moderateness and c-boundedness of $(u_\varepsilon)_\varepsilon$: By (5.8), the image of u_ε is contained in $L + \overline{B_{\frac{a}{2}(c+h)}(0)}$ for all $\varepsilon \leq \varepsilon_0$ and, hence, by our choice of h, the net $(u_\varepsilon)_\varepsilon$ is c-bounded into L_β. By an even more straightforward estimate using (5.2), the first derivative of $(u_\varepsilon)_\varepsilon$ is also uniformly bounded. By the chain rule and the uniform boundedness of $(u_\varepsilon)_\varepsilon$ and its derivative, we have

$$|u_\varepsilon''(t)| \leq |\partial_1 F_\varepsilon(t, u_\varepsilon(t))| + |\partial_2 F_\varepsilon(t, u_\varepsilon(t))| \cdot |{u_\varepsilon}'(t)| \leq C\varepsilon^{-N}$$

for $C > 0$ and some fixed $N \in \mathbb{N}$. The higher-order derivatives of u_ε are now estimated inductively by differentiating the equation

$$u_\varepsilon''(t) = \partial_1 F_\varepsilon(t, u_\varepsilon(t)) + \partial_2 F_\varepsilon(t, u_\varepsilon(t)) \cdot {u_\varepsilon}'(t).$$

Uniqueness: Let $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[J, L_\beta]$ be another solution of (5.3) and $\tilde{y}_{0\varepsilon} := v_\varepsilon(\tilde{t}_{0\varepsilon})$. Then $v_\varepsilon'(t) = F_\varepsilon(t, v_\varepsilon(t)) + n_\varepsilon(t)$ for $(n_\varepsilon)_\varepsilon \in \mathcal{N}(J)^n$ and $\tilde{y}_{0\varepsilon} = \tilde{x}_{0\varepsilon} + \tilde{n}_\varepsilon$ for $(\tilde{n}_\varepsilon)_\varepsilon \in \mathcal{N}^n$. Since J is compact and both $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ are c-bounded into L_β, there exists a compact subset K of L_β such that $u_\varepsilon(J) \subseteq K$ and $v_\varepsilon(J) \subseteq K$ for sufficiently small ε. Observe that, by Lemma 3.11 and Remark 3.12, there exist $K' \subset\subset L_\beta$ with $K \subseteq (K')^\circ$ and a constant $C_{K'} > 0$ such that for all $(t, x), (t, y) \in J \times K$ the estimate

$$|F_\varepsilon(t, x) - F_\varepsilon(t, y)| \leq C_{K'} \sup_{(t,z) \in J \times K'} (\underbrace{|F_\varepsilon(t, z)|}_{\leq a} + \underbrace{|\partial_2 F_\varepsilon(t, z)|}_{\leq C_1 |\log \varepsilon|}) \cdot |x - y|$$

holds, where $C_1 > 0$ and $C_{K'}$ only depends on K'. Therefore, for $t \in J$ it follows that

$$\begin{aligned}
&|v_\varepsilon(t) - u_\varepsilon(t)| \leq \\
&\quad \leq |\tilde{y}_{0\varepsilon} - \tilde{x}_{0\varepsilon}| + \left| \int_{\tilde{t}_{0\varepsilon}}^{t} (|F_\varepsilon(s, \underbrace{v_\varepsilon(s)}_{\in K}) - F_\varepsilon(s, \underbrace{u_\varepsilon(s)}_{\in K})| + |n_\varepsilon(s)|) ds \right| \\
&\quad = |\tilde{n}_\varepsilon| + \left| \int_{\tilde{t}_{0\varepsilon}}^{t} |n_\varepsilon(s)| ds \right| + C_{K_2}(a + C_1 |\log \varepsilon|) \cdot \left| \int_{\tilde{t}_{0\varepsilon}}^{t} |v_\varepsilon(s) - u_\varepsilon(s)|\, ds \right| \\
&\quad \leq C_2\, \varepsilon^m + (C_3 + C_4 |\log \varepsilon|) \cdot \left| \int_{\tilde{t}_{0\varepsilon}}^{t} |v_\varepsilon(s) - u_\varepsilon(s)| ds \right|
\end{aligned}$$

for suitable constants $C_2, C_3, C_4 > 0$ and arbitrary $m \in \mathbb{N}$. By Gronwall's Lemma, we obtain

$$\sup_{t \in J} |v_\varepsilon(t) - u_\varepsilon(t)| \leq C_2\, \varepsilon^m \cdot e^{(C_3 + C_4 |\log \varepsilon|) \cdot |\int_{\tilde{t}_{0\varepsilon}}^{t} 1\, ds|} \leq C \varepsilon^{m - c C_4}$$

for some constant $C > 0$. This concludes the proof of the theorem. $\square$

5.3. Remark: Let W be an open subset of U containing $\overline{L_\beta}$. Inspecting the proof of uniqueness, we note that if for all compact subsets K of W

$$\sup_{(t,x)\in\overline{B_\alpha(t_0)}\times K} |\partial_2 F_\varepsilon(t,x)| = O(|\log\varepsilon|)$$

holds, then the solution constructed in the proof is unique even in $\mathcal{G}[J,W]$.

What happens if the generalised function F does not have property (5.2)? We consider three examples.

5.4. Example: Let $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}\times\mathbb{R})$ be given by the representative $F_\varepsilon(t,x) := \frac{1}{\varepsilon}\big(2 - \frac{1}{1+x^2}\big)$, $t_0 = 0$ and $x_0 = 0$. Since $x \mapsto 2 - \frac{1}{1+x^2}$ is (globally) bounded, we have

$$\sup_{x\in[-\beta,\beta]} |F_\varepsilon(t,x)| = \frac{1}{\varepsilon}\left(2 - \frac{1}{1+x^2}\right) \to \infty \quad (\varepsilon \to 0)$$

for any $\beta > 0$, i.e. F fails to satisfy condition (5.2) on any neighbourhood of (t_0, x_0). Nevertheless, there exists a unique global solution for every ε: Integrating the differential equation $u_\varepsilon'(t) = F_\varepsilon(t, u_\varepsilon(t))$ and taking into account the initial condition $u_\varepsilon(0) = 0$, we obtain

$$\underbrace{\frac{x}{2} + \frac{1}{2\sqrt{2}}\arctan(\sqrt{2}\,x)}_{f(x):=} = \frac{1}{\varepsilon}\,t.$$

The function f is independent of ε, strictly monotonic increasing and maps $\mathbb{R}$ onto $\mathbb{R}$. Therefore, f is smoothly invertible and we denote the inverse function by f^{-1}. Since f^{-1} is a slowly increasing function, the composition with $t \mapsto \frac{1}{\varepsilon}\,t$, by Proposition 2.19, is well-defined and yields a moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R})$ where $u_\varepsilon(t) := f^{-1}(\frac{1}{\varepsilon}\,t)$. However, f^{-1} being unbounded, $(u_\varepsilon)_\varepsilon$ is not c-bounded. Hence, u_ε solves the differential equation for every ε but the generalised function $[(u_\varepsilon)_\varepsilon]$ is not a solution of the generalised initial value problem.

5.5. Example: Consider $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}\times\mathbb{R})$ that has $F_\varepsilon(t,x) := \frac{x}{\varepsilon}$ as a representative, $t_0 = 0$ and $x_0 = 1$. F does not satisfy condition (5.2) since

$$\sup_{x\in[-\beta,\beta]} |F_\varepsilon(t,x)| = \frac{x}{\varepsilon} \to \infty \quad (\varepsilon \to 0)$$

for any $\beta > 0$. For each ε, there exists a unique (even global) solution $u_\varepsilon(t) = e^{\frac{t}{\varepsilon}}$. However, $(u_\varepsilon)_\varepsilon$ is not moderate on any neighbourhood of 0.

