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LAGRANGE VERSUS EULER FOR TURBULENT FLOWS

and vice versa, with emphasis on relation(s) and dynamical aspects

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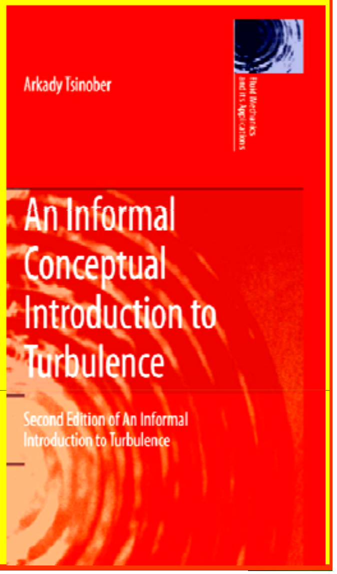
Some relevant aspects of the Euler versus Lagrange issues in turbulence

This file is based on three ingerdients:
** my short intro at the beginning,*
** my presentation at the discussion 2* (the two pages of the text for discussion 2 sent before the meeting are found on slides 23-29)
** Some additions during this discussion.*
*The latter is again mine as nobody volunteered to make any notes. For this reason (not the only one) the discussion is to some extent a discussion of "one man". * These are also the reasons I added some specific references and also below a list of relevant sections , subsections, etc., from my book, which will be referenced in the text as* T2009:

A. Tsinober 2009 An informal conceptual introduction to turbulence, xix+464 pp., Springer.

I venture to mention already here the following relevant sections which are referred to in text below:

3.6 Eulerian versus Lagrangian descriptions , pp 57-61.
4.2 Kinematic/Lagrangian chaos/advection , pp. 85-89.
4.3 On the relation between Eulerian and Lagrangian fields , pp. 89 -90 .
5.4.4 Is cascade Lagrangian or Eulerian, in some decomposition, phase space or whatever? Cascade of passive objects? pp. 118-119.
6.6 Nonlocality , pp. 163-182.**
6.7.2 The Lagrangian acceleration versus its Eulerian components, pp. 185-189 * *** As part of section 6.7 Acceleration and related matters , pp. 182-193.
9.3.2. Differences in structure(s) and 9.3.4 Issues associated with the E-L relations. Analogy between genuine turbulence and Lagrangian chaos , pp. 302- 306 as part of section 9.3 Genuine turbulence versus passive "turbulence", pp. 298 -307 in the Chapter 9 ANALOGIES, MISCONCEPTIONS AND ILL-DEFINED CONCEPTS, pp. 295-320.
13.5 Pure Lagrangian description , pp. 381-382.



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* Of course it is not late to add comments, etc. , which will be added to the WPI site .
** See also a short subsection 1.3.5 Nonlocality , pp. 28-30; subsection 5.3 Anomalous scaling , pp. 102 – 110 with subsections 5.3.1 Inertial range. Is it a well-defined concept?, pp. 103-107 and 5.3.2 On the multi-fractal models, pp 107-110; subsections 10.3.3 Nature of dissipation – is it (un)important? pp. 335- 337 and 10.3.4 Roles of viscosity/dissipation , pp. 337-338 In the subsection 10.3 Turbulence versus mathematics and vice versa , pp. 329-347
***Note that a similar subsection 6.7.1 The relation between the total acceleration and its local and convective components in the first edition (pp. 131-135) contains some info on correlations removed from the second edition , but all can be found in the paper by Tsinober, A., Vedula, P., and Yeung, P.K. (2000) Random Taylor hypothesis and the behavior of local and convective accelerations in isotropic turbulence, *Phys. Fluids,* **13**, 1974–1984.

TO HEAT UP

Correlations after experiments done is bloody bad. Only prediction is science.*

FRED HOYLE 1957, *The Black Cloud, Harper, N-Y.*

*These are "postdictions"

… *there are a variety of models of higher statistics that have meager or nonexistent deductive support from the NS equations but can be made to give good fits to experimental measurements. …Multifractal cascade models raise the general issue of distinction between what is descriptive of physical behavior and what can be used for analysis of data …Multifractal models may or may not express well the cascade physics at large but finite Reynolds numbers..* Velocity increments (let alone structure functions and their scaling if such exists) are not the only objects of interest. On top of this they *do not constitute a representation basis for a flow* (Goto and Kraichnan, 2004, Turbulence and Tsallis statistics, *Physica,* D193*, 231-244.*

Why on earth should we perform so many elaborate measurements of various scaling exponents without looking into the possible concomitant physics and/or without asking why and how more precise knowledge of such exponents, even assuming their existence, can aid our understanding of turbulent flows? The very existence of scaling exponents (in a statistical sense), which is taken for granted, is a problem by itself (T2009 pp. 214, 215).

Discussion Topic 2

Sweeping and related issues including random Taylor hypothesis

The issue is only seemingly "narrow". In reality (as was seen during the discussion) it is directly related to the problems like nonlocality and comparative discussion of the kinematical and dynamical aspects associated with pure Eulerian and pure Lagragian descriptions with the emphasis on conceptual aspects. Some specific problems requiring Lagrangian treatment along with the Eulerian one are pointed out and some nontrivial questions concerning the Lagrangian setting in turbulence are posed .

As mentioned this file is based mainly (but not only) on three ingerdients: i) my short intro at the beginning, ii) my presentaiton at the discussion 2 and iii) the additions during this discussion. The latter is again mine as nobody volonteered to make any notes. For this reason (noy the only one) the discussion is to some extent a discussion of " one man". These are also the reasons I decided to add more references including systematic references on pages, sections, etc. , from my book:

A. Tsinober 2009 An informal conceptual introduction to turbulence, xix+464 pp., Springer.

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Some generalities and important relevant questions/issues

The Lagrangian description of fluid flows is physically more natural than the Eulerian one, since it is related most directly to the motion of fluid elements. Nevertheless, mostly technical difficulties (both in physical and numerical experiments) strongly hindered use of the Lagrangian approach in most of fluid dynamical problems. The traditional problems for which Lagrangian description is considered especially appropriate are *transport and mixing* in diverse applications, e.g. geophysical and environmental, cloud formation, chemical technology, combustion and material processing, sedimentation, bio-medical and recently microfluidics, and many others. In most of the above issues the concern is with the *kinematic* aspects, i.e. with what is called today “passive turbulence”.

Another aspect is associated with the dynamics of inviscid fluids, such as theoretical problems of Euler equations, inviscid vortex dynamics and vortex methods, stability, dynamics of interfaces and surface waves, compressible flows. Though these issues seem to have little to do with genuine turbulence, there are views/beliefs that such things like possible singularity formation and collapse in Euler flows and that the infinite Reynolds number limit of turbulent flow is described by singular solutions of Euler equations . Some people regard these as "very attractive scenarios". They are definitely very attractive and mathematically beautiful (since Onsager 1949), but it is more than not clear whether they have anything to do with real turbulence at whatever large Reynolds numbers. One cannot take seriously claims like "*The existence of such near singularities for turbulent velocity fields at high Reynolds number has been confirmed by data from experiments and simulations*" or *"Observations from experiments and simulations suggest that material objects advected by such a rough velocity become fractal…"*, since all the experimental and numerical evidence is obtained at moderate Re_λ at which no singularities, fractal structure, etc. are expected and observed (if such exist at all). This evidence cannot be used as supporting any models at infinite Re_λ, which in principle cannot be confirmed or disproved by experimental or numerical evidence.

In other words the main concern is in the *evolution of passive objects* (fluid particles, passive scalars such as dispersing contaminants, chemical species, temperature, moisture; passive vectors such as material lines, (weak) magnetic field in an electrically conducting fluid; passive surfaces such as material surfaces, and in some cases reacting surfaces and turbulent flames; material volumes) in random fluid flows and more recently in any Lagrangian chaotic flows which among multitude of others* include most of laminar flows in Eulerian setting*

An essential point is that the evolution of passive objects obeys linear equations in which the velocity field does not `know' anything about the presence of these objects and therefore the velocity field is considered as given a priori be it a real fluid flow field or some artificial one. There is no involving phenomenon as pressure**. This does not mean that the problems of the evolution of passive objects are simple. The main complication and simultaneously rich variety of phenomena comes from the fact that the velocity field enters as a coefficient in front of the spatial derivatives, i.e. it is due its multiplicative character, so that statistical problems become in a sense nonlinear.

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* As mentioned the above qualification includes all artificial velocity fields both random and/or multi-scale or not . The field of particle trajectories is (can be seen) as a passive object: it is a Lagrangian signature of the underlying velocity field of any nature be it genuinely turbulent, or Lagrangian chaotic such as E-Laminar, synthetic random or not, restricted Euler, kinematic simulations of Lagrangian chaotic evolution, turbulent-like multiscale fields, including real E-laminar flows at Re≈0 from linear Stokes equations with random forcing, flows in porous media, micro-devices, to name some.

**Hence `shocks' in the form of ramp-cliff structures just like in the Burgers equation.

There is little (if any) treatment of dynamical aspects of turbulent flows (e.g. those corresponding to those described by NSE in Eulerian setting) *in Lagrangian setting* (one of our main concerns here). One of the reasons is the view that ***A principal objective of any theory of fluid motion is the prediction of the spread of matter or "tracer" within the fluid.*** BENNET 2006

But the main reasons seem to take their origin in the difficulties to handle the Lagrangian equations (with non-zero viscosity) and related issues.

In contrast , on the technical side, since in a pure Lagrangian setting the equations are intractable*(so far) in order to obtain true (not modelling!) Lagrangian information, one typically solves the problem in Eulerian setting (i.e., using NSE) and using this information together with the equation relating the two ways of description

$$\partial X(a,t)/\partial t = u[X(a,t);\, t] \qquad \{\text{E-L}\}$$

one can obtain the Lagrangian evolution of any fluid particle, i.e. the Lagrangian velocity field, $\mathbf{v}(\mathbf{a},t) = \partial \mathbf{X}(\mathbf{a},t)/\partial t$, is related to the Eulerian velocity field, $\mathbf{u}(\mathbf{x},t)$, as $\mathbf{v}(\mathbf{a},t) \equiv \mathbf{u}\,[\mathbf{X}(\mathbf{a},t);t]$.

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*but allow posing of nontrivial and important questions.

The {E-L} relation above is of utmost importance since it is not integrable even for simplest laminar Euler fields

with the exception of very simple flows such as unidirectional ones.

Thus for a wide class of (almost all) laminar flows in the Euelerian setting (i.e. with the Eulerian velocity field, **u**(**x**;t) not chaotic, regular and laminar) the Lagrangian velocity field **v**(**a**,t) ≡ **u**[**X**(**a**,t);t] (as any other property of fluid particle) is chaotic because **X**(**a**,t) is chaotic! * This fact is of utmost importance issues like the relation (s) between the Eulerian and Lagrangian characteristics of the same flow field (see below)

It has to be emphasized that this chaotic behavior is of purely kinematic nature resulting solely from the equation {E-L} (and various equations for passive objects - reminding again - linear in Euler setting) and has nothing to do with dynamics, i.e. genuine (as NSE) turbulence. This concerns also all problems with prescribed velocity fields in Eulerian setting – synthetic, Gaussian , etc. Similarly, all randomly forced flows with low Reynolds number (including multicale ones) belong to this category.

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*The field of particle trajectories is a passive object: it is a Lagrangian signature of the underlying velocity field of any nature be it genuinely turbulent, or Lagrangian chaotic such as E-Laminar, synthetic random or not, kinematic simulations of Lagrangian chaotic evolution, turbulent-like multiscale fields, including real E-laminar flows at Re≈0 from linear Stokes equations with random forcing, flows in porous media, microdevices, to name some.

The important points as concerns turbulence are as follows

• Whereas the E-turbulence is a dynamical phenomenon this is not necessarily the case with the L-turbulence which may be a purely kinematic one . In other words the flow can be purely L-turbulent (i.e. E-laminar) as metioned above and illustrated in the examples below. However, if the flow is E-turbulent (i.e. Re >> 1) it is L-turbulent as well.

Two important consequences:

* studying Lagrangian statistics only may not provide adequate information of the L-statistics of genuine turbulence as not necessarily containing its pure dynamical "stochasticity"

* the structure and evolution of passive objects (including fluid particles) in genuine turbulent flows arises from two (essentially and unfortunately inseparable) contributions: one due to the Lagrangian chaos and the other due to the random nature of the (Eulerian) velocity field itself.

All the above brings in the questions listed below.

* Is it true that dynamical issues *per se* can be treated satisfactory in Eulerian setting only?
* Is there any need to use for this purpose the Lagrangian setting too?
* Are there problems which require such an approach.
* In what sense are the E- and L-settings equvivalent (if they are? And what is (the meaning of) the relation between the two?

More specific questions are the theme of present Discussion (2) , see below

A bit of history

Is Lagrangian setting Lagrange's or Euler's?

One owes to Euler the first general formulas for fluid motion … presented in the simple and luminous notation of partial differences… By this discovery, all fluid mechanics was reduced to a single point analysis, and if the equations involved were integrable, one could determine completely, in all cases the motion of a fluid moved by any forces… LAGRANGE *Mécanique analitique*, Paris, 1788, Sec X. , 271

Of course, fluid mechanics can, in principle, be worked entirely in the Lagrangian * *frame…even neglecting viscous forces… yield awkward moment equations.* CORRSIN 1962.

The use of the viscous Lagrangian equations in turbulence theory is still a matter for the future. MONIN AND YAGLOM 1971

Though the Lagrangian description of the flow . . . has many attractions . . . it is generally unwieldy to work with. Even the kinematic task of determining closed-form solutions for the particle paths . . . from an initial position . . . is generally intractable. SOWARD AND ROBERTS 2008

It is clear that some aspects of the fluid motion are easier to understand in the Eulerian framework while others are easier to describe in the Lagrangian framework. FRIEDLANDER & LIPTON-LIFSCHITZ 2003

What one sees is real.The problem is interpretation

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*In fact what is called "Lagrangian desription is also due to Euler, see Lamb, 1932. A detailed account on the 'misnomer' by which the'Lagrangian' equations are ascribed to Lagrange is found in Truesdell, 1954. *Kinematics of vorticity*, Indiana University Press, Bloomington.

H.LAMB 1932, *Hydrodynamics*, Cambridge Univ. Press, pp 2-3

3. The equations of motion of a fluid have been obtained in two different forms, corresponding to the two ways in which the problem of determining the motion of a fluid mass, acted on by given forces and subject to given conditions, may be viewed. We may either regard as the object of our investigations a knowledge of the velocity, the pressure, and the density, at all points of space occupied by the fluid, for all instants; or we may seek to determine the history of every particle. The equations obtained on these two plans are conveniently designated, as by German mathematicians, the 'Eulerian' and the 'Lagrangian' forms of the hydrokinetic equations, although both forms are in reality due to Euler†.

† "Principes généraux du mouvement des fluides," *Hist. de l'Acad. de Berlin*, 1755. "De principiis motus fluidorum," *Novi Comm. Acad. Petrop.* xiv. 1 (1759).

P. FRANK 1935, *Die differential- und integral Gleichungen der Mechanik und Physik,* 2nd ed., Part 2 Vieweg; L.D.LANDAU AND L.D.LANDAU AND E.M.LIFSHITS 1959 *Fluid Mechanics*, Pergamon and many others.

A detailed account on the 'misnomer' by which the 'Lagrangian' equations are ascribed to Lagrange is found in C. TRUESDELL 1954, *The Kinematics of Vorticity*, Indiana University Press, pp. 30-32 and references therein (see three next slides)

INDIANA UNIVERSITY PUBLICATIONS
SCIENCE SERIES NO. 19

The Kinematics of Vorticity

by

C. TRUESDELL

Indiana University Press
BLOOMINGTON 1954
$6.00

See pp. 30-32

In part shown at the next two slides

[2] In this work we eschew the general misnomer by which X, Y, Z are called "Lagrangian" co-ordinates, while x, y, z are called "Eulerian" co-ordinates. The origin of this incorrect usage is as follows.

By the middle nineteenth century the history of fluid dynamics in the eighteenth century had apparently sunk into obscurity. Euler's papers were not often read, of his results which were not forgotten several were attributed to more recent authors who had appropriated them without acknowledgement or discovered them afresh, and indeed his supreme achievements in mathematics, mechanics, and mathematical physics were undervalued then, though not so much as now. The erroneous terminology still current was introduced in the posthumous memoir of Dirichlet [1860, 1, Introd.], edited by Dedekind, where [1757, 2] was quoted as the source of the "Eulerian" method, while it was stated that Lagrange in the *Méchanique Analitique* [1788, 1, Part II, Sect. II, ¶¶4–7] had introduced the "Lagrangian" method, but had immediately converted the resulting equations to "Eulerian" form. Although in the next year Hankel [1861, 1, §1] stated that his teacher Riemann had told him that Euler had introduced the "Lagrangian" method in [1770, 1], one year's priority has been sufficient to perpetuate the error.

Riemann's attribution is correct, but the references quoted are not the earliest, either for Euler or for Lagrange. Subsequent writers on hydrodynamics have followed Hankel in adopting the printer's error on the title page by which [1770, 1] is dated 1759, while the correct date is 1769; Lagrange's first exposition of the "Lagrangian" description is not in the *Méchanique Analitique* but actually in [1762, **3**, Chs. XL, XLIV, XLVIII, LII]. The whole matter is easily clarified, however. In a letter [1862, 2], written to Lagrange under the date 27 October 1759, Euler after expressing his admiration for Lagrange's first memoir on the propagation of sound stated that one had reason to doubt that propagation in two or three dimensions would follow the same law as in the one dimensional case, since he had already found the fundamental equations to be of different form. The equations he gives are the linearized equations of plane flow of a perfect fluid expressed in terms of the variables $\dot{X}$, Y. (That the date of Euler's discovery of the material description is 1759 or earlier is shown also by [1766, 1, §§4–13, 31–40], a memoir dated 1759. In [1767, 1], written in 1750–1751, Euler for plane motions had used a description partly spatial and partly material.)

SUR LA REDUCTION A UN PRINCIPE VARIATIONNEL DES EQUATIONS DU MOUVEMENT D'UN FLUIDE VISQUEUX INCOMPRESSIBLE

par R. GERBER (Grenoble).

1. On sait qu'il n'est pas possible par la considération du champ des vitesses à un instant, d'opérer dans le cas général la réduction à un principe variationnel des équations du mouvement non lent d'un fluide visqueux incompressible (1).

Il semble intéressant de voir si un tel principe ne pourrait pas être obtenu en envisageant non plus le fluide à une époque, mais en suivant une certaine masse $\mathcal{M}$ dans son mouvement.

Ayant défini un ensemble $\mathcal{E}$ de mouvements virtuels de $\mathcal{M}$ entre deux instants t_0 et t_1 on tentera de construire une fonction $\mathcal{L}$, définie sur $\mathcal{E}$, et telle que $\int_{t_0}^{t_1} \mathcal{L} dt$ soit stationnaire pour tout élément de $\mathcal{E}$ qui vérifie les équations indéfinies du mouvement.

On montrera qu'on aboutit également dans cette voie à un résultat négatif pour le cas général, du moins si on se limite à une certaine classe de fonctions $\mathcal{L}$.

De plus la méthode suivie nous conduira à exprimer avec les variables de Lagrange les équations de Navier pour un fluide visqueux incompressible et on indiquera une méthode rapide pour obtenir ces équations.

2. Soit un mouvement réel d'une masse $\mathcal{M}$ du fluide entre les époques t_0 et t_1, correspondant à certaines conditions initiales et aux limites et sous l'action de forces extérieures dépendant d'un potentiel. Notons $\mathcal{D}_t$ le domaine occupé à l'instant t, Σ_t sa frontière (on pourra supprimer l'indice t) ; $\mathcal{D}_0$ et Σ_0 pour $\mathcal{D}_{t_0}$ et Σ_{t_0}. En particulier Σ pourra

(1) Voir H. VILLAT. *Leçons sur les fluides visqueux*, p. 103.

Gerber, R., (1949) Sur la réduction à un principe variationnel des équations du mouvement d'un fluide visqueux incompressible, *Ann. Inst. Fourier,* 1, 157–162.

CORRSIN, S. 1962 Theories of turbulent dispersion, in: Favre, A., editor, *Mécanique de la turbulence*, Proceedings of the *Colloques Internationaux du CNRS, Marseille, 28 Aug.–2 Sept. 1961*, Publ. CNRS No 108, Paris, pp. 27–52.

MONIN, A.S. AND YAGLOM, A.M. 1971 S*tatistical fluid mechanics, vol.* 1, Ch.9 , MIT Press; 2*nd* Russian edition 1992 Gidrometeoizdat, St. Petersburg,

.