5.6. Example: Let $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}\big(\mathbb{R} \times (\mathbb{R}\backslash\{-1\})\big)$ be defined by the representative $F_\varepsilon(t,x) := -\frac{t}{x+1} \cdot g(\varepsilon)$ where $g : (0,1] \to \mathbb{R}$ is a smooth map satisfying $g(\varepsilon) \to \infty$ for $\varepsilon \to 0$. Let $t_0 = 0$ and $x_0 = 0$. Then

$$\sup_{(t,x)\in[-\alpha,\alpha]\times[-\beta,\beta]} |F_\varepsilon(t,x)| = \frac{\alpha}{1-\beta} \cdot g(\varepsilon) \to \infty \quad (\varepsilon \to 0)$$

for any $\alpha > 0$ and $\beta \in (0,1)$. For every ε we obtain (unique) solutions

$$u_\varepsilon(t) = \sqrt{1 - g(\varepsilon)\, t^2} - 1$$

that are defined, at most, on the open interval $\big(-\frac{1}{\sqrt{g(\varepsilon)}}, \frac{1}{\sqrt{g(\varepsilon)}}\big)$. Hence, there is not even a common domain on which to check the net $(u_\varepsilon)_\varepsilon$ for moderateness.

In the last example, F failing to satisfy condition (5.2) leads to shrinking of the solutions' domains as $\varepsilon \to 0$. Note that this result is not a consequence of the rate of growth of $|F_\varepsilon(t,x)|$ on any compact set; rather the only factor that matters is that $|F_\varepsilon(t,x)|$ does increase infinitely (as $\varepsilon \to 0$). So, Example 5.6 suggests that a relaxation of condition (5.2) (e.g. ε-dependence of the bound) without more detailed knowledge of the structure of F is not possible. Unfortunately, this means that e.g. the (in $\mathcal{G}[\mathbb{R},\mathbb{R}]$ solvable) initial value problem

$$u'(t) = (\iota\delta)(t), \quad u(0) = 0, \tag{5.10}$$

is not covered by Theorem 5.2. Actually, specific types of ODEs containing δ-like objects have already been treated (e.g. see Chapter 4 or cf. [GKOS01], Sections 1.5 and 5.3, and [Ste98]). The proof of existence always relies on the particular characteristics of the ODE concerned—quite in contrast to a general F being given. Nevertheless, Theorem 5.2 can handle jumps as the following example will show.

5.7. Example: Let I be an open interval in $\mathbb{R}$ and U an open subset of $\mathbb{R}^n$. Consider the initial value problem

$$u'(t) = f(t,u(t)) \cdot (\iota H)(t) + g(t,u(t)), \quad u(t_0) = x_0, \tag{5.11}$$

where ιH denotes the embedding of the Heaviside function H into the Colombeau algebra, the mappings f and g are in $\mathrm{C}^\infty(I \times U, \mathbb{R}^n)$ and $t_0 \in I$, $x_0 \in U$. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ be a mollifier. Then, for $H_\varepsilon(t) = \int_{-\infty}^t \rho_\varepsilon(s)\,ds$ (cf. Example 2.16),

$$|H_\varepsilon(t)| \leq \int\limits_{-\infty}^{t} |\rho_\varepsilon(s)|\,ds = \int\limits_{-\infty}^{t} \frac{1}{\varepsilon^n} \Big|\rho\Big(\frac{s}{\varepsilon}\Big)\Big|\,ds = \int\limits_{-\infty}^{\frac{t}{\varepsilon}} |\rho(s)|\,ds \leq \|\rho\|_{L^1(\mathbb{R}^n)}$$

holds for all t and for all ε. Fix some $\alpha > 0$ such that $B_\alpha(t_0)$ is still contained in I and choose an open subset W of U with $x_0 \in W \subseteq \overline{W} \subset\subset U$. f and g being continuous, there exist constants $a_1, a_2 > 0$ such that

$$\sup_{(t,x)\in\overline{B_\alpha(t_0)}\times\overline{W}} |f(t,x)\cdot H_\varepsilon(t) + g(t,x)| \leq a_1\|\rho\|_{L^1(\mathbb{R}^n)} + a_2$$

for all ε. Hence, the initial value problem (5.11) possesses a solution u in $\mathcal{G}[J,W]$ where $J := [t_0 - h, t_0 + h]$ and $h < \min\left(\alpha, \frac{\operatorname{dist}(x_0, W^c)}{a_1\|\rho\|_{L^1(\mathbb{R}^n)} + a_2}\right)$. Since

$$\sup_{(t,x)\in\overline{B_\alpha(t_0)}\times\overline{W}} |\partial_2 f(t,x)\cdot H_\varepsilon(t) + \partial_2 g(t,x)|$$

is also uniformly bounded with respect to ε, the solution is unique in $\mathcal{G}[J,W]$.

Next, we turn our attention to generalised ODEs which are dependent on a parameter. Taking into account that we aim at proving a generalised Frobenius theorem using a generalised ODE theorem, we want the solution to be $\mathcal{G}$-dependent on the parameter. It turns out that if conditions (5.2) and (5.4) in Theorem 5.2 are only slightly modified to include the parameter, they are sufficient to guarantee the desired result.

5.8. Theorem: *Let I be an open interval in $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, P an open subset of $\mathbb{R}^l$, $\tilde{t}_0$ a near-standard point in $\tilde{I}_c$ with $\tilde{t}_0 \approx t_0 \in I$, $\tilde{x}_0 = [(\tilde{x}_{0\varepsilon})_\varepsilon] \in \tilde{U}_c$ and $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(I\times U\times P)^n$. Let $\varepsilon_0 \in (0,1]$ and L a compact subset of U such that $\tilde{x}_{0\varepsilon} \in L$ for all $\varepsilon \leq \varepsilon_0$. Let $\alpha, \beta > 0$ such that*

$$Q := \overline{B_\alpha(t_0)} \times \overline{L_\beta} \subseteq I \times U,$$

where $L_\beta := L + B_\beta(0)$. If there exists some $a > 0$ such that

$$\sup_{(t,x,p)\in Q\times P} |F_\varepsilon(t,x,p)| \leq a$$

for all $\varepsilon \leq \varepsilon_0$ and if for all compact subsets K of P

$$\sup_{(t,x,p)\in Q\times K} |\partial_2 F_\varepsilon(t,x,p)| = O(|\log\varepsilon|) \tag{5.12}$$

holds, then for fixed $h \in \left(0, \min\left(\alpha, \frac{\beta}{a}\right)\right)$ there exists $u \in \mathcal{G}[P\times J, L_\beta]$ such that for all $\tilde{p} \in \tilde{P}_c$ the map $u(\tilde{p}, .) \in \mathcal{G}[J, L_\beta]$ is a solution of the initial value problem

$$u'(t) = F(t, u(t), \tilde{p}), \quad u(\tilde{t}_0) = \tilde{x}_0,$$

where $J := [t_0 - h, t_0 + h]$. The solution u is unique in $\mathcal{G}[P\times J, L_\beta]$.

Furthermore, there exist represenatives $(u_\varepsilon)_\varepsilon$, $(\tilde{t}_{0\varepsilon})_\varepsilon$, $(\tilde{x}_{0\varepsilon})_\varepsilon$ of u, $\tilde{t}_0$, $\tilde{x}_0$, respectively, such that

$$u_\varepsilon'(p,t) = F_\varepsilon(t, u_\varepsilon(p,t), p), \quad u_\varepsilon(p, \tilde{t}_{0\varepsilon}) = \tilde{x}_{0\varepsilon},$$

holds for all $(p,t) \in P\times J$ and ε sufficiently small.