We mention already here the issue of special interest (for the forthcoming discussion) on L - E relation and in what sense are the E- and L-settings equvivalent (if they are!)?
Indeed, the ‘more chaotic’ nature of the Lagrangian setting (" the relative orderliness of Eulerian representation over Lagrangian“) , is traced back to early Lagrangian simulations by Amsden and Harlow 1964, see also Harlow, 2004.
Therefore it is a natural conjecture that the pure Lagrangian dynamical equations (so far intractable for viscous flows) are more rich than their (E)Navier–Stokes counterpart. The former being equivalent to the latter plus the equation relating the Eulerian and Lagrangian descriptions.

Amsden, A.A. and Harlow, F.H. (1964) Slip instability, *Phys. Fluids, 7,* 327–334
Harlow, F.H. (2004) Fluid dynamics in Group T-3 Los Alamos National Laboratory (LA-UR-03-3852), *J. Comp. Phys.*, **195,** 414–433.

Part I of the discussion 2.

Sweeping decorrelation hypothesis (SDH) and/or Random Taylor hypothesis (RTH) and related issues

The basis of SDH/RTH is comprised by the *hypothesis* which essentially originates from K41, in words of Kraichnan 1959 : *Kolmogorov's basic assumption (Kolmogorov 1941) is essentially that the internal dynamics of the sufficiently fine-scale structure (in x-space) at high Reynolds numbers should be independent of the large-scale motion. The latter should, in effect, merely convect, bodily*, regions small compared to the macro scale.* ** Consequently, it is assumed that (Tennekes 1975) *Taylor's "frozen-turbulence" approximation should be valid for the analysis of the consequences of large-scale advection of the turbulent microstructure* and that *the microstructure is statistically independent of the energy containing eddies.* The latter seems too strong as compared to the statement that *the microstructure (whatever this means) is statistically decorrelated from the energy containing eddies.* The important point is that all the above remain hypotheses and never have been proven. There is some recent experimental evidence that these hypotheses are conceptually incorrect. Results like $k^{-5/3}$ spectra are insufficient for validation these as any theories and may well be (and some really are) the RRWR.

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* Note that this is what is called sweeping which is claimed to have purely kinematic nature (which is erroneous) – this is why Lagrangian in the first place and even claims that it preserves the shapes of the advected small scale eddies and thus has no effect on the turbulence energy spectrum in the Eulerian frame.

**Similar statements were made by Kraichnan 1964: *An underlying assumption of Kolmogorov theory is that very large spatial scales of motion convect very small scales without directly causing significant internal distortion of the small scales. The assumption usually is considered to be consistent with, and to imply, statistical independence of small and large scales,* Tennekes 1975 and many others; for a good list of references see Gkioulekas 2007.

In view of the above there are a number of questions which the "simpleton Wilson" would like to ask and discuss.

1. Is the sweeping really kinematic? Is it true that small scales are statistically independent of the small scales? Or even more rebelliously – do small scales have any impact on the large scales except of overall dissipation? Do large-scale motions *merely convect, bodily**, regions small compared to the macro scale, i.e. it preserve the shapes of the advected small scale eddies . A short answer is that the claims above are erroneous due to direct and bidirectional couling on smal l and large scales. This is essentially nonlocality . Some examples are given below , for more see (T2009 , pp 163-182.*) and references therein
2. Theoreticians have reasons to "remove" in some sense the sweeping (hence the crucial function of the hypotheses) . Do they really remove the sweeping or they just think that they do so? After all the physical system does not care about how WE do represent it., ALL the scales including the large ones are there whatever the representation.

   The question is in what sense the equations with removed sweeping are equivalent to the original ones and/or how the small scales do know about the removed large scales (or they shouldn't and/or are not supposed to) or are the small scales in the equations with removed large scales equivalent to the small scales in the original equations? Or is SDH/RTH the only "answer" to the latter question?
3. Related to 2. on the need of removing the sweeping : why nature does not need this removal in order to produce the right resullt?
4. The SDH/RTH - which is kind of decomposition - ignores too much from the interaction (in the first place dynamical) between the large and small scales, i.e. it is `too kinematic'. It is really justified except being convenient for theoreticians?

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* See also a short subsection 1.3.5 Nonlocality , pp. 28-30; subsection 5.3 Anomalous scaling , pp. 102 – 110 with subsections 5.3.1 Inertial range. Is it a well-defined concept?, pp. 103-107 and 5.3.2 On the multi-fractal models, pp 107-110; subsections 10.3.3 Nature of dissipation – is it (un)important? pp. 335- 337 and 10.3.4 Roles of viscosity/dissipation , pp. 337-338 In the subsection 10.3 Turbulence versus mathematics and vice versa , pp. 329-347

Part II of the discussion 2. Relation(s) between Eulerian and Lagrangian descriptions - representations of turbulent flows

1. Except of the formal kinematic relation

$$\partial \mathbf{X}(\mathbf{a},t))/\partial t = \mathbf{u}[\mathbf{X}(\mathbf{a},t); t] \qquad \text{(E-L)}$$

where $\mathbf{u}(\mathbf{x},t)$ is the Eulerian velocity field and $\mathbf{X}(\mathbf{a},t)$ is the fluid particle trajectory and $\mathbf{a}$ is its label , the common question(s) include in the first place statistics in the broad sense.

The usual question is whether "simple" relations do exist between the E – statistics and L – statistics. This is a long-standing and most difficult problem. The general reason is because the Lagrangian field $\mathbf{X}(\mathbf{a},t)$ (and Lagrangian velocities $\mathbf{v}[\mathbf{X}(\mathbf{a}, t); t]$) is impossibly complicated functional of the Euler velocity field $\mathbf{u}(\mathbf{x},t)$. Roughly, there is a general relationship in terms of path (Feynman, functional) integrals, but this does not help much, if at all. For more on these issues see Monin and Yaglom (1971, 1, Ch. 9, pp. 568--578), also Bennet (2006, pp. 21-24). The start was made by Corrsin (1959a,b) and Lumley (1962a,b)

One can even claim that, generally there cannot be a simple relation and in a sense even any relation as seen from the following counter-example.

Most of laminar flows in the Eulerian setting (E-laminar) are Lagrangian chaotic (L-turbulent) due to non-integrability of the relation $\partial \mathbf{X}(\mathbf{a},t))/\partial t = \mathbf{u}[\mathbf{X}(\mathbf{a},t); t]$

In other words, though the Eulerian velocity field, u(x;t) is not chaotic and is regular and laminar, the Lagrangian velocity field $\mathbf{v}(\mathbf{a},t) = \mathbf{u}[\mathbf{X}(\mathbf{a}, t); t])$ is chaotic because $\mathbf{X}(\mathbf{a},t)$ is chaotic. Thus almost in all E-laminar but L-turbulent flows the Lagrangian statistics has no Eulerian counterpart. This can be seen as an indication that the pure Lagrangian dynamical equations (LNSE) are more rich ("more chaotic") than their Navier--Stokes counterpart (ENSE) so that one is tempted to conjecture that LNSE is equivalent to ENSE + E-L. This is not trivial.

2. The differences are exhibited also in structure(s) and flow visualization (what do we see?) – the L-fields may exhibit different flow patterns for the same E-field.

I will bring some results for the item I. and examples as concerns the item II.

References to both parts, see also references below – those mentioned in a short form are found in full in T2009

Corrsin, S. 1959a Progress Report on some turbulent diffusion research, *Adv. Geophys.*, 6, 161–164.

Corrsin, S. 1959b Lagrangian correlations and some difficulties in turbulent diffusion experiments, *Adv. Geophys.*, 6, 441–448.

Gkioulekas, E. 2007 On the elimination of the sweeping interactions from theories of hydrodynamic turbulence, *Physica D*, 226, 151–172.

Kraichnan 1959 The structure of isotropic turbulence at very high Reynolds numbers, *J. Fluid Mech.*, 5, 497—543.

Kraichnan, R.H. 1964 Komogorov's hypotheses and Eulerian turbulence theory, *Phys. Fluids*, 7, 1723–1734.

Lumley, J.L. (1962a) The mathematical nature of the problem of relating Lagrangian and Eulerian statistical functions in turbulence, in: Favre, A., editor *M' ecanique de la turbulence*, Proceedings of the *Colloques Internationaux du CNRS, Marseille, 28 Aug.–2 Sept. 1961*, Publ. CNRS N*o* 108, Paris, pp. 17–26.

Lumley, J.L. (1962b) An approach to the Eulerian–Lagrangian problem, *J. Math. Phys.*, 3, 309–312.

Monin, A.S. and Yaglom, A.M. 1971 *Statistical fluid mechanics, vol.* 1 MITPress.

Tennekes, H. 1975 Eulerian and Lagrangian time microscales in isotropic turbulence, *J. Fluid Mech.*, 67, 561–567.

Tsinober A. 2001 An Informal Introduction to Turbulence, Kluwer, xix+324.

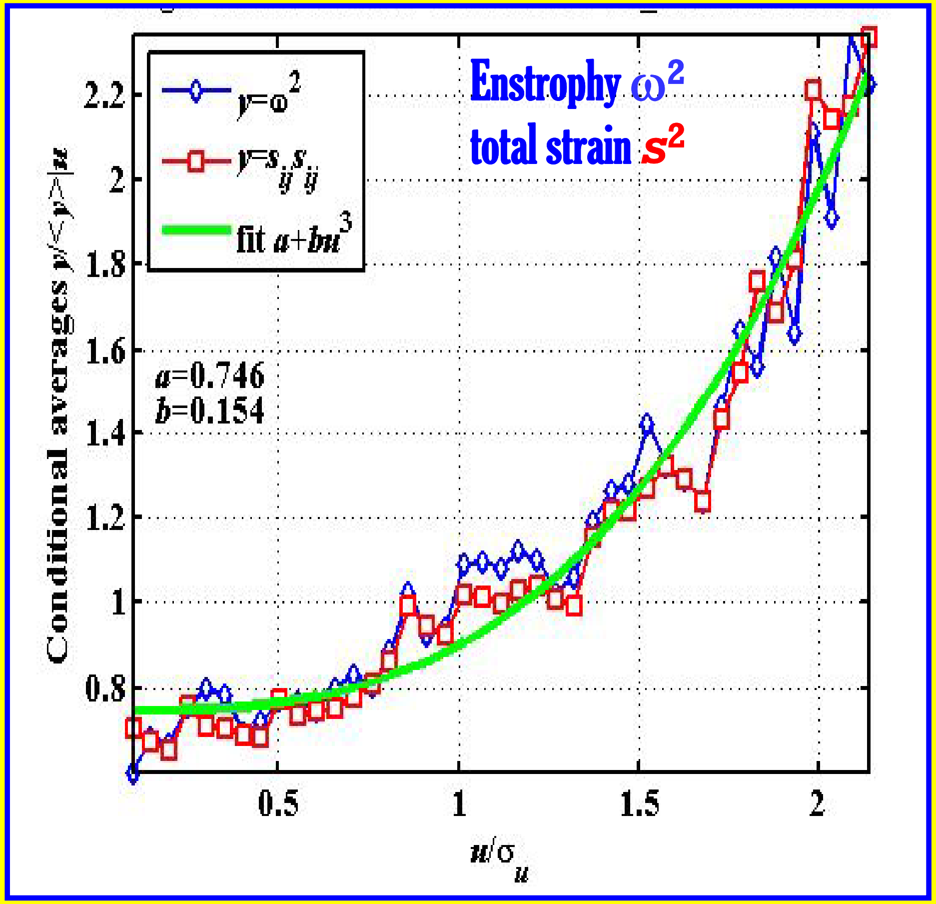
Tsinober A. 2009 An Informal Conceptual Introduction to Turbulence, Springer, xix+464.

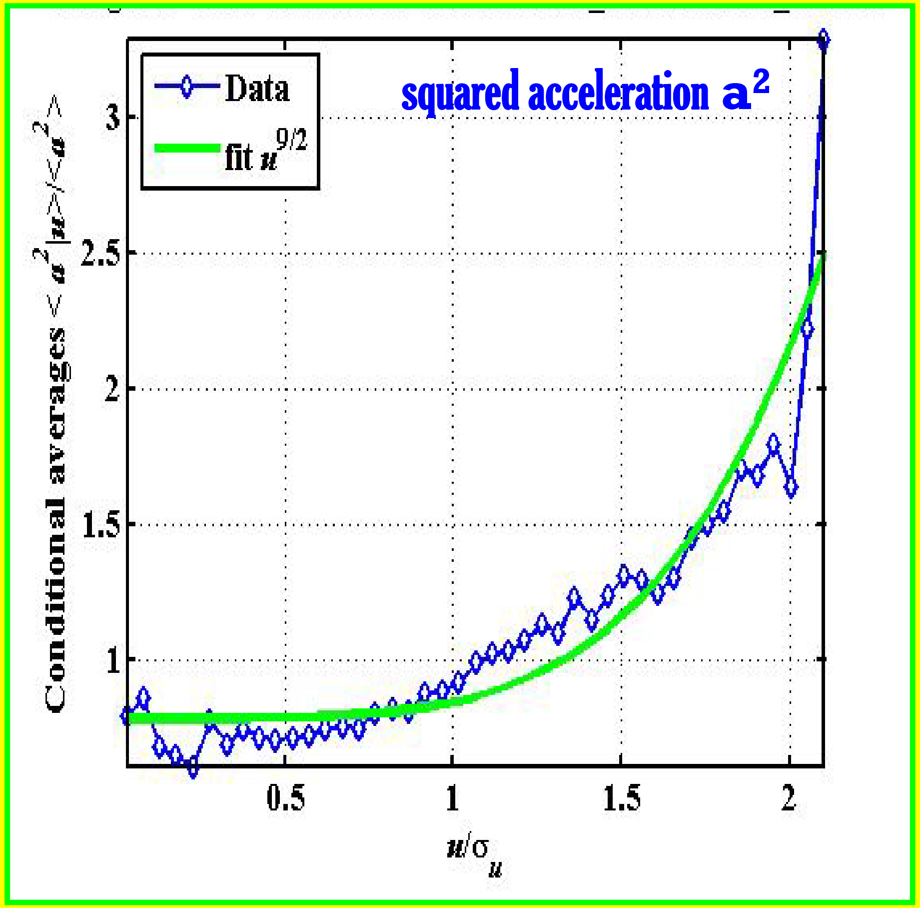
Tsinober, A., Vedula, P., and Yeung, P.K. (2001) Random Taylor hypothesis and the behaviour of local and convective accelerations in isotropic turbulence, *Phys. Fluids,* **13**, 1974–1984.

1. Is the sweeping really kinematic? Is it true that small scales are statistically independent of the small scales? Or even more rebelliously – do small scales have any impact on the large scales except of overall dissipation? Do large-scale motions merely convect, bodily *, regions small compared to the macro scale, i.e. it preserve the shapes of the advected small scale eddies

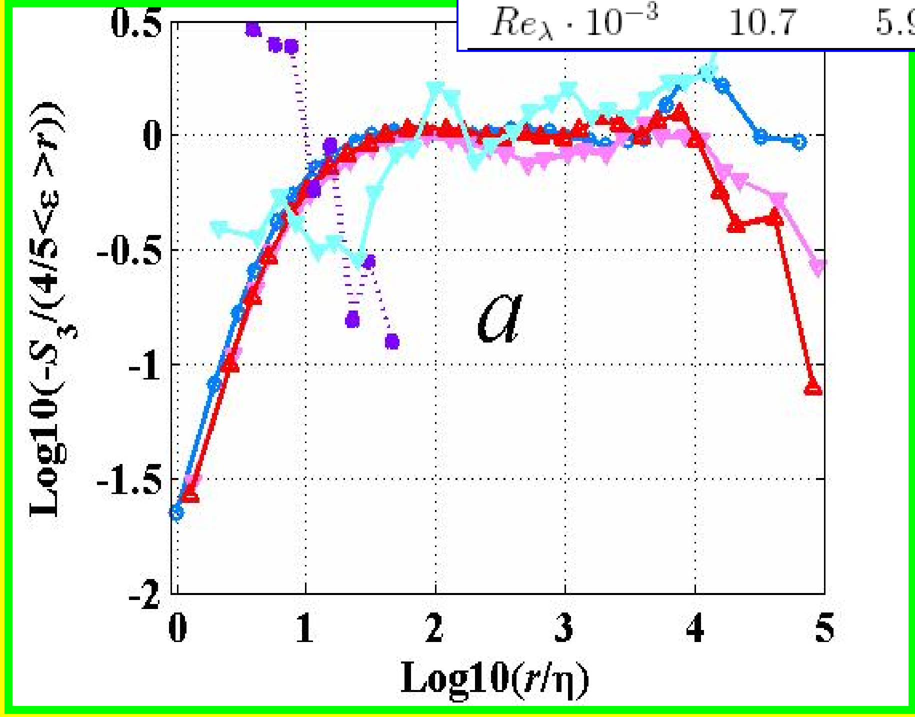
Statistical dependence of small on large scales (frequently assumed not to exist)

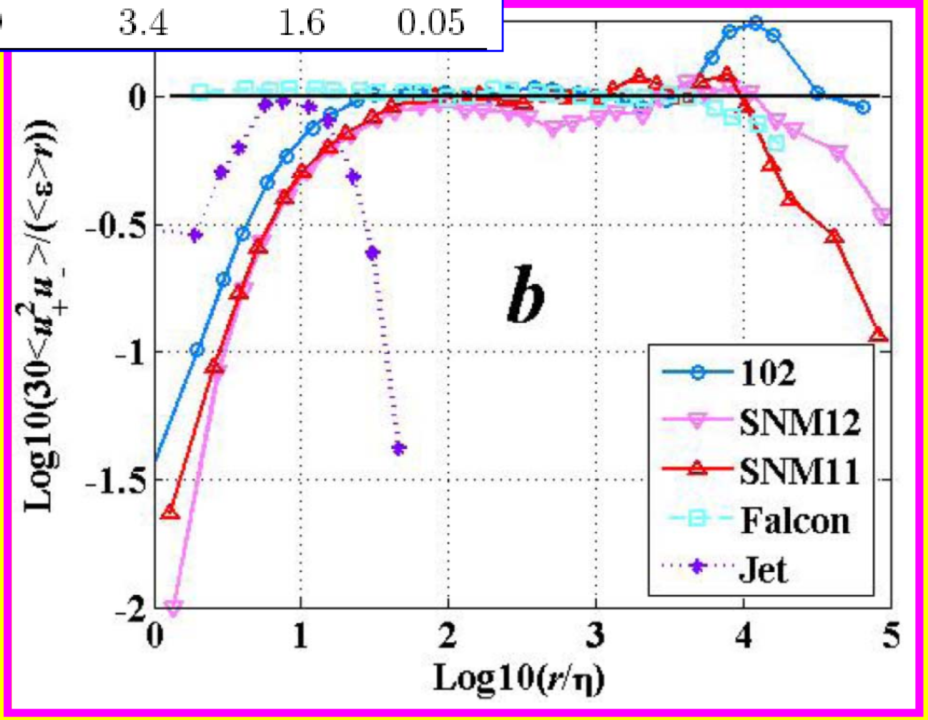
Enstrophy ω^2, total strain s^2 and squared acceleration a^2 conditioned on magnitude of the velocity fluctuation vector, Field experiment, Sils-Maria, Switzerland, 2004, $Re_\lambda = 6800$ (Gulitskii et al. 2007, *J. Fluid Mech.*, **589**, parts 1-3, 57-123)





Experiment	102,	SNM12	SNM11	Falcon	Jet
$Re_\lambda \cdot 10^{-3}$	10.7	5.9	3.4	1.6	0.05





a Conventional Kolmogorov 4/5 law 1941b

$$\langle(\Delta u_1)^3\rangle = -4/5\varepsilon r$$

b Equivalent Hosokawa's relation 2007

$$\langle u^2_+ u_-\rangle = \varepsilon r/30$$

NOTE HERE THE CORRELATION BETWEEN THE LARGE AND SMALL SCALE QUANTITIES !

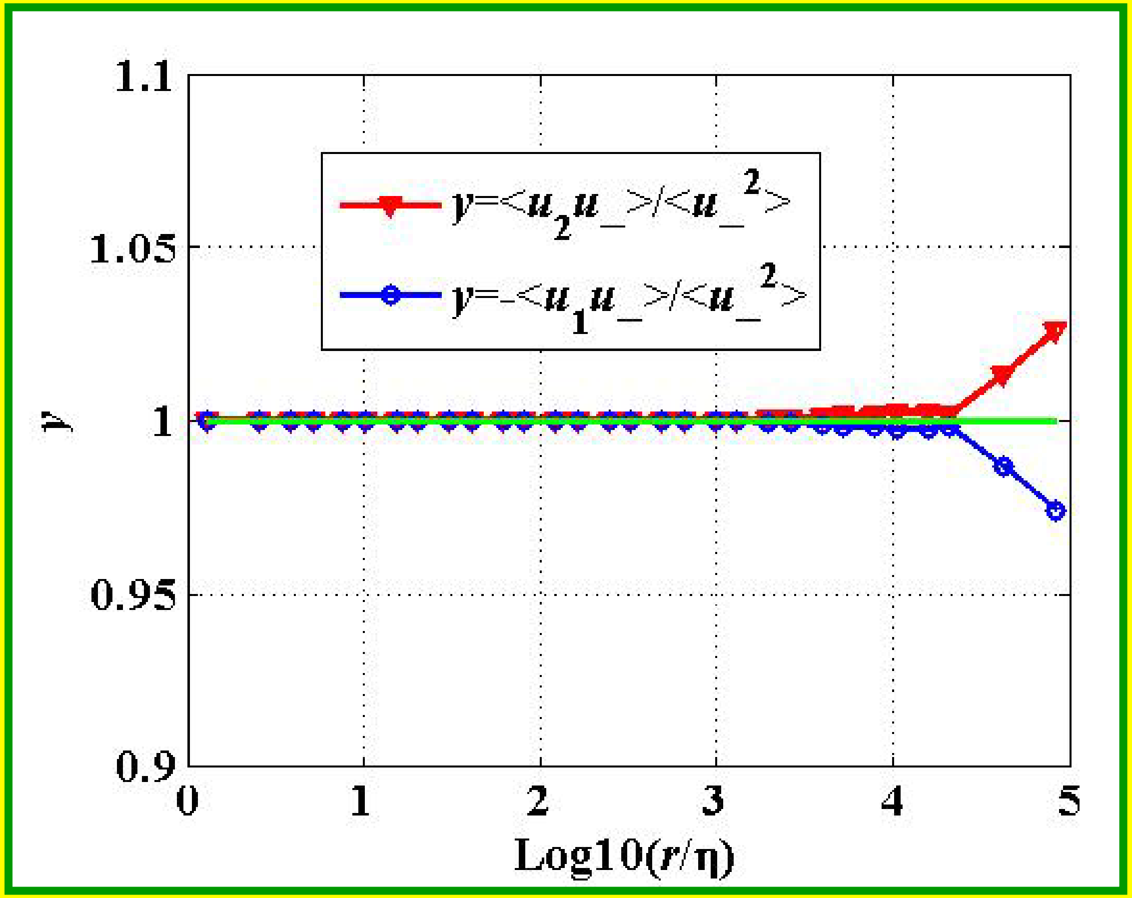
$$2u_+ = u_1(x + r) + u_1(x),\ 2u_- = u_1(x + r) - u_1(x) \equiv \Delta u_1$$

$u_1(x)$ is the longitudinal velocity component

M. Kholmyansky, and A. Tsinober 2008 Kolmogorov 4/5 law, non-locality and sweeping decorrelation hypothesis, *Physics of Fluids*, 20, *Physics of Fluids*, 20, 041704/1-4

The role of kinematic relations in the issue of nonlocality goes far beyond their use in the nonlocal interpretation of the Kolmogorov 4/5 law. There exist many kinematic relations of several types*

e.g. $<(\Delta u)^n> = -2<u_1(\Delta u)^{n-1}> = 2<u_2(\Delta u)^{n-1}>; n \geq 2$



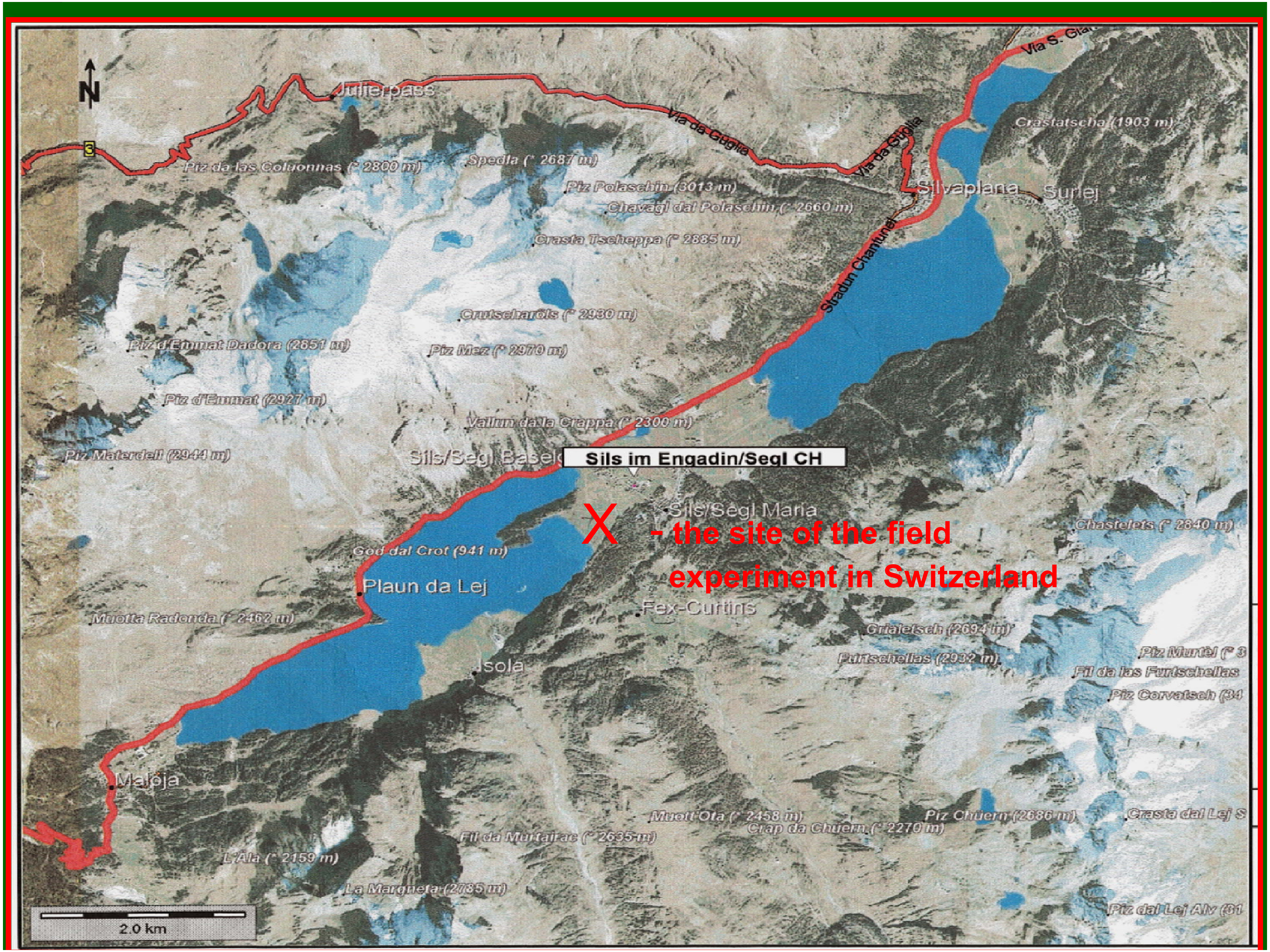
$n = 2$

$$\langle(\Delta u)^2\rangle = -2\langle u_1 \Delta u\rangle = 2\langle u_2 \Delta u\rangle$$

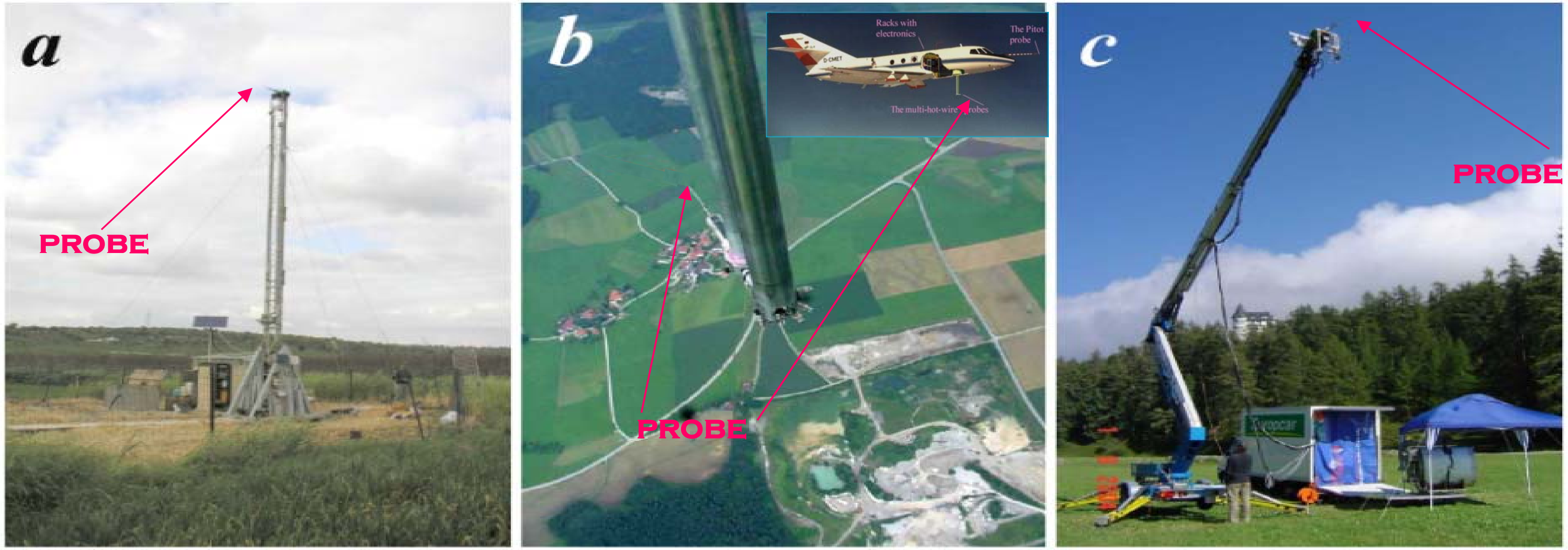
*Kholmyansky M., , Sabelnikov V. and Tsinober A 2010) Local versus Nonlocal Processes in Turbulent Flows, Kinematic Coupling and General Stochastic Processes, in: *Turbulence and Interactions Proceedings the TI 2009 Conference* ,Michel Deville ,Thien-Hiep Le, Pierre Sagaut (Editors) pp 216-221, Springer.

The inertial range (IR) is not a well defined concept

The conventionally defined inertial range is contaminated by both larger scales and strong dissipative events from the conventionally defined dissipative range . The emphasis below is on the latter



N
Julierpass
Via da Güglia
Piz da las Coluonnas (* 2800 m)
Spedla (* 2687 m)
Piz Polaschin (3013 m)
Chavagl dal Polaschin (* 2660 m)
Crasta Tscheppa (* 2885 m)
Crastatscha (1903 m)
Silvaplana
Surlej
Stradun Chantunel
Crutscharöls (* 2930 m)
Piz d'Emmat Dadora (2851 m)
Piz Mez (* 2970 m)
Piz d'Emmat (2927 m)
Vallun da la Crappa (* 2300 m)
Piz Materdell (2944 m)
Sils im Engadin/Segl CH
Sils/Segl Maria
X - the site of the field experiment in Switzerland
Chastelets (* 2840 m)
God dal Crot (941 m)
Plaun da Lej
Muotta Radonda (* 2462 m)
Fex-Curtins
Isola
Maloja
Muott Ota (* 2458 m)
Crap da Chüern (* 2270 m)
Piz Chüern (2686 m)
Fil da Murtairac (* 2635 m)
L'Ala (* 2159 m)
La Margneta (2785 m)
2.0 km



KFAR GLIKSON MEASUREMENT STATION, ISRAEL, THE PROBE ON THE MAST (A), 1999

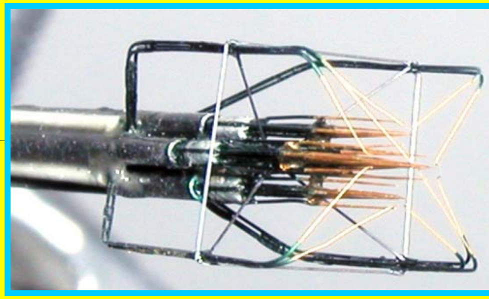
AIRBORNE EXPERIMENT, GERMANY, THE PROBE IN THE FLIGHT (B) , 2000

SILS-MARIA EXPERIMENT, SWITZERLAND, THE PROBE ON THE LIFTING MACHINE (C), 2004

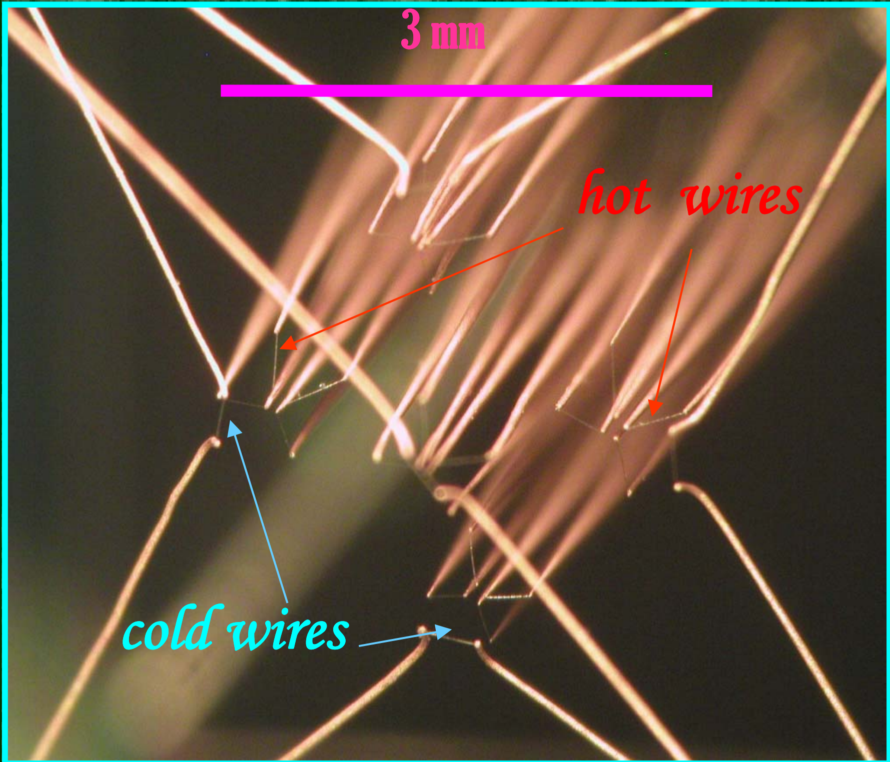
Experiment	102,	SNM12	SNM11	Falcon	Jet
$Re_\lambda \cdot 10^{-3}$	10.7	5.9	3.4	1.6	0.05

Israel Switzerland Airborn

The Taylor micro-scale Reynolds numbers, Re_λ, for the experiments.



THE PROBE

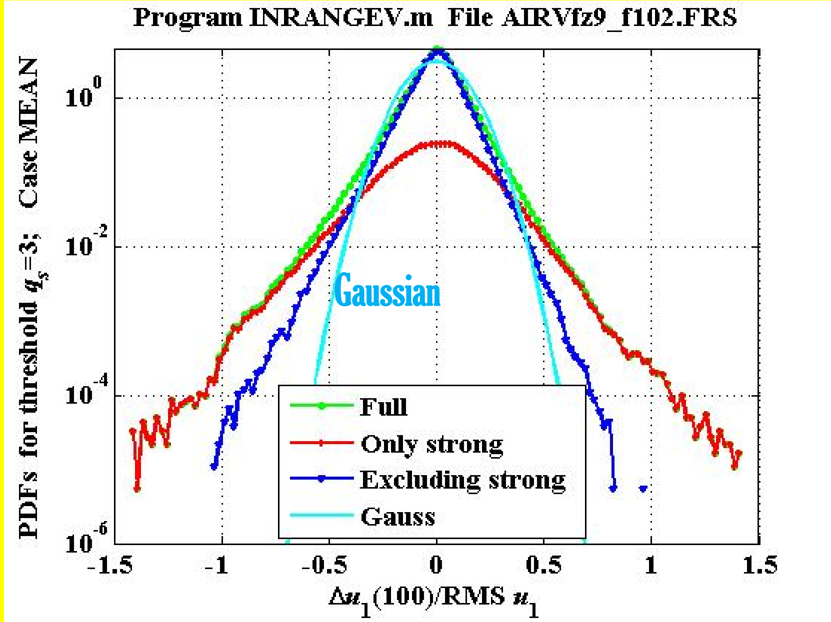


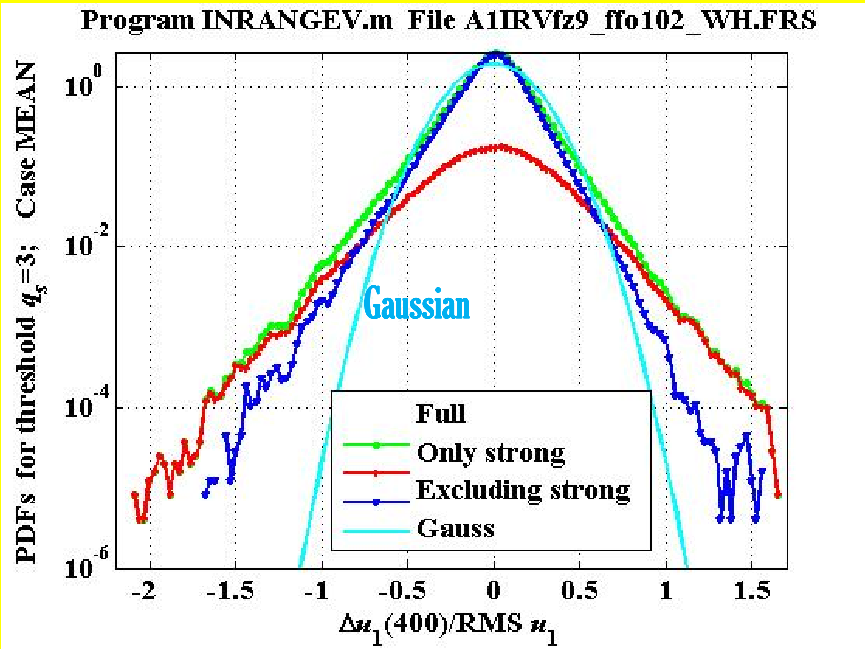
The tip of the probe

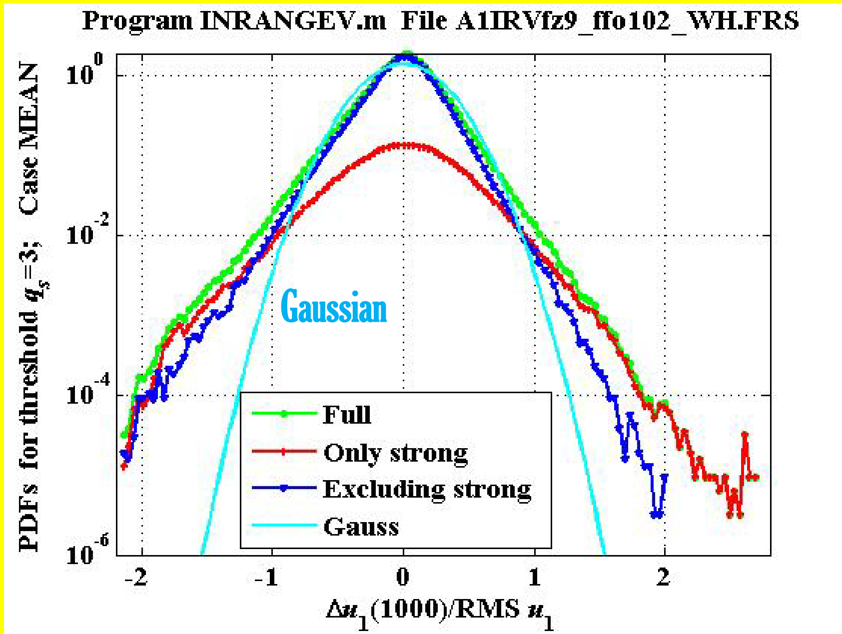
Manganin is used as a material for the sensor prongs instead of tungsten because the temperature coefficient of the electrical resistance of manganin is 400 times smaller than that of tungsten.