Proof: *Existence*: Let $(\tilde{t}_{0\varepsilon})_\varepsilon$ be a representative of $\tilde{t}_0$. Set $c := \min\left(\alpha, \frac{\beta}{a}\right)$ and choose some $h \in (0, c)$. Let $\varepsilon_1 \leq \varepsilon_0$ such that $|\tilde{t}_{0\varepsilon} - t_0| < \frac{1}{2}(c - h)$ for all $\varepsilon \leq \varepsilon_1$. From now on, we always let $\varepsilon \leq \varepsilon_1$. As before, we have $|t - \tilde{t}_{0\varepsilon}| < c$. Since the upper bound a is independent of p, we obtain—as in the proof of Theorem 5.2—for all $\tilde{p} = [(\tilde{p}_\varepsilon)_\varepsilon] \in \tilde{P}_c$ nets of classical solutions $u_\varepsilon(\tilde{p}_\varepsilon, .) : J \to L_\beta$ of the initial value problem

$$u_\varepsilon'(t) = F_\varepsilon(t, u_\varepsilon(t), \tilde{p}_\varepsilon), \quad u_\varepsilon(\tilde{t}_{0\varepsilon}) = \tilde{x}_{0\varepsilon}. \tag{5.13}$$

By the classical Existence and Uniqueness Theorem for ODEs 1.7, for all ε the mapping $(p, t) \mapsto u_\varepsilon(p, t)$ is C^∞ since F_ε is smooth.
The moderateness of $(u_\varepsilon)_\varepsilon$ will be shown in three steps: First we consider derivatives with respect to t, then only derivatives with respect to p and, finally, mixed derivatives.
The $\mathcal{E}_M$-estimates for $u_\varepsilon(p, t)$, $\partial_2 u_\varepsilon(p, t)$ and all its higher-order derivatives with respect to t are obtained in the same way as in the proof of Theorem 5.2.
Next, we consider the derivatives with respect to p. The initial value problem (5.13) with $\tilde{p}_\varepsilon = p$ is equivalent to the integral equation

$$u_\varepsilon(p, t) = \tilde{x}_{0\varepsilon} + \int\limits_{\tilde{t}_{0\varepsilon}}^{t} F_\varepsilon\big(s, u_\varepsilon(p, s), p\big)\, ds. \tag{5.14}$$

Differentiating equation (5.14) with respect to p yields

$$\partial_1 u_\varepsilon(p, t) = \int\limits_{\tilde{t}_{0\varepsilon}}^{t} \Big(\partial_2 F_\varepsilon\big(s, u_\varepsilon(p, s), p\big) \cdot \partial_1 u_\varepsilon(p, s) + \partial_3 F_\varepsilon\big(s, u_\varepsilon(p, s), p\big)\Big)\, ds. \tag{5.15}$$

Let $K_1 \times K_2 \subset\subset P \times J$ and $(p, t) \in K_1 \times K_2$. Since $u_\varepsilon(p, .)$ maps into the compact set $\overline{L_\beta}$ for all $p \in P$ and by the additional assumption on $(\partial_2 F_\varepsilon)_\varepsilon$, we obtain

$$\begin{aligned} &|\partial_1 u_\varepsilon(p, t)| \leq \\ &\quad \leq \left| \int\limits_{\tilde{t}_{0\varepsilon}}^{t} \partial_3 F_\varepsilon\big(s, u_\varepsilon(p, s), p\big) ds \right| + \left| \int\limits_{\tilde{t}_{0\varepsilon}}^{t} \left|\partial_2 F_\varepsilon\big(s, u_\varepsilon(p, s), p\big)\right| \cdot \left|\partial_1 u_\varepsilon(p, s)\right| ds \right| \\ &\quad \leq c\, C_1 \varepsilon^{-N_1} + \left| \int\limits_{\tilde{t}_{0\varepsilon}}^{t} C_2 |\log \varepsilon| \cdot |\partial_1 u_\varepsilon(p, s)| ds \right| \end{aligned}$$

for constants $C_1, C_2 > 0$ and some fixed $N \in \mathbb{N}$. By Gronwall's Lemma, it follows that

$$|\partial_1 u_\varepsilon(p,t)| \leq c\, C_1 \varepsilon^{-N_1} \cdot e^{|\int_{\tilde{t}_{0\varepsilon}}^{t} C_2 |\log\varepsilon| ds|} \leq (c\, C_1)\, \varepsilon^{-(N_1 + c\, C_2)}.$$

Differentiating (5.15) $i-1$ times with respect to p ($i \in \mathbb{N}$) gives an integral formula for $\partial_1^i u_\varepsilon(p,t)$. Observe that in this formula $\partial_1^i u_\varepsilon(p,t)$ itself appears on the right hand side only once, namely with $\partial_2 F_\varepsilon(s, u_\varepsilon(p,s), p)$ as coefficient, and that the remaining terms contain only ∂_1-derivatives of u_ε of order less than i. Thus, we may estimate the higher-order derivatives with respect to p inductively by differentiating equation (5.15) and applying Gronwall's Lemma.
Finally, it only remains to show that the $\mathcal{E}_M$-estimates are also satisfied for the mixed derivatives. For arbitrary $i \in \mathbb{N}$ we have

$$\partial_1^i \partial_2\, u_\varepsilon(p,t) = \frac{\partial^i}{\partial p^i}\frac{\partial}{\partial t}\left(\tilde{x}_{0\varepsilon} + \int\limits_{\tilde{t}_{0\varepsilon}}^{t} F_\varepsilon\big(s, u_\varepsilon(p,s), p\big)ds\right) = \frac{\partial^i}{\partial p^i} F_\varepsilon\big(t, u_\varepsilon(p,t), p\big). \tag{5.16}$$

By carrying out the i-fold differentiation on the right hand side of equation (5.16), we obtain a polynomial expression in $\partial_2^k F_\varepsilon\big(t, u_\varepsilon(p,t), p\big)$, $\partial_3^k F_\varepsilon\big(t, u_\varepsilon(p,t), p\big)$ and $\partial_1^k u_\varepsilon(p,t)$ for $1 \leq k \leq i$ all of which satisfy the $\mathcal{E}_M$-estimates. The estimates for $\partial_1^i \partial_2^j\, u_\varepsilon(p,t)$ with $j \geq 2$ are now obtained inductively by differentiating equation (5.16) with respect to t.

Uniqueness: By Proposition 2.30, it suffices to show that for every near-standard point $\tilde{p} \in \tilde{P}_c$ the solution $u(\tilde{p}, \,.\,)$ is unique in $\mathcal{G}[J, L_\beta]$. For a fixed near-standard point $\tilde{p} = [(\tilde{p}_\varepsilon)_\varepsilon] \in \tilde{P}_c$, condition (5.12) implies the condition for uniqueness (5.4) in Theorem 5.2 with respect to $(F_\varepsilon(\,.\,,\,.\,,\tilde{p}_\varepsilon))_\varepsilon$. Therefore, $u(\tilde{p}, \,.\,)$ is unique in $\mathcal{G}[J, L_\beta]$ for all near-standard points $\tilde{p} \in \tilde{P}_c$. $\square$

5.9. Remark: Again, let W be an open subset of U containing $\overline{L_\beta}$. As before, the solution u is unique even in $\mathcal{G}[P \times J, W]$ if for all compact subsets K_1 of W and K_2 of P

$$\sup_{(t,x,p) \in \overline{B_\alpha(t_0)} \times K_1 \times K_2} |\partial_2 F_\varepsilon(t,x,p)| = O(|\log\varepsilon|)$$

holds.

If we restrict the generalised point values in the initial condition to near-standard points, we can also prove $\mathcal{G}$-dependence of the solution on the initial value.