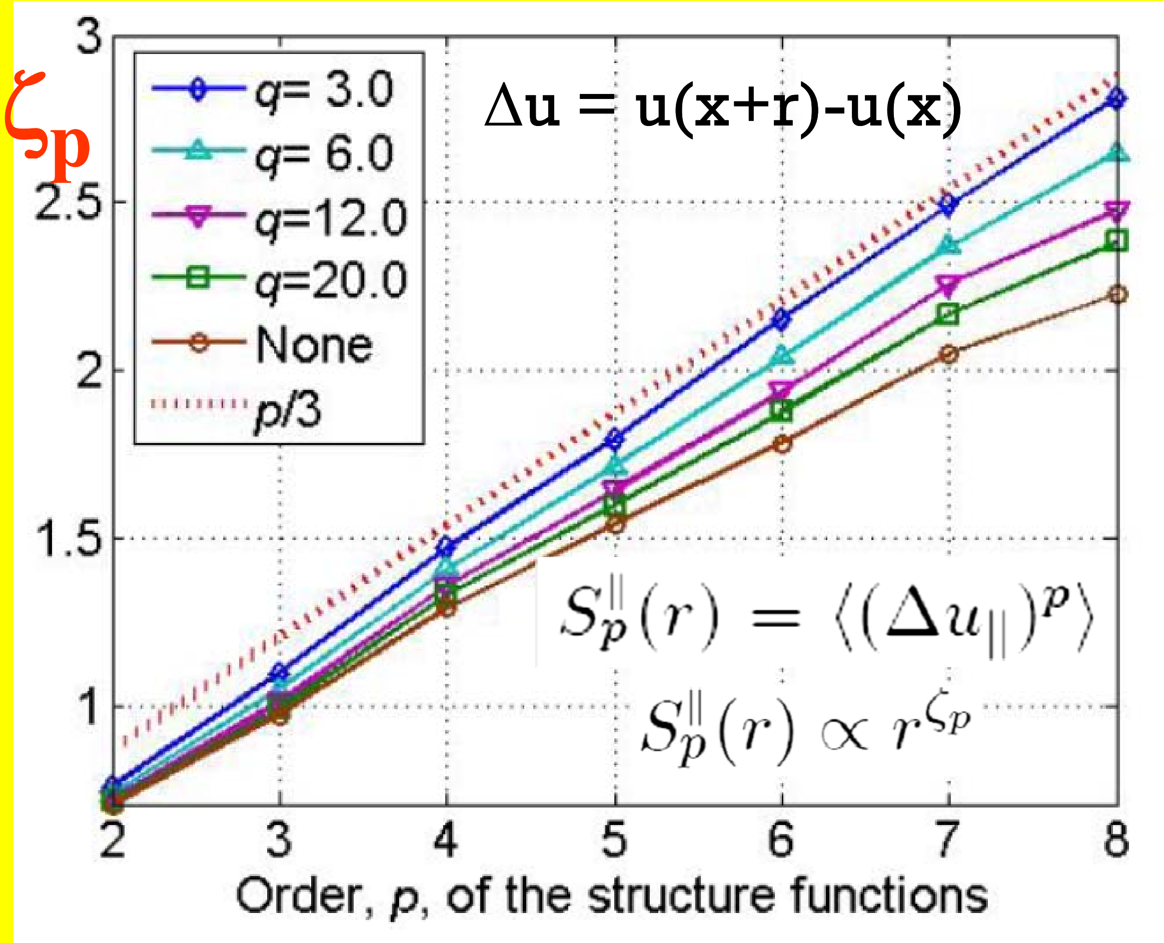
HISTOGRAMS of the increments of the longitudinal velocity component for the full data and the same data in which the strong dissipative events with different thresholds were removed. **a)** $r/\eta = 100$ corresponds to the lower edge of the inertial range, **b)** $r/\eta = 400$, 1000 deep in the inertial range . Note that the PDFs with removed strong dissipative events (dark blue ones) are **not** close to the Gaussian curve as claimed in some later publications.

An event $\Delta u = u(x+r)-u(x)$ *is qualified as a strong dissipative if at least at one of its ends* $(x, x+r)$ *the instantaneous dissipation* $\varepsilon > q\,\langle\varepsilon\rangle$ *for* $q > 1$









SCALING EXPONENTS, ζ_p, of structure functions for the longitudinal velocity component for the full data and the same data in which the strong dissipative events with different thresholds were removed

An event $\Delta u = u(x+r)-u(x)$ is qualified as a strong dissipative if at least at one of its ends (x, x+r) the instantaneous dissipation

$\varepsilon > q \langle \varepsilon \rangle$ for $q > 1$

The 'anomalous scaling' is not and attribute of the conventionally defined inertial range (CDIR) and is not a manifestation of IR intermittency. It is due to the strong dissipative events <u>within !</u> the CDIR.

No need for MF "formalism" ?!?

For more see T2009 , pp. 102-110 and references therein, also Borisenkov, Y., M. Kholmyansky, M., Krylov, S., Liberzon, A. and Tsinober, A. (2011) Super-miniature multi-hot-film probe for sub-Kolmogorov resolution in high-Re turbulence, *Journal of Physics: Conference Series*, **318**, 072004.

The 4/5 law is not a pure inertial relation at large Re

$$S_3^{\|}(r) = -(4/5)\langle\varepsilon\rangle r + 6\nu\, dS_2^{\|}(r)/dr,$$

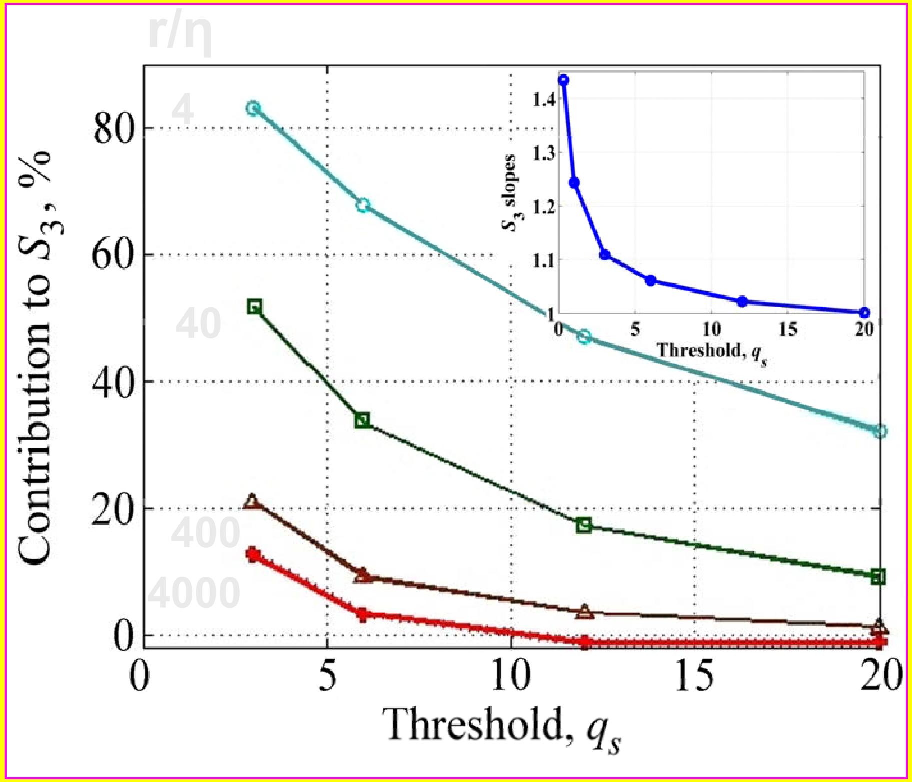
$S_p^{\|}(r) = \langle(\Delta u_{\|})^p\rangle$ $\Delta u_{\|} \equiv [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})]\cdot \mathbf{r}/r,$

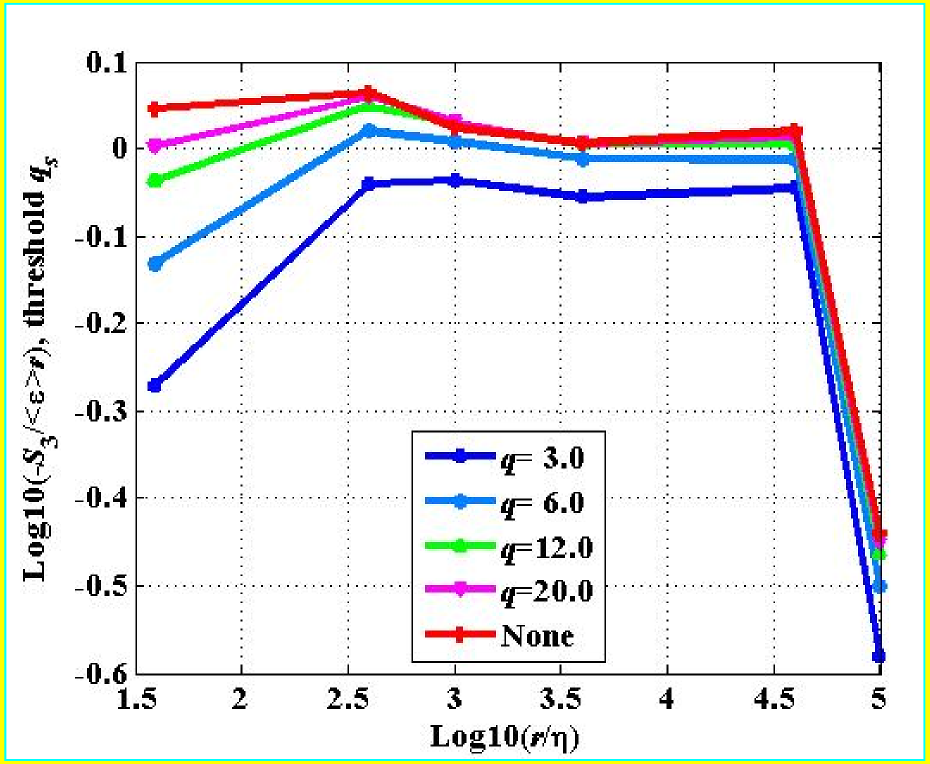
A crucial point here is that the negligible (and neglected at large Re) viscous term $6\nu\, dS_2^{\|}(r)/dr,$ in the Kolmogorov 4/5 law at large Re does not contain ALL the viscous contributions. Namely, **strong dissipative events present in the structure function** $S_3^{\|}(r)$ **itself in the IR do contribute to the 4/5 law and keep the 4/5 valid: without the dissipative events just mentioned the 4/5 law does not hold! Removal of the strong dissipative events (STE) leads to *i)* a decrease of** $-(4/5)\langle\varepsilon\rangle r$ **below unity, which means that the STE make a positive contribution to the energy transfer and *ii)* an increase of the scaling exponent above unity**, see next slide

In this sense the 4/5 law is not a purely inertial law.

$$S_3(r,t) = -\frac{4}{5}\varepsilon r + 6\nu\frac{\partial S_2}{\partial r} - \frac{3}{r^4}\int_0^r r'^4 \frac{\partial S_2}{\partial t}\, dr',$$

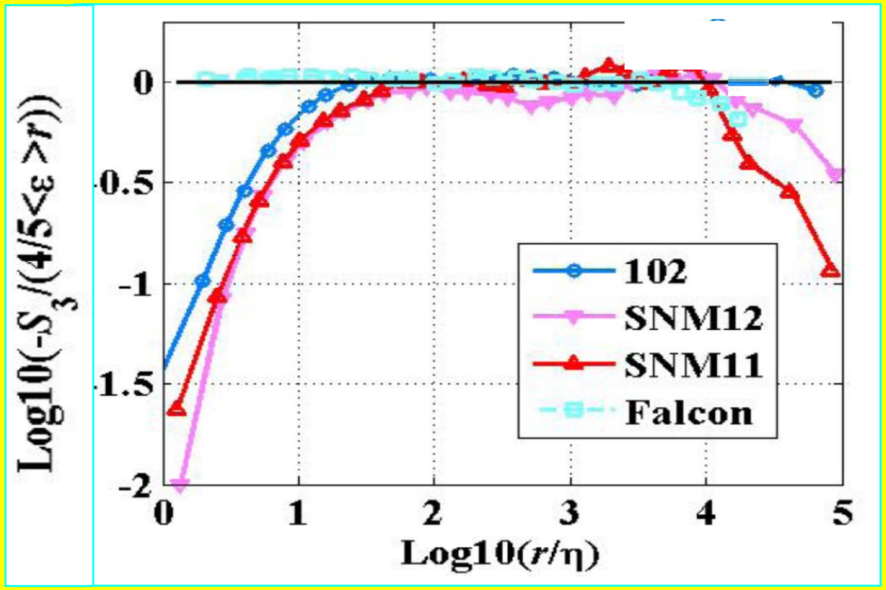
Eq (34.20)p.139 Landau and Lifshits, 1987

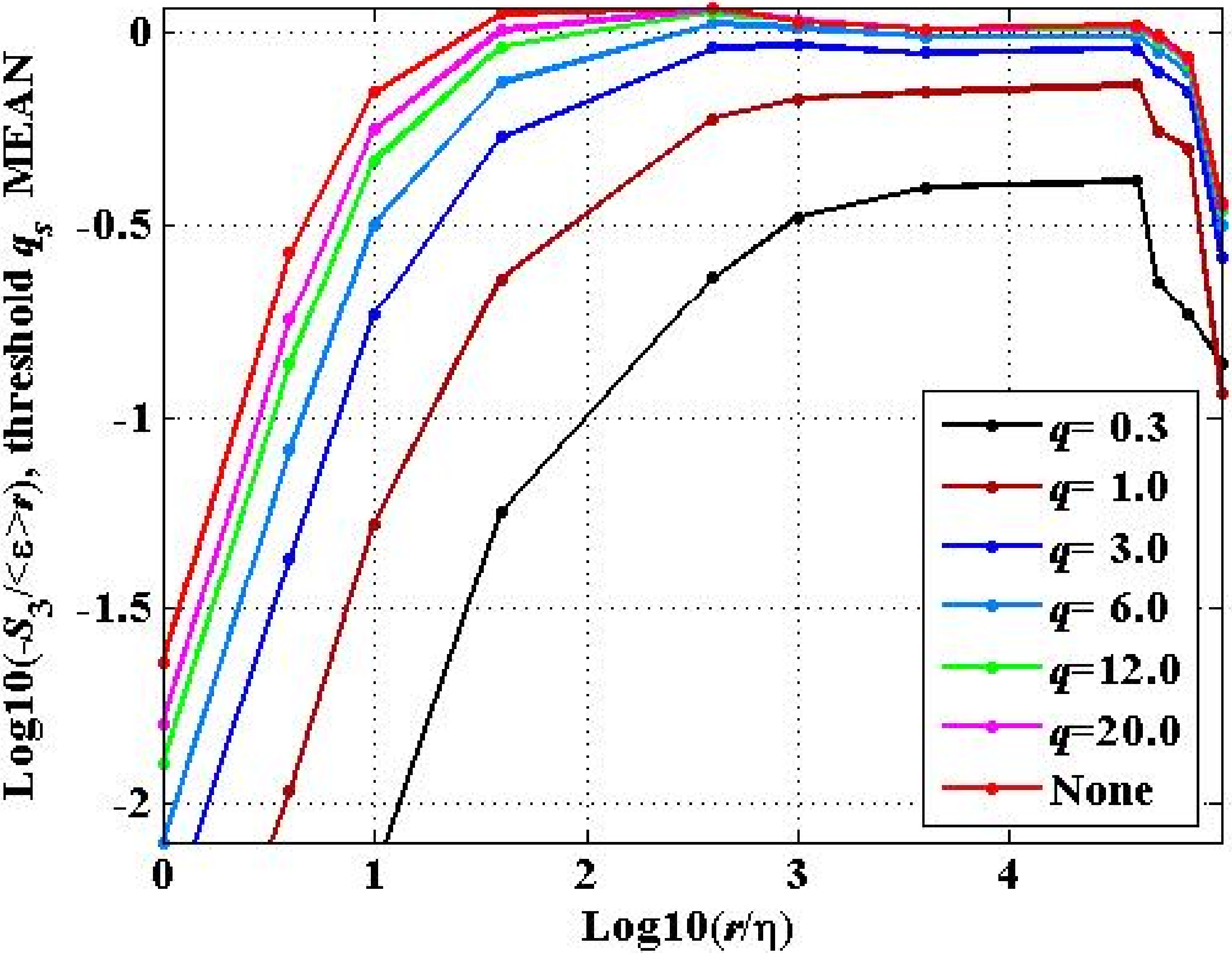




Top: Contributions of the strong dissipative events to the third-order structure function as a function of the threshold q_s for various separations r.
In the insert: scaling exponents of the third-order structure function as a function of the threshold.

Top right: Zoom in a plot of the 4/5 law (shown at bottom) with removed strong dissipative events.





0
-0.5
-1
-1.5
-2
0
1
2
3
4
Log10(-S3/<ε>r), threshold qs MEAN
Log10(r/η)
q= 0.3
q= 1.0
q= 3.0
q= 6.0
q=12.0
q=20.0
None

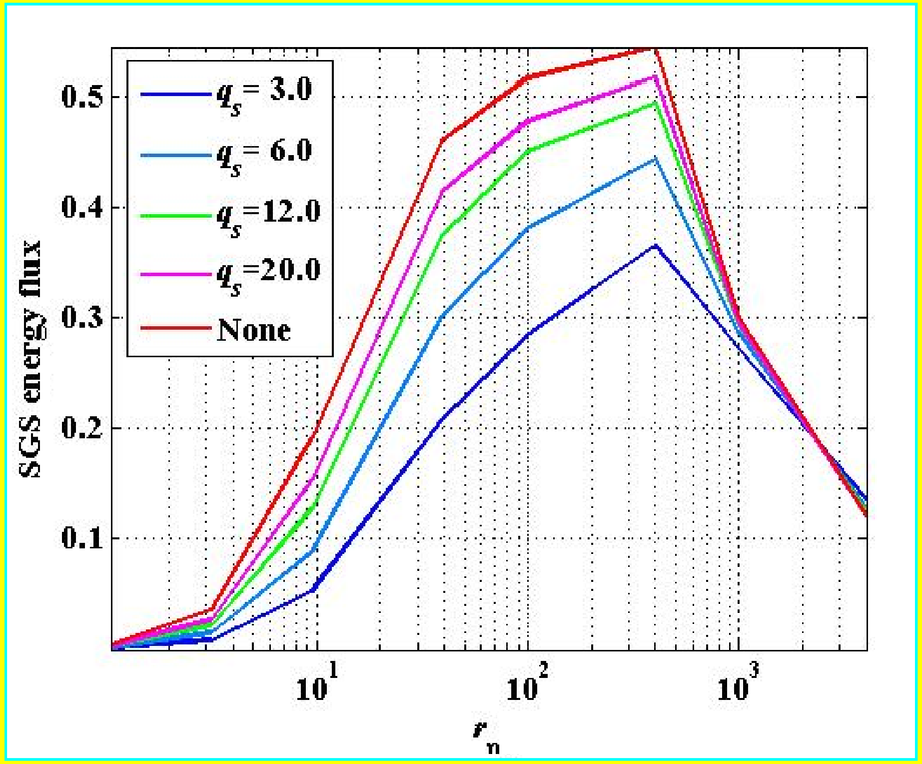
The subgrid scale energy flux Π

$$\Pi(x;r) = -\tau_{ik}\,[s_{ik}];\ \tau_{ik} = [u_i u_k] - [u_i][u_k]$$

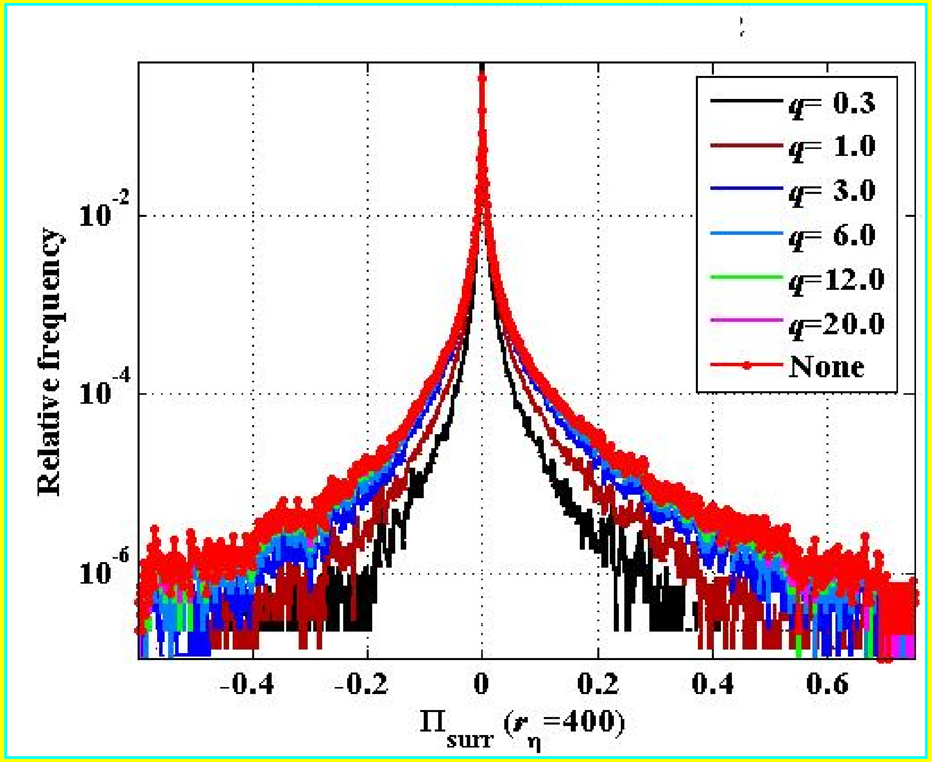
[...]- a Gaussian one-dimensional filter of width **r**

The results shown here are by necessity of qualitative nature as we used one-dimensional filter which was a standard Gaussian filter of width r.

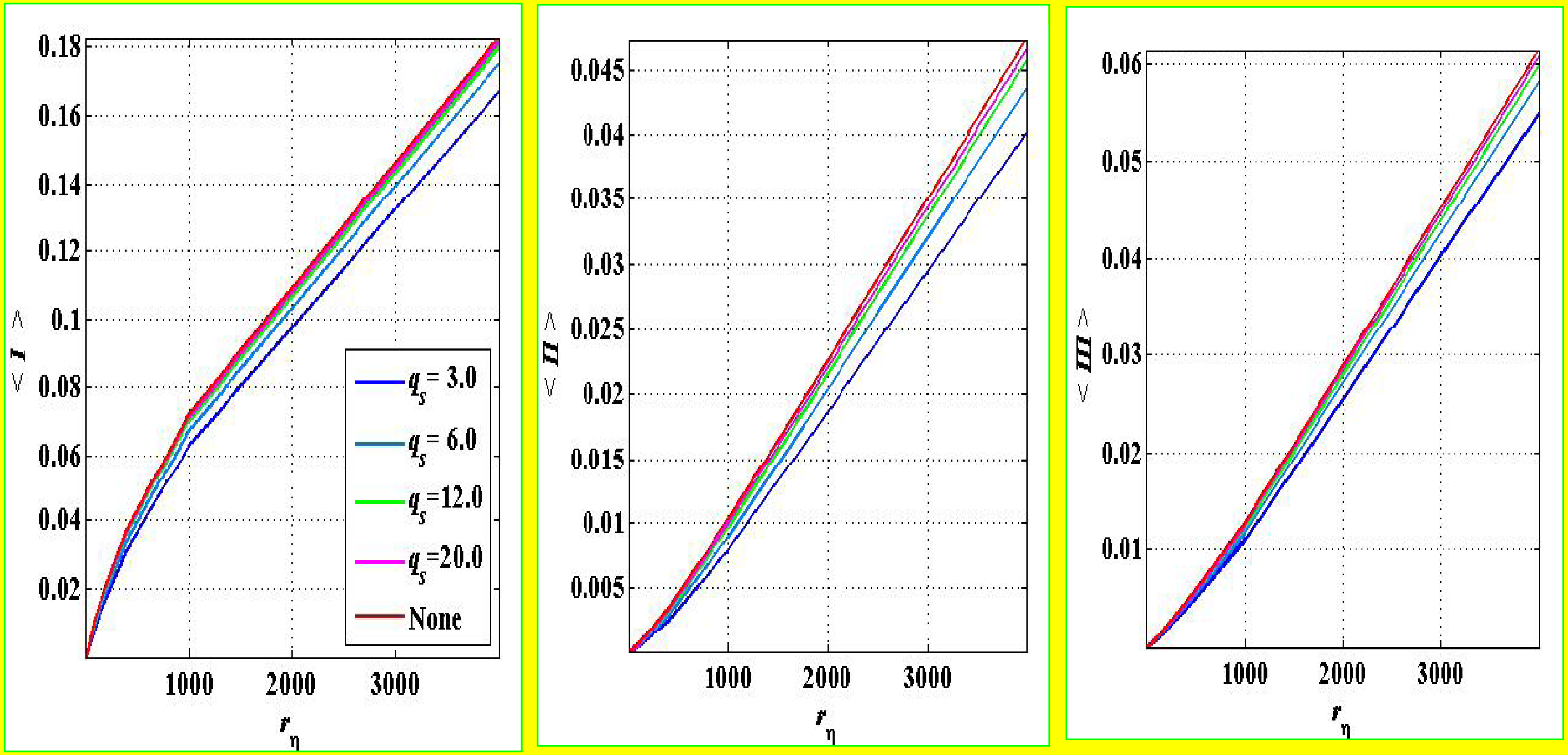
$\langle\Pi\rangle$



PDF



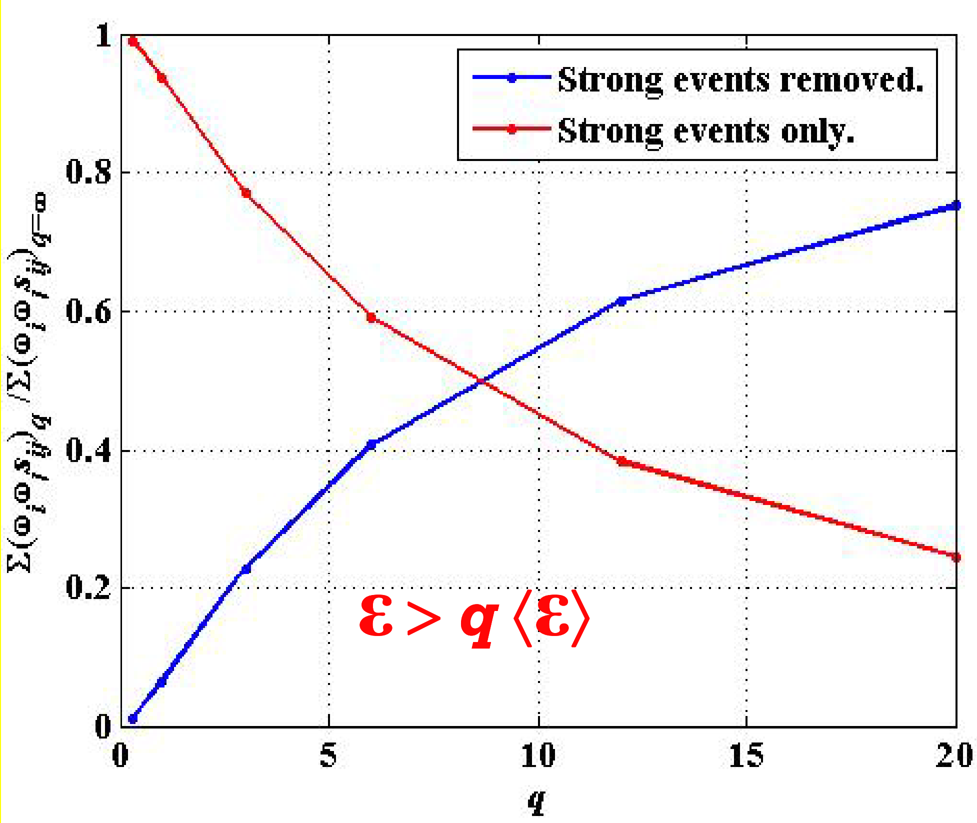
The invariants of the subgrid stress $\tau_{ik}=[u_i u_k]-[u_i][u_k]$

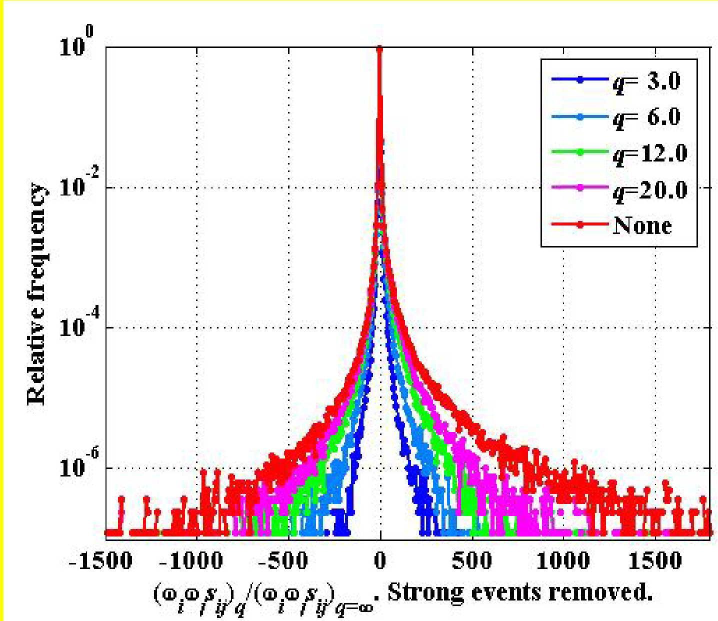


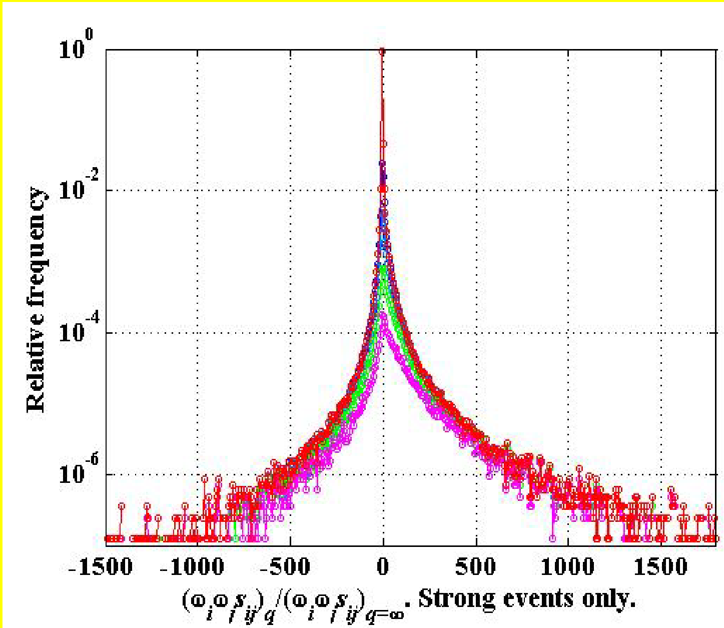
Contributions of strong dissipative events to enstrophy production $\omega_i\omega_j s_{ij}$

The common view is that the origin of enstrophy production and similar processes is purely inertial. However, it appears that it contains a substantial contribution from the dissipative events.

$$d\omega^2/dt = \omega_i\omega_j s_{ij} + \nu\omega_i\Delta\omega_i.$$







Borisenkov, Y., M. Kholmyansky, M., Krylov, S., Liberzon, A. and Tsinober, A. (2011) Super-miniature multi-hot-film probe for sub-Kolmogorov resolution in high-Re turbulence, *Journal of Physics: Conference Series*, 318, 072004.

All the above shows that the conventionally defined inertial range is not well defined . On the simplest level this means that it is contaminated by strongly dissipative events (along with large scale events either) so the one cannot employ in dimensional anlysis the standart variables from the K41.

A BET

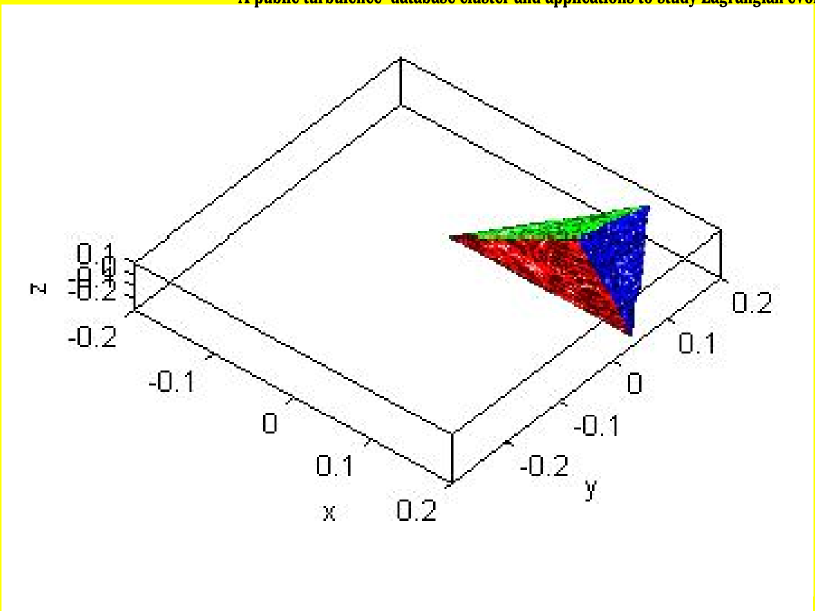
It is likely, therefore, that the same is true of Lagrangian setting, so that removing (from the statistics) will result in the scaling $\varepsilon\tau$ so far not observed for the second order Lagrangian structure. This is the bet I proposed during the discussion at May 10.

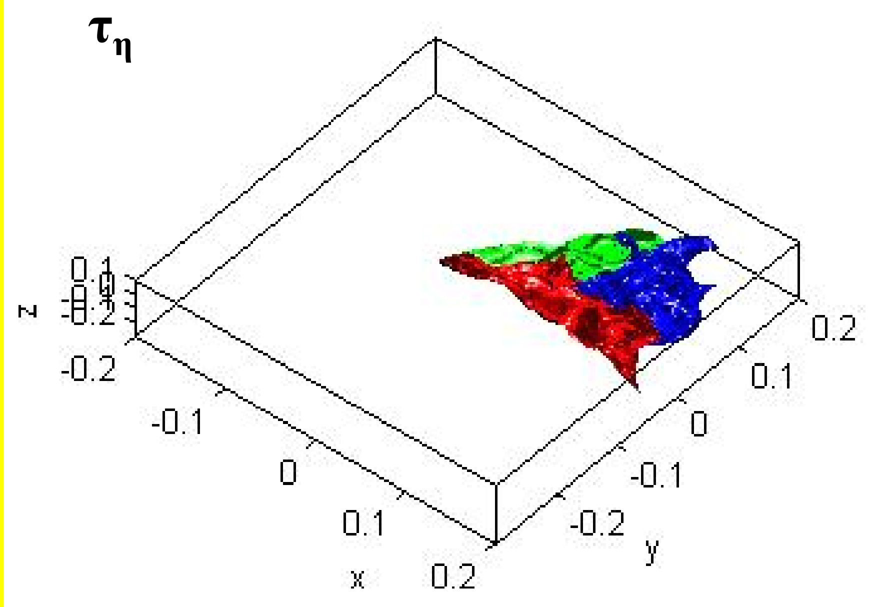
1. Is the sweeping really kinematic? Is it true that small scales are statistically independent of the small scales? Or even more rebelliously – do small scales have any impact on the large scales except of overall dissipation? **Do large-scale motions merely convect, bodily *, regions small compared to the macro scale, i.e. it preserves the shapes of the advected small scale eddies ?**

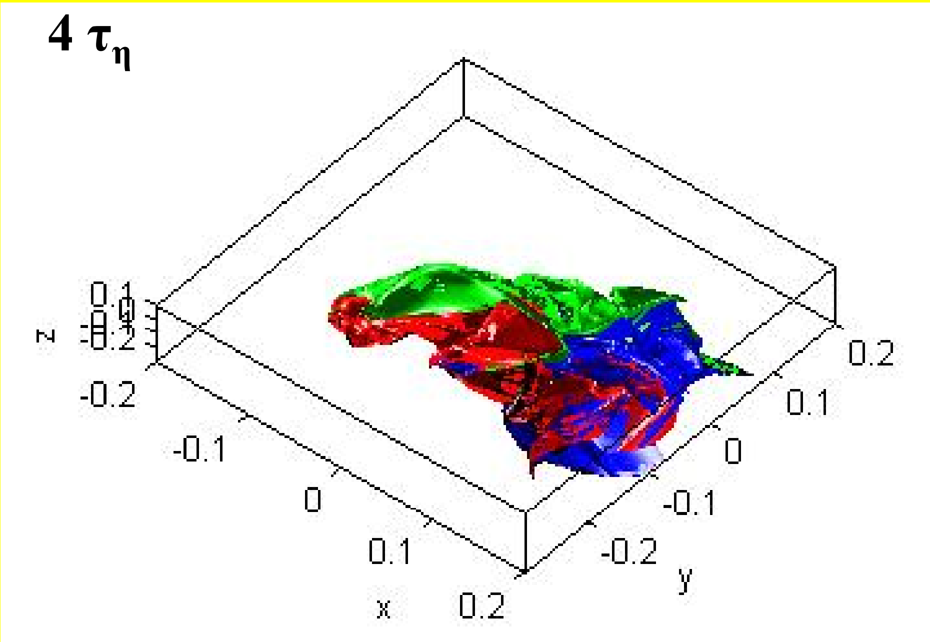
In view of the above the answer is negative , see also the two next slides.

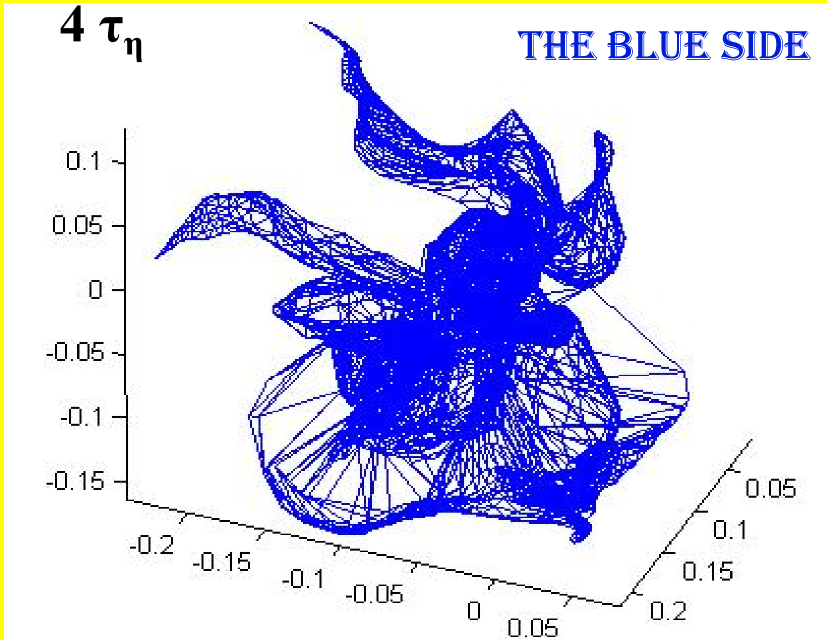
Evolution of a tetrahedron with edge of $\approx 4\eta$ *at* $t = 0$ *(courtesy Beat Luethi)*

Li, Y., Perlman, E., Wan, M., Yang, Y., Burns, R., Meneveau, C., Burns, R., Chen, S., Szalay, A. & Eyink, G. 2008
A public turbulence database cluster and applications to study Lagrangian evolution of velocity increments in turbulence. *J. Turbulence,* **9** (31) 1–29.



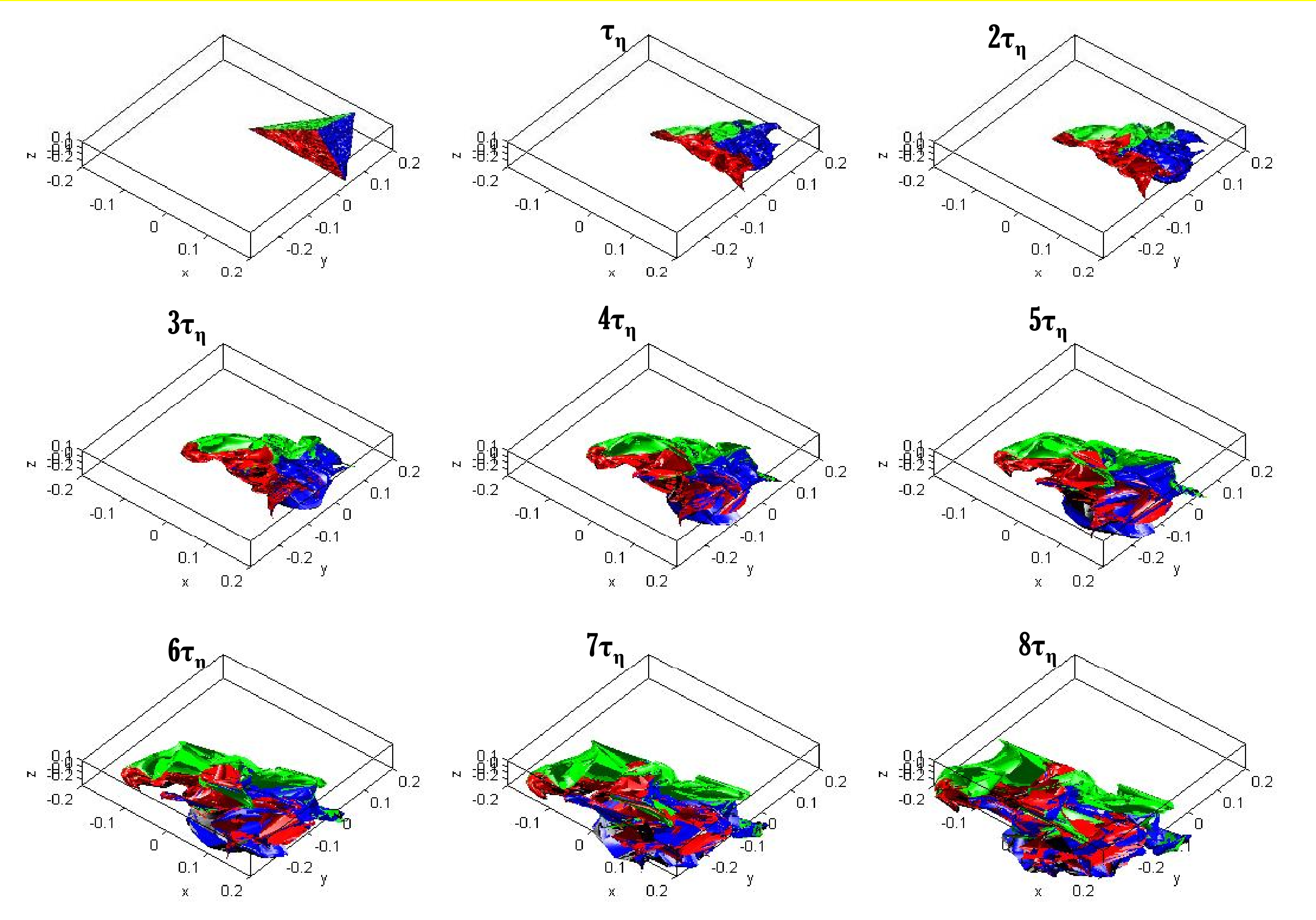






Evolution of a tetrahedron with edge of 4η *at* $t = 0$ *(courtesy Beat Luethi)*

Li, Y., Perlman, E., Wan, M., Yang, Y., Burns, R., Meneveau, C., Burns, R., Chen, S., Szalay, A. & Eyink, G. 2008
A public turbulence database cluster and applications to study Lagrangian evolution of velocity increments in turbulence. *J. Turbulence,* **9** (31) 1–29.