5.10. Theorem: *Let I be an open interval in $\mathbb{R}$, U an open subset of $\mathbb{R}^n$, P an open subset of $\mathbb{R}^l$, $\tilde{t}_0$ a near-standard point in $\tilde{I}_c$ with $\tilde{t}_0 \approx t_0 \in I$, $\tilde{x}_0$ a near-standard point in $\tilde{U}_c$ with $\tilde{x}_0 \approx x_0 \in U$ and $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(I \times U \times P)^n$. Let $\alpha, \beta > 0$ such that*

$$Q := \overline{B_\alpha(t_0)} \times \overline{B_\beta(x_0)} \subseteq I \times U.$$

If there exist $a > 0$ and $\varepsilon_0 \in (0,1]$ such that

$$\sup_{(t,x,p) \in Q \times P} |F_\varepsilon(t,x,p)| \leq a \tag{5.17}$$

for all $\varepsilon \leq \varepsilon_0$ and if for all compact subsets K of P

$$\sup_{(t,x,p) \in Q \times K} |\partial_2 F_\varepsilon(t,x,p)| = O(|\log \varepsilon|)$$

holds, then for fixed $h \in \left(0, \min\left(\alpha, \frac{\beta}{a}\right)\right)$ there exist neighbourhoods J_1 of t_0 in $J := [t_0 - h, t_0 + h]$ and U_1 of x_0 in U and a generalised function $u \in \mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)]$, where $\gamma \in (0, \beta)$ with $\beta - \gamma > 0$ sufficiently small, such that for all $(\tilde{t}_1, \tilde{x}_1, \tilde{p}) \in \tilde{J}_{1c} \times \tilde{U}_{1c} \times \tilde{P}_c$ the map $u(\tilde{t}_1, \tilde{x}_1, \tilde{p}, .) \in \mathcal{G}[J, B_\gamma(x_0)]$ is a solution of the initial value problem

$$u'(t) = F(t, u(t), \tilde{p}), \quad u(\tilde{t}_1) = \tilde{x}_1. \tag{5.18}$$

The solution u is unique in $\mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)]$.

Furthermore, there exists a represenative $(u_\varepsilon)_\varepsilon$ of u such that

$$u'_\varepsilon(t_1, x_1, p, t) = F_\varepsilon(t, u_\varepsilon(t_1, x_1, p, t), p), \quad u_\varepsilon(t_1, x_1, p, t_1) = x_1,$$

holds for all $(t_1, x_1, p, t) \in J_1 \times U_1 \times P \times J$ and ε sufficiently small.

Proof: *Existence*: The basic strategy of the proof is to consider $(\tilde{t}_0, \tilde{x}_0)$ as part of the parameter and apply Theorem 5.8. However, we will have to deal with several technical details.

Let $(\tilde{t}_{0\varepsilon})_\varepsilon$ and $(\tilde{x}_{0\varepsilon})_\varepsilon$ be representatives of $\tilde{t}_0$ resp. $\tilde{x}_0$. From now on, we always let $\varepsilon \leq \varepsilon_0$. Let $\lambda \in (0,1)$ and set

$$\hat{I} := B_{\lambda\alpha}(0), \quad I_1 := B_{(1-\lambda)\alpha}(t_0).$$

Choose $\mu \in \left(0, \frac{\beta}{3}\right)$, set $\gamma := \beta - 2\mu$ and define

$$\hat{U} := B_{\gamma+\mu}(0), \quad U_1 := B_\mu(x_0).$$

Then

$$\hat{I} + I_1 = B_\alpha(t_0) \subseteq I \quad \text{and} \quad \hat{U} + U_1 = B_\beta(x_0) \subseteq U$$

hold. Hence, we may define $G_\varepsilon : \hat{I} \times \hat{U} \times (I_1 \times U_1 \times P) \to \mathbb{R}^n$ by

$$G_\varepsilon(t, x, (t_1, x_1, p)) := F_\varepsilon(t + t_1, x + x_1, p).$$

Obviously, $(G_\varepsilon)_\varepsilon$ is moderate and, therefore, $G := [(G_\varepsilon)_\varepsilon]$ is in $\mathcal{G}(\hat{I} \times \hat{U} \times (I_1 \times U_1 \times P))^n$. Now let $\delta \in (0, \lambda\alpha)$ and $\eta \in (0, \gamma - \mu)$. By assumption (5.17) on $(F_\varepsilon)_\varepsilon$, we obtain

$$\begin{aligned}
\sup_{(t,x,(t_1,x_1,p))\in D} |G_\varepsilon(t, x, (t_1, x_1, p))| &= \sup_{(t,x,(t_1,x_1,p))\in D} |F_\varepsilon(t + t_1, x + x_1, p)| \\
&\le \sup_{(t,x,p)\in \overline{B_\alpha(t_0)}\times\overline{B_\gamma(x_0)}\times P} |F_\varepsilon(t, x, p)| \\
&\le a,
\end{aligned}$$

where $D := \overline{B_\delta(0)} \times \overline{B_\eta(0)} \times (I_1 \times U_1 \times P) \subseteq \hat{I} \times \hat{U} \times (I_1 \times U_1 \times P)$. Since

$$\partial_2 G_\varepsilon(t, x, (t_1, x_1, p)) = \frac{\partial}{\partial x} F_\varepsilon(t + t_1, x + x_1, p) = \partial_2 F_\varepsilon(t + t_1, x + x_1, p),$$

it follows immediately that for all $K \subset\subset I_1 \times U_1 \times P$

$$\sup_{(t,x,(t_1,x_1,p))\in\overline{B_\delta(0)}\times\overline{B_\eta(0)}\times K} |\partial_2 G_\varepsilon(t, x, (t_1, x_1, p))| = O(|\log\varepsilon|). \tag{5.19}$$

By Theorem 5.8, there exists $v \in \mathcal{G}[(I_1 \times U_1 \times P) \times \hat{J}, B_\eta(0)]$ such that for all $(\tilde{t}_1, \tilde{x}_1, \tilde{p}) \in \tilde{I}_{1c} \times \tilde{U}_{1c} \times \tilde{P}_c$ the map $v(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \,.\,) \in \mathcal{G}[\hat{J}, B_\eta(0)]$ is a solution of the initial value problem

$$v'(t) = G(t, v(t), (\tilde{t}_0, \tilde{x}_0, \tilde{p})), \quad v(0) = 0, \tag{5.20}$$

where $\hat{h} < \min\left(\delta, \frac{\eta}{a}\right)$ and $\hat{J} := [-\hat{h}, \hat{h}]$. Note that, since $B_\gamma(0) + B_{2\mu}(x_0) = B_\beta(x_0)$, the estimate (5.19) still holds if $B_\eta(0)$ is replaced by $B_{\gamma+\mu}(0)$. Therefore, it follows from Remark 5.9 that the solution v is unique in $\mathcal{G}[(I_1 \times U_1 \times P) \times \hat{J}, B_{\gamma+\mu}(0)]$ (we will need that in the proof of uniqueness). Let $(v_\varepsilon)_\varepsilon$ be the representative of v that satisfies

$$v_\varepsilon'(t_1, x_1, p, t) = G_\varepsilon(t, v_\varepsilon(t_1, x_1, p, t), (t_1, x_1, p)), \quad v(0) = 0,$$

for all $(t_1, x_1, p, t) \in I_1 \times U_1 \times P \times \hat{J}$. Let $\sigma \in \left[\frac{1}{2}, 1\right)$, $h := \sigma\hat{h}$ and $h_1 := \min(h, (1-\sigma)\hat{h}, (1-\lambda)\alpha)$. Set $J := [t_0 - h, t_0 + h]$ and $J_1 := (t_0 - h_1, t_0 + h_1)$. Then $J_1 \subseteq J \subseteq \hat{J}$. We now define $u_\varepsilon : J_1 \times U_1 \times P \times J \to \mathbb{R}^n$ by

$$u_\varepsilon(t_1, x_1, p, t) := v_\varepsilon(t_1, x_1, p, t - t_1) + x_1.$$

The map u_ε is well-defined since $J_1 \subseteq I_1$ by the choice of h_1 and

$$|t - t_1| \le |t - t_0| + |t_0 - t_1| \le h + h_1 \le \sigma\hat{h} + (1 - \sigma)\hat{h} = \hat{h}. \tag{5.21}$$

The moderateness of $(u_\varepsilon)_\varepsilon$ is an immediate consequence of the moderateness of $(v_\varepsilon)_\varepsilon$. Moreover, since $t - t_1 \in \hat{J}$ by (5.21) for all $t \in J$, $t_1 \in J_1$ and $x_1 - x_0 \in B_\mu(0)$ for all $x_1 \in U_1$, it follows that

$$\begin{aligned} u_\varepsilon(t_1, x_1, p, J) &\subseteq v_\varepsilon(t_1, x_1, p, \hat{J}) + x_1 \\ &\subseteq B_\eta(0) + x_1 \\ &\subseteq B_\eta(x_0) - x_0 + x_1 \\ &\subseteq \overline{B_\eta(x_0) + B_\mu(0)} \\ &\subseteq B_\gamma(x_0) \end{aligned}$$

for all $(t_1, x_1, p) \in J_1 \times U_1 \times P$, i.e. u_ε is c-bounded from $J_1 \times U_1 \times P \times J$ into $B_\gamma(x_0)$. Therefore, $u := [(u_\varepsilon)_\varepsilon]$ is an element of $\mathcal{G}[J_1 \times U_1 \times P \times J, B_\gamma(x_0)]$. Furthermore, the function $u_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, \,.\,)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} u_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, t) &= \frac{\partial}{\partial t}\Big(v_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, t - \tilde{t}_{1\varepsilon}) + \tilde{x}_{1\varepsilon}\Big) \\ &= \frac{\partial}{\partial t}\, v_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, t - \tilde{t}_{1\varepsilon}) \\ &= G_\varepsilon(t - \tilde{t}_{1\varepsilon}, v_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, t - \tilde{t}_{1\varepsilon}), (\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon)) \\ &= F_\varepsilon(t, v_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, t - \tilde{t}_{1\varepsilon}) + \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon) \\ &= F_\varepsilon(t, u_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, t), \tilde{p}_\varepsilon) \end{aligned}$$

and

$$u_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, \tilde{t}_{1\varepsilon}) = v_\varepsilon(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}, \tilde{p}_\varepsilon, 0) + \tilde{x}_{1\varepsilon} = \tilde{x}_{1\varepsilon}$$

for all $(\tilde{t}_1, \tilde{x}_1, \tilde{p}) = ([(\tilde{t}_{1\varepsilon})_\varepsilon], [(\tilde{x}_{1\varepsilon})_\varepsilon], [(\tilde{p}_\varepsilon)_\varepsilon]) \in \tilde{J}_{1c} \times \tilde{U}_{1c} \times \tilde{P}_c$ and $t \in J$. Thus, $u(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \,.\,)$ is indeed a solution of the initial value problem (5.18).
Note that for any $h \in \Big(0, \min\big(\alpha, \frac{\beta}{a}\big)\Big)$ the constants λ, μ, δ, η, $\hat{h}$ and σ can be chosen within their required bounds such that all the necessary inequalities in the construction of $(u_\varepsilon)_\varepsilon$ are satisfied.

Uniqueness: By Proposition 2.30, it suffices to show that for every near-standard point $(\tilde{t}_1, \tilde{x}_1, \tilde{p}) \in \tilde{J}_{1c} \times \tilde{U}_{1c} \times \tilde{P}_c$ the solution $u(\tilde{t}_1, \tilde{x}_1, \tilde{p}, \,.\,)$ is unique in $\mathcal{G}[J, B_\gamma(x_0)]$. Let $\tilde{p} \in \tilde{P}_c$ and let $(\tilde{t}_1, \tilde{x}_1) = ([(\tilde{t}_{1\varepsilon})_\varepsilon], [(\tilde{x}_{1\varepsilon})_\varepsilon])$ be near-standard in $\tilde{J}_{1c} \times \tilde{U}_{1c}$ with $(\tilde{t}_{1\varepsilon}, \tilde{x}_{1\varepsilon}) \to (t_1, x_1) \in J_1 \times U_1$ for $\varepsilon \to 0$. Assume that $w(\tilde{t}_1, \tilde{x}_1, \tilde{p}) \in \mathcal{G}[J, B_\gamma(x_0)]$ is another solution of (5.18). For brevity's sake we simply write u resp. w in place of $u(\tilde{t}_1, \tilde{x}_1, \tilde{p})$ resp. $w(\tilde{t}_1, \tilde{x}_1, \tilde{p})$.
We will show that $w|_{(t_0-a,t_0+a)} = u|_{(t_0-a,t_0+a)}$ holds for any $a \in (0, h)$. Since $\mathcal{G}$ is a sheaf, the equality of w and u also holds on J°.Then, by the continuity of representatives, w and u are also equal on J.
Now, let $a \in (0, h)$ and set $\tau := \frac{1}{2}(a + h)$. Define $\bar{w} : B_\tau(t_0 - t_1) \to B_{\gamma+\mu}(0)$ by $\bar{w}(t) := w(t + \tilde{t}_1) - \tilde{x}_1$. By Proposition 2.21, $\bar{w}$ is well-defined since, by

$$|t + \tilde{t}_{1\varepsilon} - t_0| \le |t - (t_0 - t_1)| + |\tilde{t}_{1\varepsilon} - t_1| < \tau + h - \tau = h$$

for $t \in B_\tau(t_0 - t_1)$ and $|\tilde{t}_{1\varepsilon} - t_1| < h - \tau = \tau - a$, the map $t \mapsto t + \tilde{t}_1$ is c-bounded from $B_\tau(t_0 - t_1)$ into J. Moreover, $\bar{w}$ is a solution of

$$v'(t) = G(t, v(t), (\tilde{t}_1, \tilde{x}_1, \tilde{p})), \quad v_\varepsilon(0) = 0. \tag{5.22}$$

Since $B_\tau(t_0 - t_1) \subseteq \hat{J}$ and solutions of (5.22) are unique in $\mathcal{G}[\hat{J}, B_{\gamma+\mu}(0)]$ (as proved earlier), it follows that $\bar{w} = v(\tilde{t}_1, \tilde{x}_1, \tilde{p})|_{B_\tau(t_0-t_1)}$. From

$$|t - \tilde{t}_{1\varepsilon} - (t_0 - t_1)| \leq |t - t_0| + |t_1 - \tilde{t}_{1\varepsilon}| < a + \tau - a = \tau$$

for $t \in B_a(t_0)$, it follows that $t \mapsto t - \tilde{t}_1$ is c-bounded from $B_a(t_0)$ into $B_\tau(t_0 - t_1)$. Hence, we may calculate

$$w(t) = \bar{w}(t - \tilde{t}_1) + \tilde{x}_1 = v(\tilde{t}_1, \tilde{x}_1, \tilde{p})(t - \tilde{t}_1) + \tilde{x}_1 = u(t),$$

establishing $w|_{(t_0-a,t_0+a)} = u|_{(t_0-a,t_0+a)}$, and we are done. $\square$

5.2 A Frobenius theorem in generalised functions

In order to prove a generalised Frobenius theorem we need to solve a generalised first order linear system of ODEs.

5.11. Proposition: *Let I be an open interval, $t_0 \in I$ and $A \in \mathcal{G}(I)^{n^2}$ satisfying*

$$\sup_{t\in I} \|A_\varepsilon(t)\| = O(|\log \varepsilon|).$$

Then the initial value problem

$$u'(t) = A(t) \cdot u(t), \quad u(t_0) = 0 \tag{5.23}$$

has only the trivial solution $u = 0$ in $\mathcal{G}(I)^n$.

Proof: Obviously, $u = 0$ is a solution of (5.23). The uniqueness of this solution follows from a slight modification of the proof of uniqueness in Theorem 5.2. $\square$

Note that in the above proposition a solution of the initial value problem need not to be c-bounded since $A(t)$ is a generalised matrix for all $t \in I$.