In view of the above there are a number of questions which the "simpleton Wilson" would like to ask and discuss.

1. Is the sweeping really kinematic? Is it true that small scales are statistically independent of the small scales? Or even more rebelliously – do small scales have any impact on the large scales except of overall dissipation? Do large-scale motions merely convect, bodily *, regions small compared to the macro scale, i.e. it preserve the shapes of the advected small scale eddies

2. Theoreticians have reasons to "remove" in some sense the sweeping (hence the crucial function of the hypotheses) . The question is in what sense the equations with removed sweeping are equivalent to the original ones and/or how the small scales do know about the removed large scales (or they shouldn't and/or are not supposed to) or are the small scales in the equations with removed large scales equivalent to the small scales in the original equations? Or is SDH/RTH the only "answer" to the latter question?

3. Related to 2. on the need of removing the sweeping : why nature does not need this removal in order to produce the right result?

4. The SDH/RTH - which is kind of decomposition - ignores too much from the interaction (in the first place dynamical) between the large and small scales, i.e. it is `too kinematic'. It is really justified except being convenient for theoreticians?

More on

In what sense are SDH/RTH valid?

Or why - being ***approxmately*** *correct -*

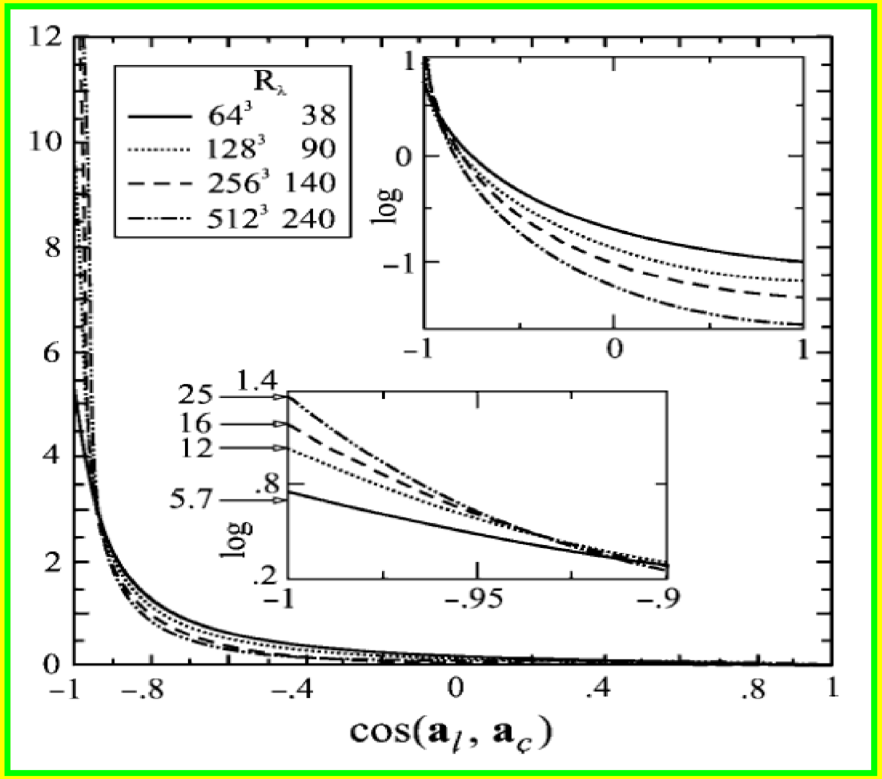
they are ***erroneous conceptually***

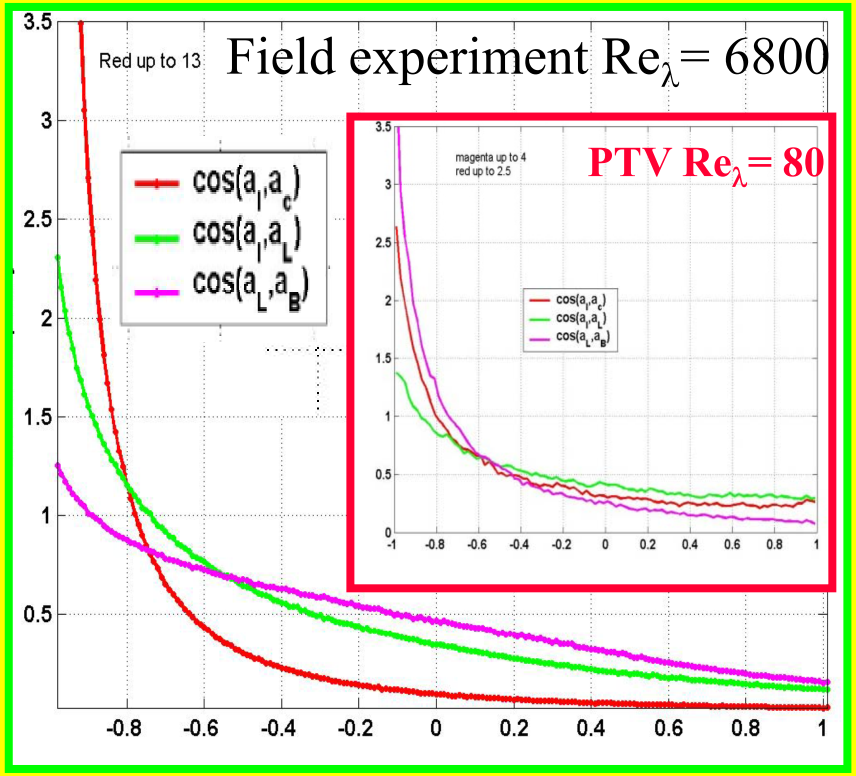
It appears that for many quantities (scalars, vectors, tensors) the Lagrangian derivative $DQ/Dt = \partial Q/\partial t + u_k \partial Q/\partial x_k$ is much smaller in some sense than its Eulerian components, $\partial Q/\partial t$ and $u_k \partial Q/\partial x_k$ *. This is true of fluid particle accelerations, e.g., $\langle a^2\rangle/\langle a^2{}_l\rangle$ and $\langle a^2\rangle/\langle a^2{}_c\rangle << 1$ with $\mathbf{a} = \mathbf{a}_l + \mathbf{a}_c$ and $\mathbf{a}_l = \partial \mathbf{u}/\partial t$, $\mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$, and comprizes the basis of the Random Taylor hypothesis in which Tennekes (1975) put just $\mathbf{a} = 0$. This assumption is local point-wise in space/time and is not a statistical one. The second assumption made by Tennnekes is of statistical nature, namely, that ***the microstructure is statistically independent of the energy containing eddies.*** The equality $\mathbf{a} = 0$ should and cannot not be understood literally (just as the microstructure is not statistically independent of and even not decorrelated from the energy containing eddies). It is obvious that the acceleration of fluid particles cannot be vanishing, since othrewise one one may arrive to the conclusion that both equations $\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0$ and $-\rho^{-1}\nabla p + \nu\nabla^2\mathbf{u} = 0$ are valid , which is trivially incorrect !

---

*Galanti, B., Gulitsky, G., Kholmyansky, M., Tsinober, A. & Yorish, S. 2003 Velocity derivatives in turbulent flow in an atmospheric boundary layer without Taylor hypothesis. In : Turbulence and Shear Flow Phenomena (ed. N. Kasagi, J. Eaton, R. Friedrich, J. Humphrey, M.Leschziner & T. Miyauchi), , vol. II, pp. 745–750.

The smallness of **a**, e.g. $\langle a^2\rangle/\langle a^2{}_l\rangle$ and $\langle a^2\rangle/\langle a^2{}_c\rangle << 1$ is possible if there is mutual (statistical) cancellation between the local acceleration, $\mathbf{a}_l$, and convective acceleration, $\mathbf{a}_c$; $\mathbf{a}_l = \partial \mathbf{u}/\partial t$, $\mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$. Since these quantities are vectors, the degree of this mutual cancellation should be studied both in terms not only of their magnitude but also of the geometry of vector alignments. Indeed there is a strong anti-alignment between the two .





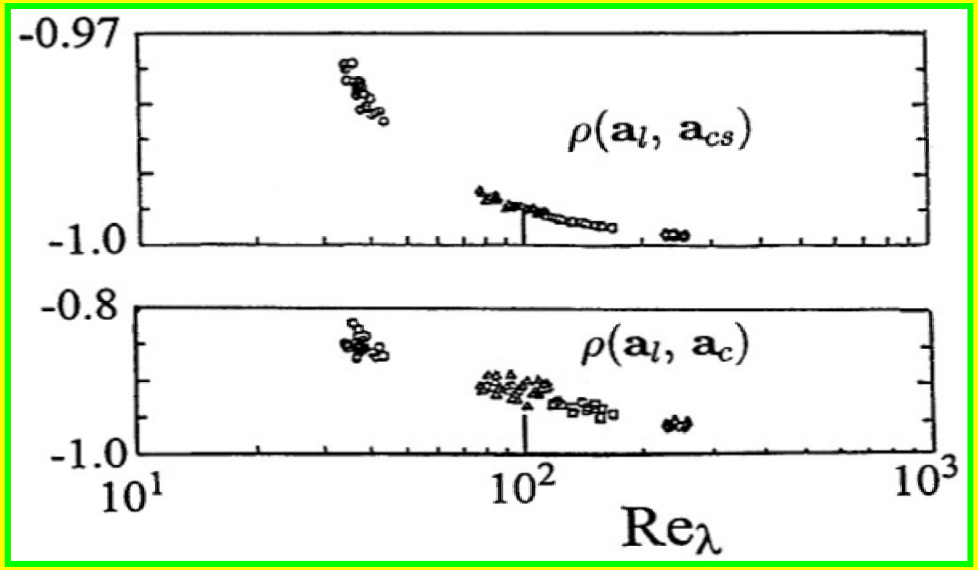
PDFs of the cosine of the angle between $\mathbf{a}_l$ and $\mathbf{a}_c$. The insets show this dependence with the vertical in log and in the proximity of $\cos(\mathbf{a}_l, \mathbf{a}_c) \approx -1$. This alignment was observed first in DNS (Tsinober et al 2001) - left, and in laboratory experiments (Lüthi et al., 2005) and in the atmospheric surface layer Gulitski et al. (2007) – right.

An important observation is very high values of correlation between

$a_l = \partial u/\partial t$ and $a_c = (u \cdot \nabla)u$

In other words the approximation $a \approx 0$ is very good and becomes better with increasing Reynolds numbers. This is true also of the validity of the Random Taylor Hypothesis (RTH) or sweeping decorrelation hypothesis (SDH).

The important point is though both are *approximately* kinematic, this - as mentioned above - does not mean that the non-kinematic "small difference" justifies the validity of the equations $\partial u/\partial t + (u \cdot \nabla)u = 0$ and $\rho^{-1}\nabla p + \nu\nabla^2 u = 0$. It is this "small non-kinematic difference" which is mostly responsible for all the dynamics in the Eulerian representation.



Correlation coefficients between a_l and a_c, and a_l and a_{cs}, DNS (Tsinober 2001, Tsinober et al 2001) The latter is the solenoidal part of a_c. Simlar results for a_l and a_c obtained in laboratory experiments (Lüthi et al., 2005) and in the atmospheric surface layer Gulitski et al. (2007)

Thus sweeping cannot be considered as just a kinematic effect. The dynamics involved is of utmost importance!

A critical remark on the nature of Kolmogorov-Kraichnan-Tennekes 'decomposition': The sweping decorrelation and/or random Taylor hypothesis

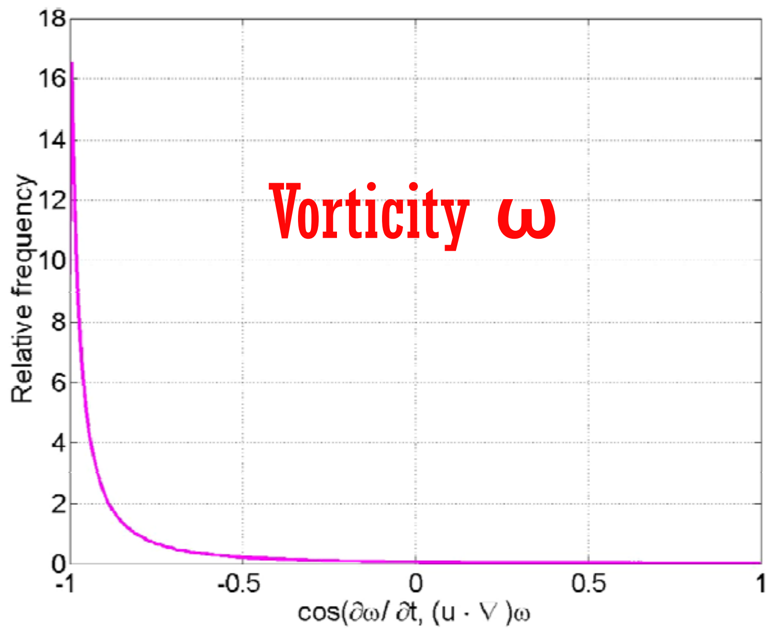
There are two main ingredients in the (Eulerian) decorrelation: i) – the sweeping of microstucture by the large-scale motions (and associated kinematic nonlocality), ii) – and the local straining (which is roughly pure Lagrangian). As seen from above this kind of "decomposition' is insufficient as it is missing an essential dynamical aspect – the interaction between the two*. The random Taylor hypothesis (and, of course, the conventional Taylor hypothesis) lack/discard this aspect at the outset (this does not mean that these hypotheses are useless): both are 'too kinematic' (while acceleration is a dynamic quantity in the first place). A closely related issue is with the rather popular assumption that choosing an appropriate 'local' system of reference one can get rid (mostly) of the sweeping of the small scales by the large-scale motions. The underlying assumption is that small scales are 'passive' and just 'swept' by the large scales without any participation in the process, i.e., 'slaved' without any reaction back. This is a major misconception: there is a rich direct an bidirectional coupling between large and small scales.

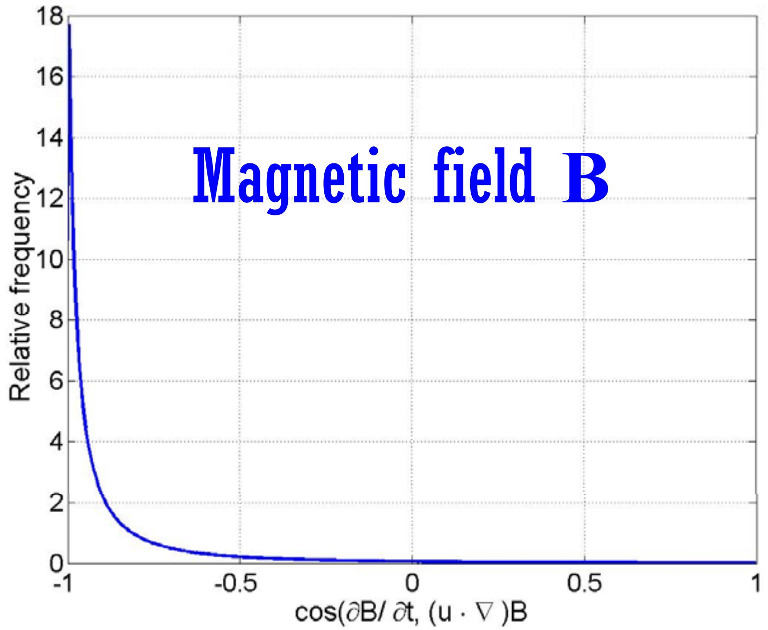
---

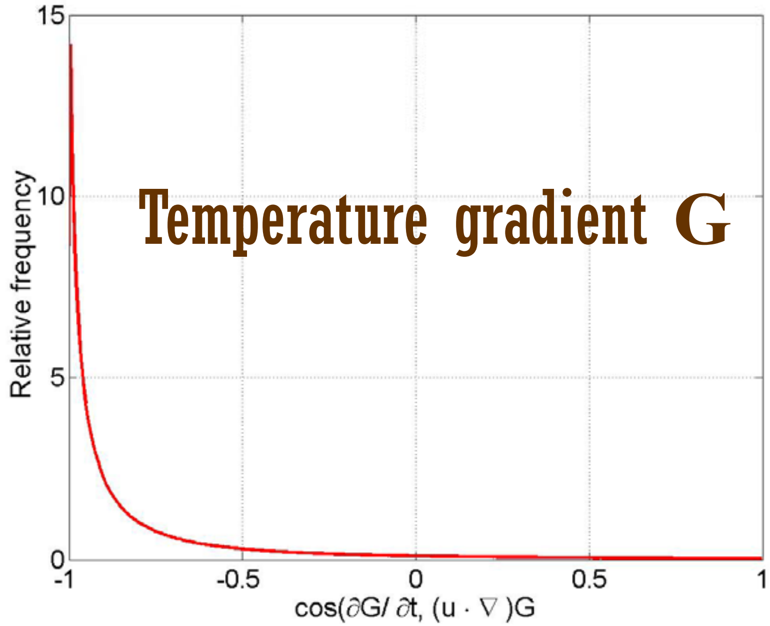
* The 'random sweeping decorrelation hypothesis' means that the microstructure (whatever this means) is *statistically decorrelated* from the energy containing eddies. This is different from the original assumption made by Tennekes (1975) and before , in which he held that *the microstructure is statistically independent* of the energy containing eddies. The large and small scales are statistically not independent, though they are weakly correlated. Indeed, there is a variety of manifestations of direct and bidirectional impact/coupling of large and small scales. The issue of sweeping is closely related to the comparative aspects of Lagrangian versus Eulerian descriptions – an issue of utmost importance and difficulty.

Examples of alignments between $\partial Q/\partial t$ and $u_k \partial Q/\partial x_k$ for vorticity, temperature gradient and magnetic field from DNS of NSE

Galanti, B., Gulitsky, G., Kholmyansky, M., Tsinober, A. & Yorish, S. 2003 Velocity derivatives in turbulent flow in an atmospheric boundary layer without Taylor hypothesis. In : Turbulence and Shear Flow Phenomena (ed. N. Kasagi, J. Eaton, R. Friedrich, J. Humphrey, M.Leschziner & T. Miyauchi), , vol. II, pp. 745–750.







Part II. Relation(s) between Eulerian and Lagrangian descriptions -representations of turbulent flows.

This is a long-standing and most difficult problem posed by Corrsin in 1957. The general reason is because the Lagrangian field is an extremely complicated non-linear functional of the Eulerian field and vice versa (there is also a problem of invertibility). The complexity of this relation can be seen in the example of Lagrangian turbulence (chaotic advection) with *a priori* prescribed and not random Eulerian velocity field (E-laminar) among others. In this extreme example (which in reality is a set containing almost all E-laminar flows) the Lagrangian statistics has no Eulerian counterpart. In other words, generally, it may be meaningless to look for such a relation. In a sense it belongs to the category of THE questions:

... the possession of such relationship would imply that one had (in some sense) solved the general turbulence problem. Thus it seems arguable that such an aim, although natural, may be somewhat illusory. Nevertheless attempts to realize this aim can teach us about the subject... McCOMB, 1990

1. Except of the formal kinematic relation

$$\partial \mathbf{X}(\mathbf{a},t))/\partial t = \mathbf{u}[\mathbf{X}(\mathbf{a},t); t] \qquad \{\mathbf{E\text{-}L}\}$$

where $\mathbf{u}(\mathbf{x},t)$ is the Eulerian velocity field and $\mathbf{X}(\mathbf{a},t)$ is the fluid particle trajectory and $\mathbf{a}$ is its label , the common question(s) include in the first place statistics in the broad sense.