Now we are ready to prove a generalised version of Frobenius' Theorem 1.8.

5.12. Theorem: *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$ and $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(U \times V)^{mn}$. If for all $\tilde{x}_0 \in \tilde{U}_c$ with $\tilde{x}_0 \approx x_0 \in U$ and $\tilde{y}_0 = [(\tilde{y}_{0\varepsilon})_\varepsilon] \in \tilde{V}_c$ there exist $\varepsilon_0 \in (0,1]$, $\alpha, \beta > 0$ and $a > 0$ such that*

$$\sup_{(x,y)\in Q} |F_\varepsilon(x,y)| \leq a \tag{5.24}$$

for all $\varepsilon \leq \varepsilon_0$ and

$$\sup_{(x,y)\in Q} |\partial_2 F_\varepsilon(x,y)| = O(|\log \varepsilon|), \tag{5.25}$$

where $Q := \overline{B_\alpha(x_0)} \times \overline{L_\beta}$, $L_\beta := L + B_\beta(0)$ and L is a compact subset of V such that $\tilde{y}_{0\varepsilon} \in L$ for all $\varepsilon \leq \varepsilon_0$, then the following are equivalent:

(1) For all $(\tilde{x}_0, \tilde{y}_0) \in \tilde{U}_c \times \tilde{V}_c$ with $\tilde{x}_0 \approx x_0 \in U$ the initial value problem

$$\mathrm{D}u(x) = F(x, u(x)), \quad u(\tilde{x}_0) = \tilde{y}_0 \tag{5.26}$$

has a unique solution $u(\tilde{x}_0, \tilde{y}_0)$ in $\mathcal{G}[U(\tilde{x}_0, \tilde{y}_0), L_\beta]$, where $U(\tilde{x}_0, \tilde{y}_0)$ is an open neighbourhood of x_0 in U.

(2) The integrability condition is satisfied, i.e. the mapping

$$(x, y, v_1, v_2) \mapsto \mathrm{D}F(x,y)(v_1, F(x,y) \cdot v_1) \cdot v_2$$

is symmetric in $v_1, v_2 \in \mathbb{R}^n$ as a generalised function in $\mathcal{G}(U \times V \times \mathbb{R}^n \times \mathbb{R}^n)^m$.

5.13. Remark: Note that if for all $(x_0, y_0) \in U \times V$ there exist ε_0, α, β and a such that condition (5.24) holds, this property is equivalent to F being c-bounded.

Proof: The proof uses the same line of argument as in the classical case 1.12. However, we have to be much more careful when it comes to composing and pointwise characterisation of generalised functions. We will make good use of several results of Chapter 2.

(1) $\Rightarrow$ (2): By Proposition 2.30, we only have to check if

$$\mathrm{D}F(\tilde{x},\tilde{y})(\tilde{v}_1, F(\tilde{x},\tilde{y}) \cdot \tilde{v}_1) \cdot \tilde{v}_2 = \mathrm{D}F(\tilde{x},\tilde{y})(\tilde{v}_2, F(\tilde{x},\tilde{y}) \cdot \tilde{v}_2) \cdot \tilde{v}_1$$

for all near-standard points $\tilde{v}_1, \tilde{v}_2 \in \tilde{\mathbb{R}}^n_c$ and $(\tilde{x},\tilde{y}) \in \tilde{U}_c \times \tilde{V}_c$. Therefore, let $\tilde{x}$ and $\tilde{y}$ be near-standard points in $\tilde{U}_c$ resp. $\tilde{V}_c$. By (1), there exists a solution u of the initial value problem

$$\mathrm{D}u(x) = F(x, u(x)), \quad u(\tilde{x}) = \tilde{y}.$$

Writing $\mathrm{D}u$ as $\mathrm{D}u = F \circ (\mathrm{id}, u)$, then, by Proposition 2.21 and Corollary 2.29, we obtain

$$\begin{aligned}
\mathrm{D}^2u(\tilde{x})(\tilde{v}_1, \tilde{v}_2) &= (\mathrm{D}^2u(\tilde{x}) \cdot \tilde{v}_1) \cdot \tilde{v}_2 \\
&= \mathrm{ev}_{\tilde{v}_2}\big(\mathrm{D}(\mathrm{D}u)(\tilde{x}) \cdot \tilde{v}_1\big) \\
&= \mathrm{ev}_{\tilde{v}_2}\big(\mathrm{D}(F \circ (\mathrm{id}, u))(\tilde{x}) \cdot \tilde{v}_1\big) \\
&= \mathrm{ev}_{\tilde{v}_2}\Big(\big(\mathrm{D}F(\tilde{x}, u(\tilde{x})) \circ (\mathrm{id}, \mathrm{D}u(\tilde{x}))\big) \cdot \tilde{v}_1\Big) \\
&= \mathrm{ev}_{\tilde{v}_2}\Big(\mathrm{D}F\big(\tilde{x}, u(\tilde{x})\big)\big(\tilde{v}_1, F(\tilde{x}, u(\tilde{x})) \cdot \tilde{v}_1\big)\Big) \\
&= \mathrm{D}F(\tilde{x}, \tilde{y})\big(\tilde{v}_1, F(\tilde{x}, \tilde{y}) \cdot \tilde{v}_1\big) \cdot \tilde{v}_2
\end{aligned}$$

for all near-standard points $\tilde{v}_1, \tilde{v}_2 \in \tilde{\mathbb{R}}^n_c$. The last expression is symmetric in $\tilde{v}_1$ and $\tilde{v}_2$ since, by Schwarz's Theorem, $\mathrm{D}^2u(\tilde{x})$ has this property.

(2) $\Rightarrow$ (1): Let $\tilde{x}_0 = [(\tilde{x}_{0\varepsilon})_\varepsilon]$ be a near-standard point in $\tilde{U}_c$ with $\tilde{x}_0 \approx x_0$ and let $\tilde{y}_0 \in \tilde{V}_c$.

Existence: Choose $\delta \in (0, \alpha)$ and set $\gamma := \alpha - \delta$. There exists some $\varepsilon_1 \leq \varepsilon_0$ such that $\tilde{x}_{0\varepsilon}$ is in $B_\delta(x_0)$ for all $\varepsilon \leq \varepsilon_1$. From now on, we always let $\varepsilon \leq \varepsilon_1$. Since for $|t| < \gamma$ and $v \in B_1(0) \subseteq \mathbb{R}^n$

$$|\tilde{x}_{0\varepsilon} + tv - x_0| \leq |\tilde{x}_{0\varepsilon} - x_0| + |t||v| < \delta + \gamma = \alpha$$

holds, we have $\tilde{x}_{0\varepsilon} + tv \in B_\alpha(x_0) \subseteq U$ and the function

$$\begin{array}{rccc}
G_\varepsilon: & (-\gamma, \gamma) \times V \times B_1(0) & \to & \mathbb{R}^m \\
& (t, y, v) & \mapsto & F_\varepsilon(\tilde{x}_{0\varepsilon} + tv, y) \cdot v
\end{array}$$

is well-defined. By Propositions 2.21 and 2.32, $G := [(G_\varepsilon)_\varepsilon]$ is a well-defined generalised function in $\mathcal{G}\left((-\gamma, \gamma) \times V \times B_1(0)\right)^m$. Now consider the initial value problem

$$f'(t) = G(t, f(t), v), \quad f(0) = \tilde{y}_0, \tag{5.27}$$

with parameter $v \in B_1(0)$. We will show that the conditions of Theorem 5.8 are satisfied. Choose $\eta \in (0, \gamma)$. Then, by (5.24),

$$\begin{aligned}
\sup_{(t,y,v) \in \overline{B_\eta(0)} \times \overline{L_\beta} \times B_1(0)} |G_\varepsilon(t, y, v)| &= \sup_{(t,y,v) \in \overline{B_\eta(0)} \times \overline{L_\beta} \times B_1(0)} |F_\varepsilon(\tilde{x}_{0\varepsilon} + tv, y) \cdot v| \\
&\leq \sup_{(x,y) \in Q} |F_\varepsilon(x, y)| \cdot \sup_{v \in B_1(0)} |v| \\
&\leq a.
\end{aligned}$$