The usual question is whether "simple" relations do exist between the E - statistics and L - statistics. This is a long-standing and most difficult problem. The general reason is because the Lagrangian field $\mathbf{X}(\mathbf{a},t)$ (and Lagrangian velocities $\mathbf{v}[\mathbf{X}(\mathbf{a}, t); t]$) is impossibly complicated functional of the Euler velocity field $\mathbf{u}(\mathbf{x},t)$. Roughly, there is a general relationship in terms of path (Feynman, functional) integrals, but this does not help much, if at all. For more on these issues see Monin and Yaglom (1971, 1, Ch. 9, pp. 568--578), also Bennet (2006, pp. 21-24). The start was made by Corrsin (1959a,b) and Lumley (1962a,b).

One can even claim that, generally, there cannot be a simple relation and in a sense even any relation as seen from the following counter-example(s).

Most of laminar flows in the Eulerian setting (E-laminar) are Lagrangian chaotic (L-turbulent) due to non-integrability of the relation $\partial \mathbf{X}(\mathbf{a},t))/\partial t = \mathbf{u}[\mathbf{X}(\mathbf{a},t); t]$ In other words, though the Eulerian velocity field, $\mathbf{u}(\mathbf{x};t)$ is not chaotic and is regular and laminar, the Lagrangian velocity field $\mathbf{v}(\mathbf{a},t) = \mathbf{u}[\mathbf{X}(\mathbf{a}, t); t])$ is chaotic because $\mathbf{X}(\mathbf{a},t)$ is chaotic. Thus almost in all E-laminar but L-turbulent flows the Lagrangian statistics has no Eulerian counterpart.* This can be seen as an indication that the pure Lagrangian dynamical equations (LNSE) are more rich ("more chaotic") than their Navier--Stokes counterpart (ENSE) so that one is tempted to conjecture that LNSE is equivalent to ENSE + E-L. Is this trivial? !

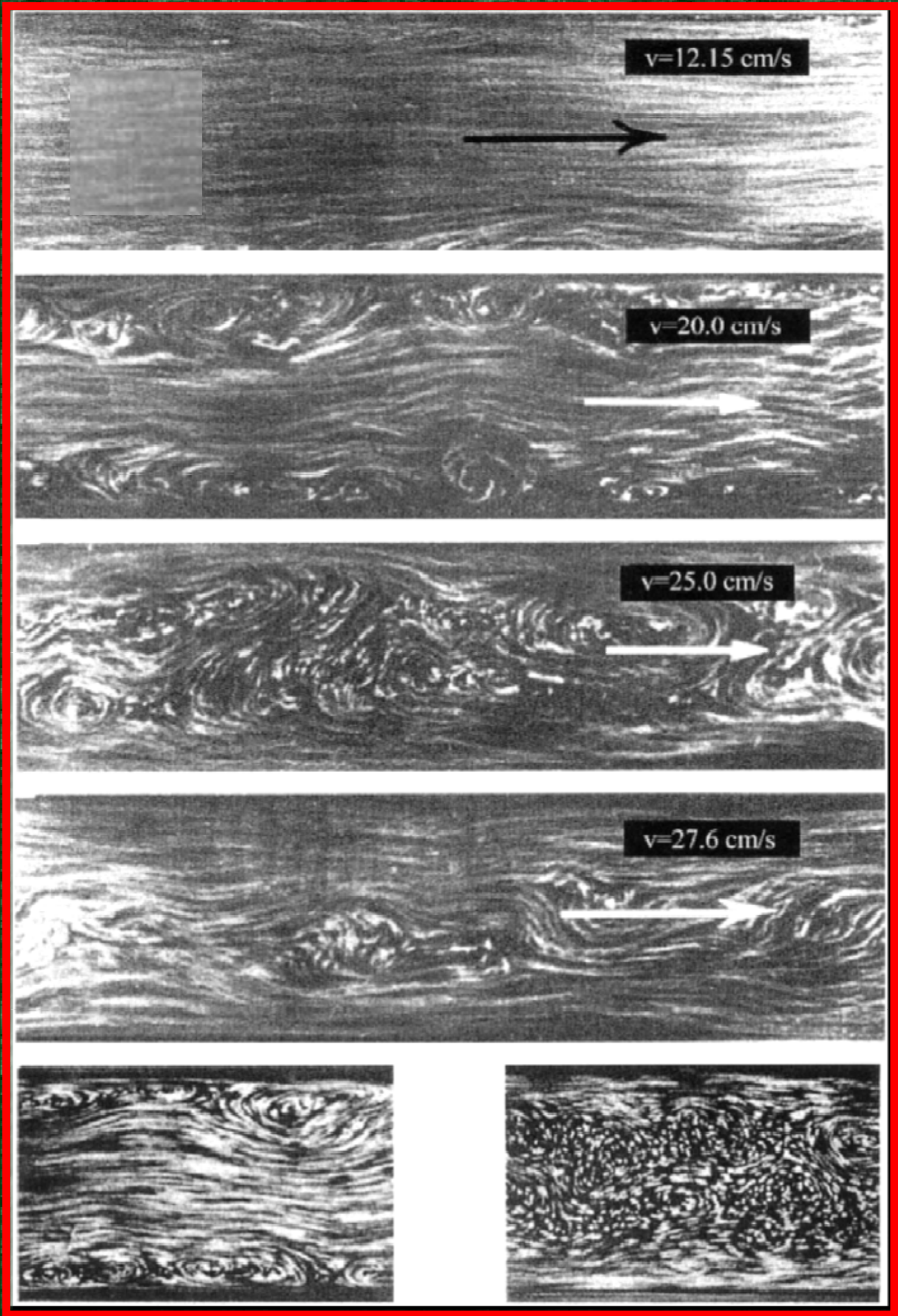
2. The differences are exhibited also in structure(s) and flow visualization (what do we see) – the L-fields may exhibit different flow patterns for the same E-field , and may have nontrivilal structure(s) in the L-setting with no/or trivial structure for the E-setting of the same flow as in examples given above and below.

What about "cascade" in Lagrangian chaotic/Eulerian laminar flows?!

Examples showing how extremely intricate is the L - E relation

Same flow - not the same pattern

Seeing is not necessarily believing



SAME FLOW - NOT THE SAME PATTERN

The four upper pictures, TOLLMIEN 1931, correspond to the visualization of a turbulent water flow in an open 6cm wide channel photographed by a moving camera at different speeds. The mean velocity of the flow is 16.7cm/s. The two lower pictures are from PRANDTL AND TIETJENS 1934. In the right picture, the camera moves with the speed equal to the velocity of water in the centre of the channel. In the left picture, the speed of the camera is small and close to the velocity of the water near the walls

All frames (the four different Lagrangian fields) correspond to the same Eulerian flow

What one sees is real

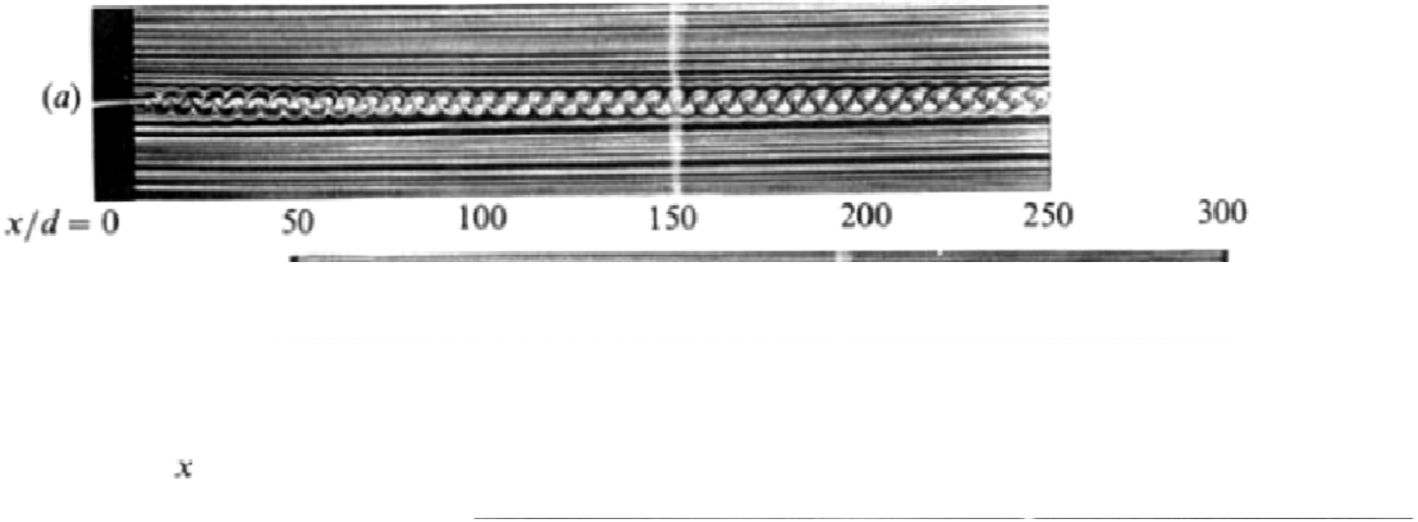


FIGURE 1. Circular-cylinder wake at $Re = 90$; smoke wire at (a) $x/d = 4$, (b) 50, (c) 100 and (d) 150.

The problem is interpretation

Cimbala, J.M., Nagib, H. M and Roshko, A. (1988) Large structures in the far wakes of two-dimensional bluff bodies, J. Fluid Mech., **190**, 265--298.

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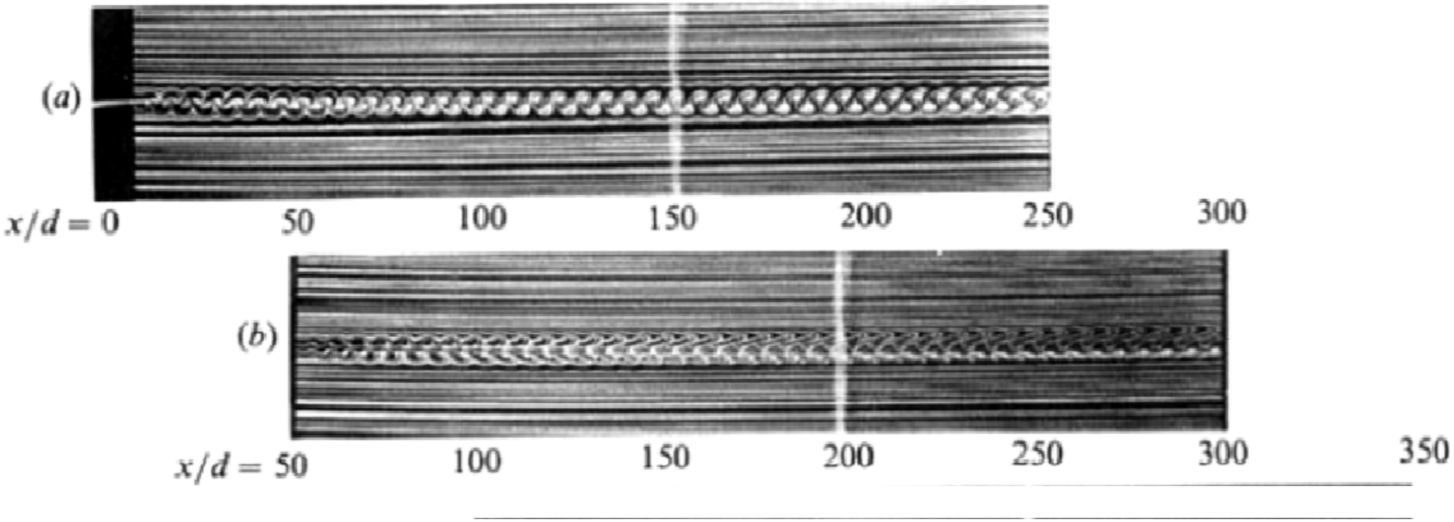

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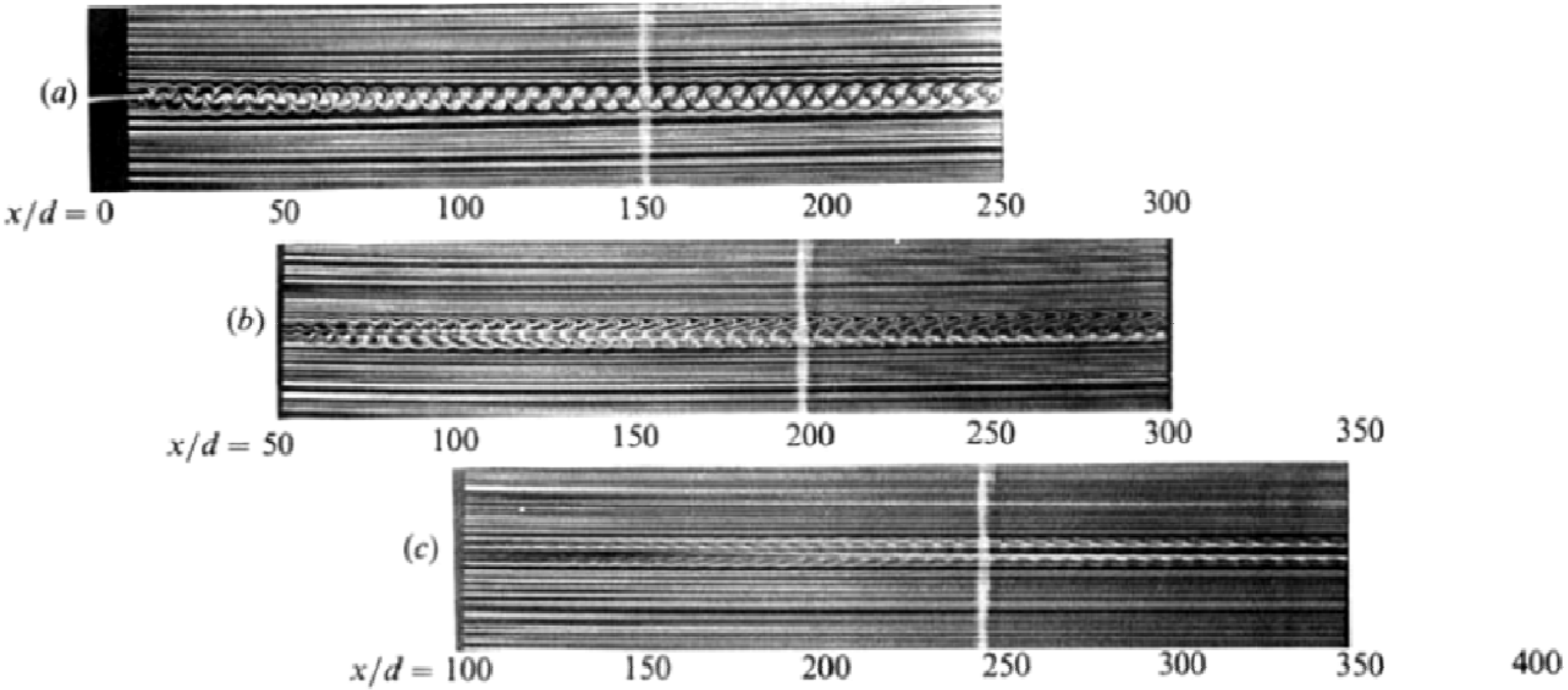

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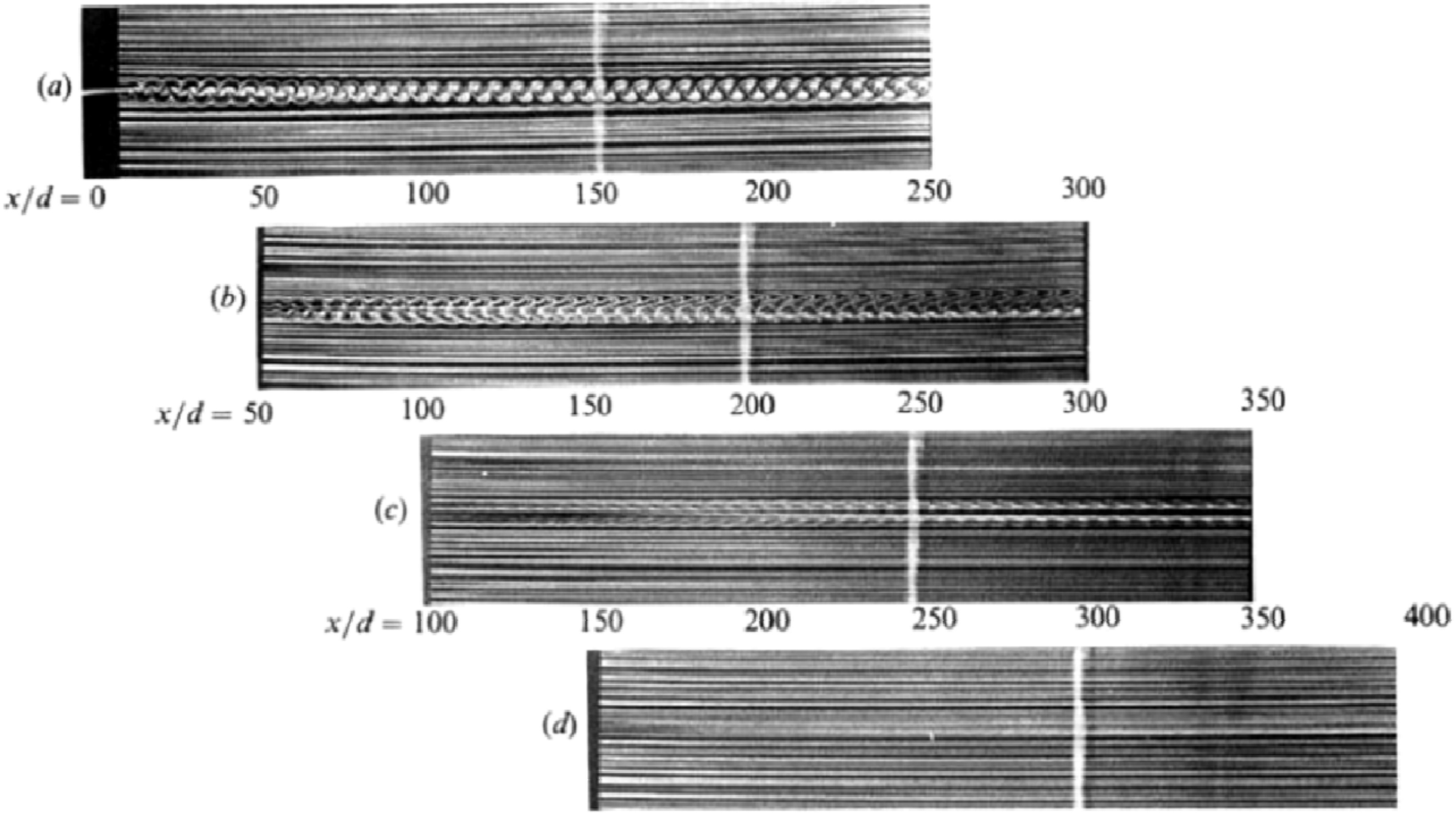

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LAMINAR EULERIAN FLOW AT Re~1
(E-LAMINAR)

BUT

CHAOTIC LAGRANGIAN
(L-TURBULENT)

OR

Kinematics versus Dynamics

E-Laminar but L-turbulent.