Furthermore, by (5.25), we obtain

$$\begin{aligned}
\sup_{(t,y,v)\in\overline{B_\eta(0)}\times\overline{L_\beta}\times B_1(0)} \partial_2 G_\varepsilon(t,y,v) = \\
&= \sup_{(t,y,v)\in\overline{B_\eta(0)}\times\overline{L_\beta}\times B_1(0)} |\operatorname{ev}_v \circ \partial_2 F_\varepsilon(\tilde{x}_{0\varepsilon}+tv,y)| \\
&\le \sup_{v\in B_1(0)} |\operatorname{ev}_v| \cdot \sup_{(x,y)\in Q} |\partial_2 F_\varepsilon(x,y)| \\
&= O(|\log\varepsilon|).
\end{aligned}$$

From Theorem 5.8, it follows that there exists a generalised function $f \in \mathcal{G}[B_1(0)\times J, L_\beta]$ such that $f(v, .\,)$ is a solution of (5.27) for all $v \in B_1(0)$ where h is in $\left(0, \min\left(\eta, \frac{\beta}{a}\right)\right)$ and $J := [-h,h]$. Fix some $r \in (0,h)$ and $\lambda \in (0,1)$ and set

$$U(\tilde{x}_0,\tilde{y}_0) := B_{\lambda r}(x_0).$$

We choose $\varepsilon_2 \le \varepsilon_1$ such that $|x_0 - \tilde{x}_{0\varepsilon}| < (1-\lambda)r$ for all $\varepsilon \le \varepsilon_2$. From now on, we always let $\varepsilon \le \varepsilon_2$. We define $u_\varepsilon(\tilde{x}_0,\tilde{y}_0) : U(\tilde{x}_0,\tilde{y}_0) \to L_\beta$ by

$$u_\varepsilon(\tilde{x}_0,\tilde{y}_0)(x) := f_\varepsilon\left(\frac{1}{r}(x-\tilde{x}_{0\varepsilon}), r\right).$$

By the choice of ε_2, the inequality

$$\left|\frac{1}{r}(x-\tilde{x}_{0\varepsilon})\right| \le \frac{1}{r}(|x-x_0| + |x_0-\tilde{x}_{0\varepsilon}|) < \frac{1}{r}(\lambda r - (1-\lambda)r) = 1$$

holds and the function $u_\varepsilon(\tilde{x}_0,\tilde{y}_0)$ is well-defined. From now on, we denote $u_\varepsilon(\tilde{x}_0,\tilde{y}_0)$ simply by u_ε. Since, obviously, the net $\left(x \mapsto \frac{1}{r}(x-\tilde{x}_{0\varepsilon})\right)_\varepsilon$ is moderate and c-bounded into $B_1(0)$, the composition with $(f_\varepsilon)_\varepsilon$ is moderate. By the c-boundedness of $(f_\varepsilon)_\varepsilon$ into L_β, also $(u_\varepsilon)_\varepsilon$ is c-bounded into L_β, i.e. $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[U(\tilde{x}_0,\tilde{y}_0), L_\beta]$.
To prove that u is indeed a solution of (5.26) we will use the equality of $(t,v,w) \mapsto \partial_1 f(v,t)\cdot w$ and $(t,v,w) \mapsto F(\tilde{x}_0+tv, f(v,t))\cdot(tw)$ in $\mathcal{G}((-h,h)\times B_1(0)\times\mathbb{R}^n)^m$. To see this we consider the net $(k_\varepsilon)_\varepsilon$ given by $k_\varepsilon : (-h,h)\times B_1(0)\times\mathbb{R}^n \to \mathbb{R}^m$,

$$k_\varepsilon(t,v,w) := \partial_1 f_\varepsilon(v,t)\cdot w - F_\varepsilon(\tilde{x}_{0\varepsilon}+tv, f_\varepsilon(v,t))\cdot(tw).$$

Note that, by Propositions 2.21 and 2.32, $k := [(k_\varepsilon)_\varepsilon]$ is a well-defined generalised function in $\mathcal{G}((-h,h)\times B_1(0)\times\mathbb{R}^n)^m$. Let $\tilde{v} \in \widetilde{B_1(0)}_c$ and $\tilde{w} \in \tilde{\mathbb{R}}^n_c$. Since $v \mapsto f(v,0) = \tilde{y}_0$ is constant in $\mathcal{G}(B_1(0))^m$ and F maps to a space of generalised linear functions, we have

$$k(0,\tilde{v},\tilde{w}) = \partial_1 f(\tilde{v},0)\cdot\tilde{w} - F_\varepsilon(\tilde{x}_0 + 0\cdot\tilde{v}, f(\tilde{v},0))\cdot(0\cdot\tilde{w}) = 0$$

in $\tilde{\mathbb{R}}^m$. By Schwarz's Theorem, the chain rule and the integrability condition (2), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}k(t,\tilde v,\tilde w) = \\
&\quad= \frac{\partial}{\partial t}\Big(\partial_1 f(\tilde v,t)\cdot\tilde w - F(\tilde x_0+t\tilde v, f(\tilde v,t))\cdot(t\tilde w)\Big) \qquad (5.28)\\
&\quad= \frac{\partial}{\partial v}\Big(\underbrace{\frac{\partial}{\partial t}f(v,t)}_{=F(\tilde x_0+tv,f(v,t))\cdot v}\Big)\Big|_{v=\tilde v}\cdot\tilde w\\
&\qquad\qquad - \Big(\partial_1 F(\tilde z)\cdot\tilde v\cdot t\tilde w + \partial_2 F(\tilde z)\cdot\Big(\underbrace{\frac{\partial}{\partial t}f(\tilde v,t)}_{=F(\tilde z)\cdot\tilde v}\Big)\cdot t\tilde w + F(\tilde z)\cdot\tilde w\Big)\\
&\quad= \frac{\partial}{\partial v}\Big(F(\tilde x_0+tv,f(v,t))\cdot v\Big)\Big|_{v=\tilde v}\cdot\tilde w\\
&\qquad\qquad - \Big(\mathrm{D}F(\tilde z)\cdot(\tilde v, F(\tilde z)\cdot\tilde v)\cdot t\tilde w + F(\tilde z)\cdot\tilde w\Big)\\
&\quad\overset{(2)}{=} \Big(\partial_1 F(\tilde z)\cdot t\tilde w\cdot\tilde v + \partial_2 F(\tilde z)\cdot(\partial_1 f(\tilde v,t)\cdot\tilde w)\cdot\tilde v + F(\tilde z)\cdot\tilde w\Big)\\
&\qquad\qquad - \Big(\mathrm{D}F(\tilde z)\cdot(t\tilde w, F(\tilde z)\cdot t\tilde w)\cdot\tilde v + F(\tilde z)\cdot\tilde w\Big)\\
&\quad= \partial_1 F(\tilde z)\cdot t\tilde w\cdot\tilde v + \partial_2 F(\tilde z)\cdot(\partial_1 f(\tilde v,t)\cdot\tilde w)\cdot\tilde v\\
&\qquad\qquad - \partial_1 F(\tilde z)\cdot t\tilde w\cdot\tilde v - \partial_2 F(\tilde z)\cdot(F(\tilde z)\cdot t\tilde w)\cdot\tilde v\\
&\quad= \partial_2 F(\tilde z)\cdot(\partial_1 f(\tilde v,t)\cdot\tilde w - F(\tilde z)\cdot t\tilde w)\cdot\tilde v\\
&\quad= \partial_2 F(\tilde z)\cdot k(t,\tilde v,\tilde w)\cdot\tilde v\\
&\quad= \Big(\mathrm{ev}_{\tilde v}\circ\partial_2 F(\tilde x_0+t\tilde v, f(\tilde v,t))\Big)\cdot k(t,\tilde v,\tilde w) \qquad (5.29)
\end{aligned}
$$