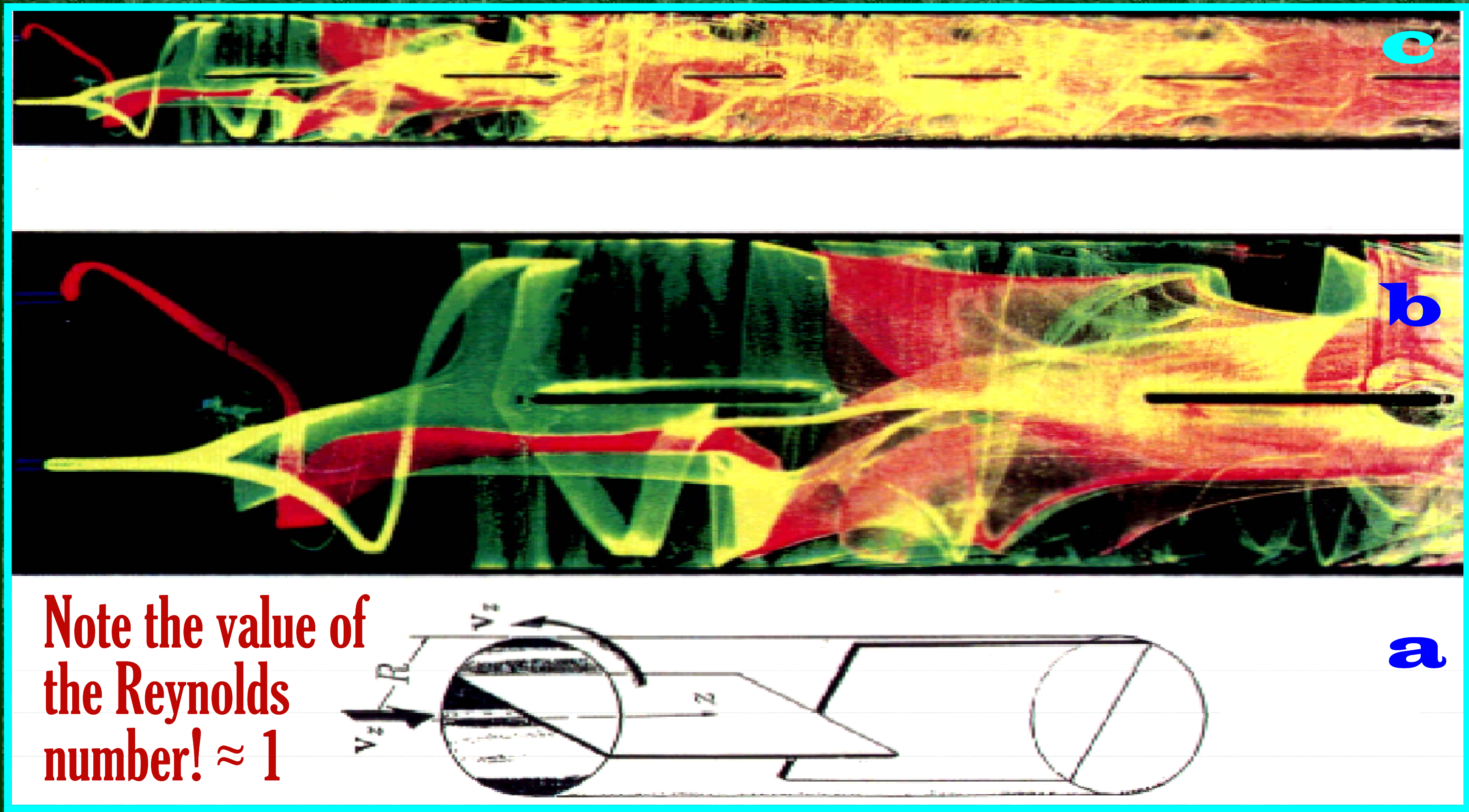
E-turbulent necessarily L-turbulent

A flow at Re$\sim$1 in Eulerian setting is laminar (**E-laminar**), but in Lagrangian setting it is chaotic (**L-turbulent**)

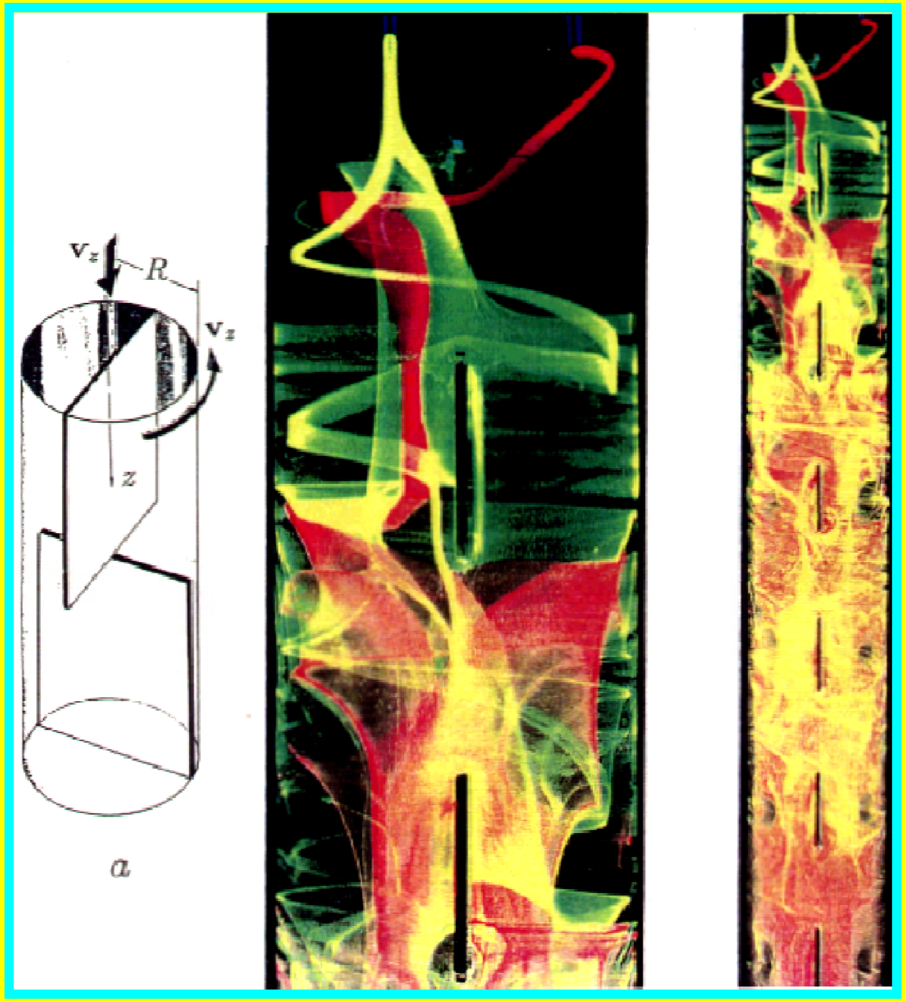
Most of E-laminar flows are L-turbulent

Thus in almost all E-laminar but L-turbulent flows, the Lagrangian statistics has no Eulerian counterpart.

Assuming that laminar flows possess no or trivial statistics.



Mixing in PPM - partitioned-pipe mixer at very low Reynolds number. $Re_{PPM:axial} = \langle v_z \rangle R/\nu = 0.3$ and $Re_{PPM:cs} = v_R R/\nu = 1.8$; here $\langle v_z \rangle$ – average axial velocity and $v_R = \frac{1}{2}(|v_{1|\,max} + |v_{1|\,min})$ – characteristic cross-sectional velocity. $0 < Re_{PPM:axial} < 0.8$ and $0 < Re_{PPM:cs} < 0.8 < 8$. a) schematic of the PPM, b) is a close up of the upper part of c). From Kusch and Ottino (1992).



MIXING IN PMM, KUSH & OTTINO (1992)

RELEVANT TO MICROFLUIDICS with Re ~ 0 (!);

Linked twist maps (LTMs), Bernoulli mixing ...

The complexity and problematic aspects of the relation between the Lagrangian and Eulerian fields is seen in the example of Lagrangian (kinematic) chaos or Lagrangian turbulence (chaotic advection) with a priori prescribed and not random Eulerian velocity field (E-laminar). This is why Lagrangian description - being physically more transparent - is much more difficult than the Eulerian description. In such E-laminar but L-turbulent flows the Lagrangian statistics has no Eulerian counterpart, as in the flow shown at the left.

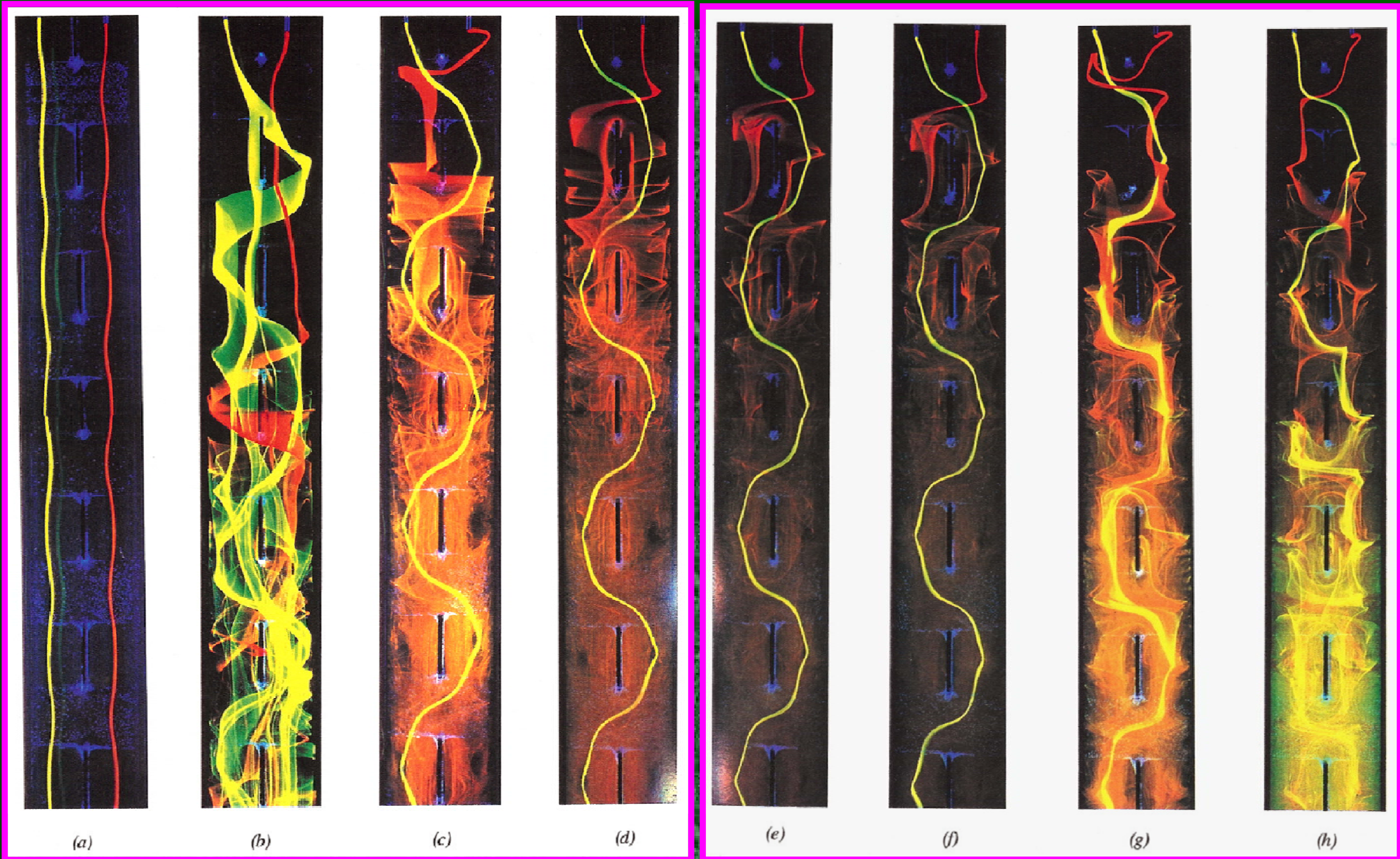
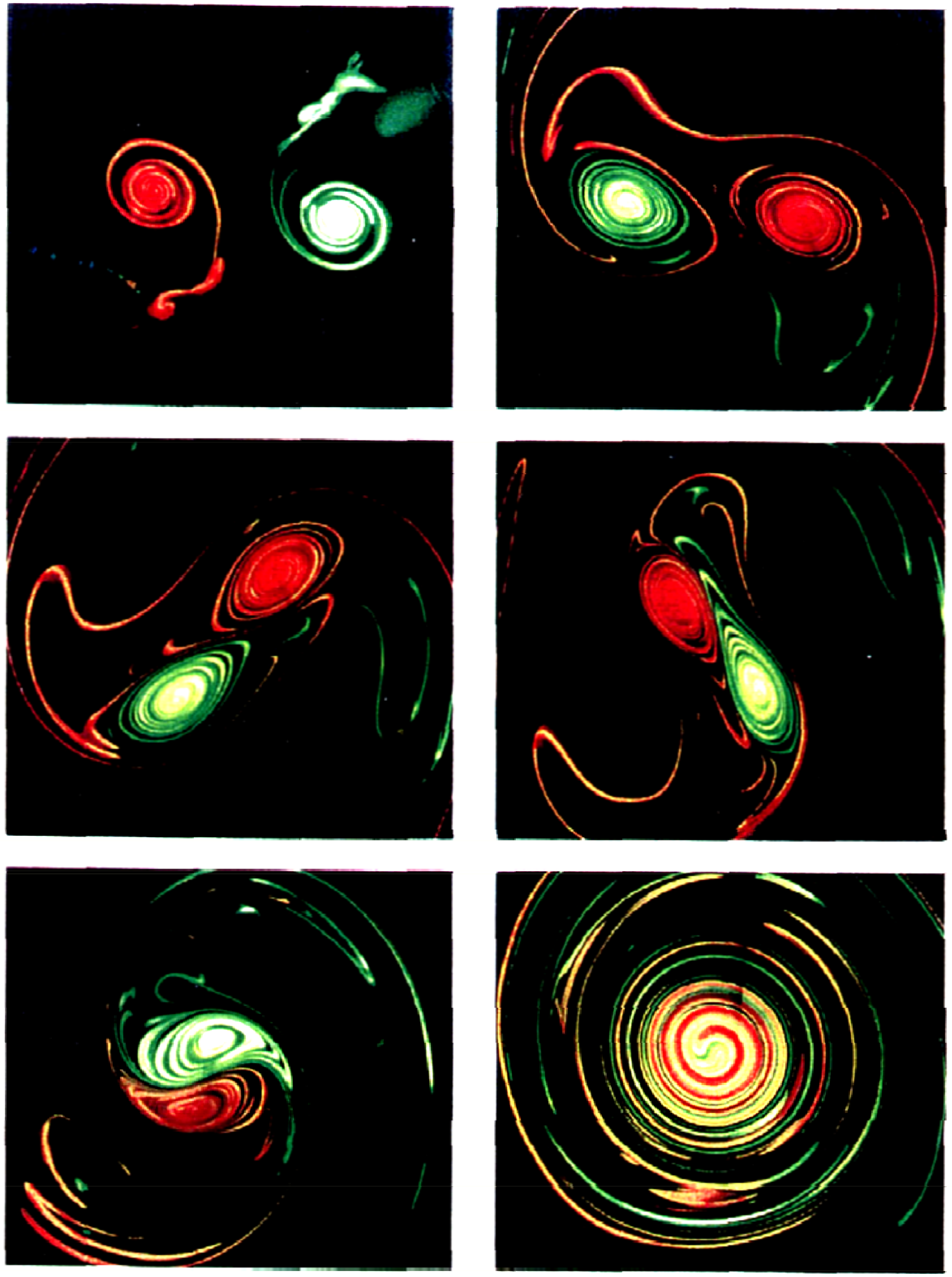


FIGURE 8. Mixing in the PPM as the mixing strength is increased. The mixing strength parameter and Reynolds numbers are *(a)* $\beta=0$, $Re_{PPM:axial}=0.6$, and $Re_{PPM:cs}=0$; *(b)* $\beta=4\pm0.1$, $Re_{PPM:axial}=0.5$, and $Re_{PPM:cs}=1.3$; *(c)* $\beta=10\pm0.1$, $Re_{PPM:axial}=0.6$, and $Re_{PPM:cs}=3.5$; *(d)* $\beta=15\pm0.2$, $Re_{PPM:axial}=0.5$, and $Re_{PPM:cs}=4.1$; *(e)* $\beta=20\pm0.3$, $Re_{PPM:axial}=0.5$, and $Re_{PPM:cs}=5.5$; *(f)* $\beta=25\pm0.4$, $Re_{PPM:axial}=0.5$, and $Re_{PPM:cs}=7.3$; *(g)* $\beta=30\pm0.6$, $Re_{PPM:axial}=0.3$, and $Re_{PPM:cs}=5.9$; and *(h)* $\beta=40\pm0.9$, $Re_{PPM:axial}=0.3$, and $Re_{PPM:cs}=7.5$. KUSH & OTTINO (1992)



Dye visualization of two simple corotating vortices merging into one vortex with simple velocity field, but not that simple field of the passive scalar(s)

LEWEKE 2000

Tip for mixing of two components of epoxy at Re~0

E- versus L-structure(s),

i.e. structure(s) in E- versus L-settings

We start by mentioning that passive objects (including fluid particles) have lots of structure in Gaussian (and other artificial) velocity fields which by definition is "structureless", but will possess quite a bit of Lagrangian coherent structures (LCS) in the spirit of Haller and followers. This is a kind of warning for searching structure(s) in Lagrangian setting when dealing with the dynamical issues of turbulence, which seems to be described better in the Eulerian setting: flow visualizations used for studying the structure of dynamical fields (velocity, vorticity, etc.) of turbulent flows may be quite misleading, making the question "what do we see?" extremely nontrivial.

Indeed , the meaning of 'seeing' turbulent flow is not so simple as the Eulerian flow structure is different from the Lagrangian one: watching the evolution of material 'colored bands' (as suggested by Reynolds 1884) in a flow may not reveal the nature of the underlying motion, and even in the case of right qualitative observations the right result may come not necessarily for the right reasons. The famous verse by Richardson belongs to this kind of observation (which is not necessarily right either).

This is because the structure of a passive marker (L – fluid particles, etc.) can be (and usually is) very complicated and may have a nontrivial structure and statistics!, whereas the corresponding (underlying) Eulerian velocity field (E) may have rather simple structure and statistics or may have none.

This is a part of a broader question. Namely, what can be learnt about the properties and especially dynamics of genine turbulence (NSE, Euler) from studies of passive objects (particles, scalars, vectors)? In particular, what can be learnt about the velocity field and other dynamical variables in real turbulence from comparison of the behaviour of passive objects in real and some 'synthetic' turbulence?

We are again back with (some aspect of) the L - E relation

The Newton law in pure Lagrangian setting

plus incompressibility

$$\partial^2 X_i/\partial^2 t = [X_j, X_k, p] + \nu[X_n, X_{n+1}, [X_n, X_{n+1}, \partial X_i/\partial t],$$
$$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1, X_2, X_3] = 1$$

It seems to be **different** from the one in the Euler setting (i.e. NSE) **not only technically, but conceptually** as it is expected to produce chaotic behavior in most cases when the flow is Laminar in Euler setting!

Here (i, j, k) means an even permutation of the indices $(1, 2, 3)$. The vector $\mathbf{X}(\mathbf{a}, t)$ is the particle position vector for a particle labeled by $\mathbf{a}$. Usually $\mathbf{a} \equiv \mathbf{X}(\mathbf{a}, \mathbf{t}_0)$, i.e. the initial positions of fluid particles are used as their labels. The expression $[A, B, C] \equiv \frac{\partial(A,B,C)}{\partial(a_1,a_2,a_3)}$ is an abbreviation for the Jacobian of the variables A, B, C in respect with variables a_1, a_2, a_3. We denote $[X_1, X_2, X_3] \equiv J$.

Thus, one is tempted to conjecture that the pure Lagrangian **dynamical** equations (so far intractable for viscous flows)

$$\partial^2 X_i/\partial^2 t = [X_j, X_k, p] + \nu[X_n, X_{n+1}, [X_n, X_{n+1}, \partial X_i/\partial t],$$

$$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1, X_2, X_3] = 1$$

are more rich than their Navier Stokes counterpart

$$Du_i/Dt \equiv \partial U_i/\partial t + U_k \partial U_i/\partial x_k = -\partial p/\partial x_i + \nu \nabla^2 U_i$$

$$\partial U_i/\partial x_i = 0$$

The former being equivalent to the latter plus the equation

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{U}[\mathbf{X}(\mathbf{a},t);t] \qquad \text{(E-L)}$$

Though such a conjecture looks plausible, there remain nontrivial issues on the relation between Lagrangian versus Eulerian settings in purely dynamical contexts. One such issue deserves special mentioning. In the Lagrangian setting the fluid particle acceleration is linear in the fuid particle displacement and the `inertial' effects are manifested only by the term containing pressure*. That is, one can hardly speak about things like Reynolds decomposition and Reynolds stresses, turbuent kinetic energy production in shear flows in pure Lagrangian setting. There is no sweeping of any kind at the outset as there are no terms like the advective terms $(u \cdot \nabla)$ in pure Eulerian setting, so one cannot speak about the interaction between advective and diffusive processes in pure Largangian setting. It seems that nonlinearity in the Lagrangian representation cannot be interpreted in terms of some cascade (as it cannot be maintained by pressure gradient alone) and it is far less clear (if at all) how one can employ decompositions even at the problematic level as done in pure Eulerian setting .

*From which it immediately follows that the inertial interactions are of nonlocal nature.

CONCLUDING REMARKS

The sweeping decorrelation and/or random Taylor hypotheses
are missing an essential dynamical aspect –
there is a rich direct an bidirectional coupling
between large and small scales –
both hypotheses are 'too kinematic'.
Thus sweeping cannot be considered as just a kinematic effect:
the dynamics involved is of utmost and primary importance!

It is (more than) plausible that

Pure Lagrangian description **(L-NSE)** $\equiv$ *Eulerian description* **(E-NSE)** + *the equation*

$$\partial \mathbf{X}(\mathbf{a},t))/\partial t = \mathbf{u}[\mathbf{X}(\mathbf{a},t);t] \qquad \textbf{(E-L)}$$

In other words, the Eulerian and Lagrangian settings are different **conceptually** not just/only technically. Eulerian setting is revealing the pure dynamical chaotic aspects of genuine turbulence as contrasted to "mixing" of kinematical with the dynamical ones in the Lagrangian setting, i.e. in genuine turbulence the latter contains both which seem to be essentially inseparable.

One can hardly expect even the existence of "simple" relation(s) between L- and E- statistics in turbulence flows