for $\tilde z = (\tilde x_0+t\tilde v, f(\tilde v,t))$. Corollary 2.29 says that $\mathrm{ev}_{\tilde v}$ is in $\mathcal{G}(\mathbb{R}^{mn})^m$. We may regard $\mathrm{ev}_{\tilde v}$ as a generalised function of $(t,A)\in(-h,h)\times\mathbb{R}^{mn}$ which is independent of t, i.e. $\mathrm{ev}_{\tilde v}\in\mathcal{G}((-h,h)\times\mathbb{R}^{mn})^m$. From Proposition 2.33, it follows that $\check{\mathrm{ev}}_{\tilde v} : t\mapsto \mathrm{ev}_{\tilde v}$ is in $\mathcal{G}((-h,h))^{m\cdot mn}$. Therefore, the expression in the brackets in the last line of (5.28) can also be written as $\check{\mathrm{ev}}_{\tilde v}(t)\circ \partial_2 F(\tilde x_0+t\tilde v, f(\tilde v,t))$. Since $t\mapsto \partial_2 F(\tilde x_0+t\tilde v, f(\tilde v,t))$ is in $\mathcal{G}((-h,h))^{mn\cdot m}$, by Proposition 2.32, the mapping $A(t) := \check{\mathrm{ev}}_{\tilde v}(t)\circ\partial_2 F(\tilde x_0+t\tilde v, f(\tilde v,t))$ is in $\mathcal{G}((-h,h))^{m^2}$. From (5.25), it follows that $\sup_{t\in(-h,h)}|A(t,\tilde v,\tilde w)| = O(|\log\varepsilon|)$ for all $\tilde v,\tilde w\in\tilde{\mathbb{R}}^n_c$. Hence, $k(\,.\,,\tilde v,\tilde w)$ is a solution of a linear initial value problem satisfying the conditions of Proposition 5.11 and, therefore, $k(\,.\,,\tilde v,\tilde w)=0$ for all $\tilde v,\tilde w\in\tilde{\mathbb{R}}^n_c$. By Proposition 2.30, we conclude that $k=0$ in $\mathcal{G}((-h,h)\times\mathbb{R}^n\times\mathbb{R}^n)^m$.

Finally, we check that u is indeed a solution of (5.26). Observe that for $v=0$ the initial value problem (5.27) is reduced to

$$f'(t)=0,\quad f(0)=\tilde y_0.$$

Therefore, $f(0,\,.\,)$ is the (in $\mathcal{G}[(-h,h),L_\beta]$) constant function $t \mapsto \tilde{y}_0$. Thus, by the definition of u, we obtain

$$u(\tilde{x}_0) = f\Big(\frac{1}{r}(\tilde{x}_0 - \tilde{x}_0), r\Big) = \tilde{y}_0.$$

At last, we have

$$\begin{aligned}
\mathrm{D}u(x)\cdot\tilde{w} &= \frac{d}{dx}\,\Big(f\Big(\frac{x-\tilde{x}_0}{r}, r\Big)\Big)\cdot\tilde{w} \\
&= \partial_1 f\Big(\frac{x-\tilde{x}_0}{r}, r\Big)\cdot\frac{1}{r}\cdot\tilde{w} \\
&= F\Big(\tilde{x}_0 + r\cdot\frac{x-\tilde{x}_0}{r}, f\Big(\frac{x-\tilde{x}_0}{r}, r\Big)\Big)\cdot r\,\frac{1}{r}\,\tilde{w} \\
&= F(x, u(x))\cdot\tilde{w}
\end{aligned}$$

for all $\tilde{w} \in \tilde{\mathbb{R}}^n_c$. Applying Proposition 2.30 to the above equation, we conclude that u is indeed a solution of the initial value problem (5.26).

Uniqueness: Let $\bar{u} \in \mathcal{G}[B_{\lambda r}(x_0), L_\beta]$ be another solution of (5.26). We will show that $\bar{u}|_{B_a(x_0)} = u|_{B_a(x_0)}$ for all $a < \lambda r$. Since $\mathcal{G}$ is a sheaf, the equality also holds on $B_{\lambda r}(x_0) = U(\tilde{x}_0, \tilde{y}_0)$.

Let $a \in (0, \lambda r)$. Observe that if $(f_\varepsilon)_\varepsilon$ is the representative of f that solves (5.27) for the representatives $(G_\varepsilon)_\varepsilon$ of G, $(\tilde{x}_{0\varepsilon})_\varepsilon$ of $\tilde{x}_0$ and $(\tilde{y}_{0\varepsilon})_\varepsilon$ of $\tilde{y}_0$ classically for small ε (such a representative exists by Theorem 5.8), then

$$f_\varepsilon(v, ct) = f_\varepsilon(cv, t)$$

holds for all c, v and t for which both sides are defined. Hence, the same is true for f as a generalised function. Now, let $\tilde{v} = [(\tilde{v}_\varepsilon)_\varepsilon] \in \widetilde{B_1(0)}_c$ and set $\tau := \frac{1}{3}(\lambda r - a)$. We define $g(\tilde{v}) : B_{a+2\tau}(0) \to L_\beta$ by $g(\tilde{v})(t) := \bar{u}(\tilde{x}_0 + t\tilde{v})$. The function $g(\tilde{v})$ is well-defined since, by

$$|\tilde{x}_{0\varepsilon} + t\tilde{v}_\varepsilon - x_0| \leq |\tilde{x}_{0\varepsilon} - x| + |t||\tilde{v}_\varepsilon| < \tau + (a + 2\tau)\cdot 1 = \lambda r$$

for $t \in B_{a+2\tau}(0)$ and $|x_0 - \tilde{x}_{0\varepsilon}| < \tau$, the map $t \mapsto \tilde{x}_0 + t\tilde{v}$ is c-bounded from $B_{a+2\tau}(0)$ into $B_{\lambda r}(x_0)$. Moreover, $g(\tilde{v})$ is an element of $\mathcal{G}[B_{a+2\tau}(0), L_\beta]$ and a solution of (5.27) for $v = \tilde{v}$. Since $B_{a+2\tau}(0) \subseteq J$ and solutions are unique in $\mathcal{G}[J, L_\beta]$, it follows that $g(\tilde{v}) = f(\tilde{v},\,.\,)|_{B_{a+2\tau}(0)}$ for all $\tilde{v} \in \widetilde{B_1(0)}_c$. By Proposition 2.30, $g : (v,t) \mapsto g(v)(t)$ is an element of $\mathcal{G}[B_1(0) \times B_{a+2\tau}(0), L_\beta]$ and equal to f on $B_1(0) \times B_{a+2\tau}(0)$. Since

$$\Big|\frac{1}{a+\tau}(x - \tilde{x}_{0\varepsilon})\Big| \leq \frac{1}{a+\tau}\cdot(|x - x_0| + |x_0 - \tilde{x}_{0\varepsilon}|) < \frac{1}{a+\tau}\cdot(a+\tau) = 1$$

holds for all $x \in B_a(x_0)$ and sufficiently small ε, the map $x \mapsto \frac{1}{a+\tau}(x - \tilde{x}_0)$ is c-bounded from $B_a(x_0)$ into $B_1(0)$. Hence, we may calculate

$$\begin{aligned}
\bar{u}(x) &= g\Big(\frac{1}{a+\tau}(x - \tilde{x}_0)\Big)(a+\tau) \\
&= f\Big(\frac{1}{a+\tau}(x - \tilde{x}_0)\Big)(a+\tau) \\
&= f\Big(\frac{1}{r}(x - \tilde{x}_0)\Big)(r) \\
&= u(x),
\end{aligned}$$

establishing the claim. □

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