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Eulerian-Lagrangian bridge for the energy and dissipation spectra in homogeneous turbulence

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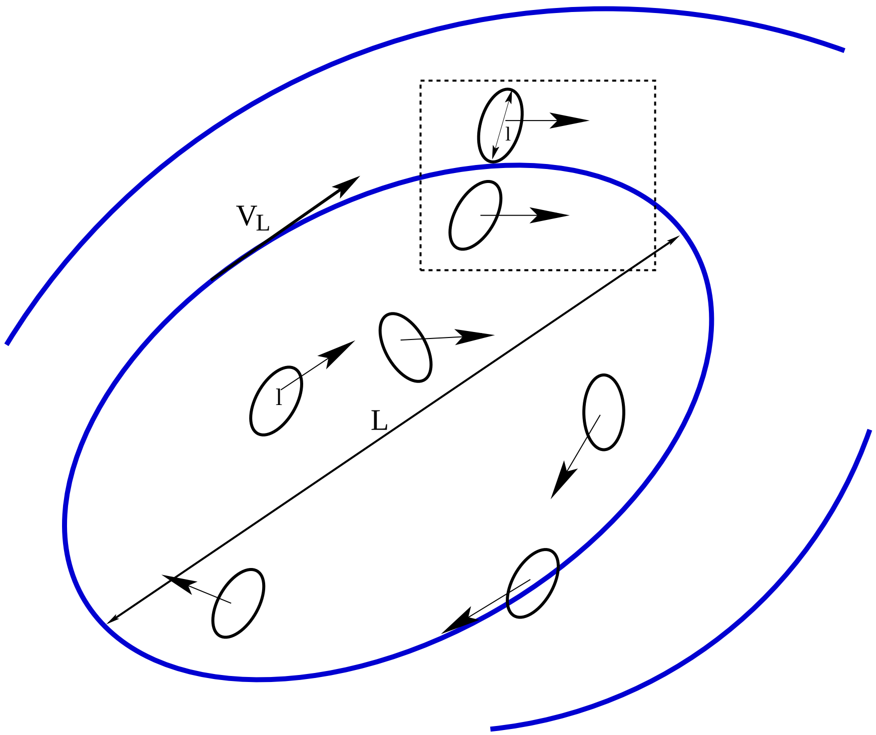
We derive first-principles equations that *bridge* the Eulerian and Lagrangian energy spectra, $E_{\rm E}(k)$ and $E_{\rm L}(\omega)$, as well as the Eulerian and Lagrangian dissipation, $\varepsilon_{\rm E}(k) = 2\nu k^2 E_{\rm E}(k)$ and $E_{\rm L}(\omega)$ $\varepsilon_{\rm L}(\omega)$ for homogeneous isotropic hydrodynamic turbulence. We demonstrate that both analytical relationships, $E_{\rm L}(\omega) \Leftrightarrow E_{\rm E}(k)$ and $\varepsilon_{\rm L}(\omega) \Leftrightarrow \varepsilon_{\rm E}(k)$, are in very good quantitative agreement with our DNS results, which show that not only $E_{\rm L}(\omega, t)$ but also the Lagrangian spectrum of the dissipation rate $\varepsilon_{\rm L}(\omega, t)$ has its maximum at low frequencies (about the turnover frequency of energy containing eddies) and vanishes at large frequencies ω (about a half of the Kolmogorov microscale frequency) for both stationary and decaying isotropic turbulence.

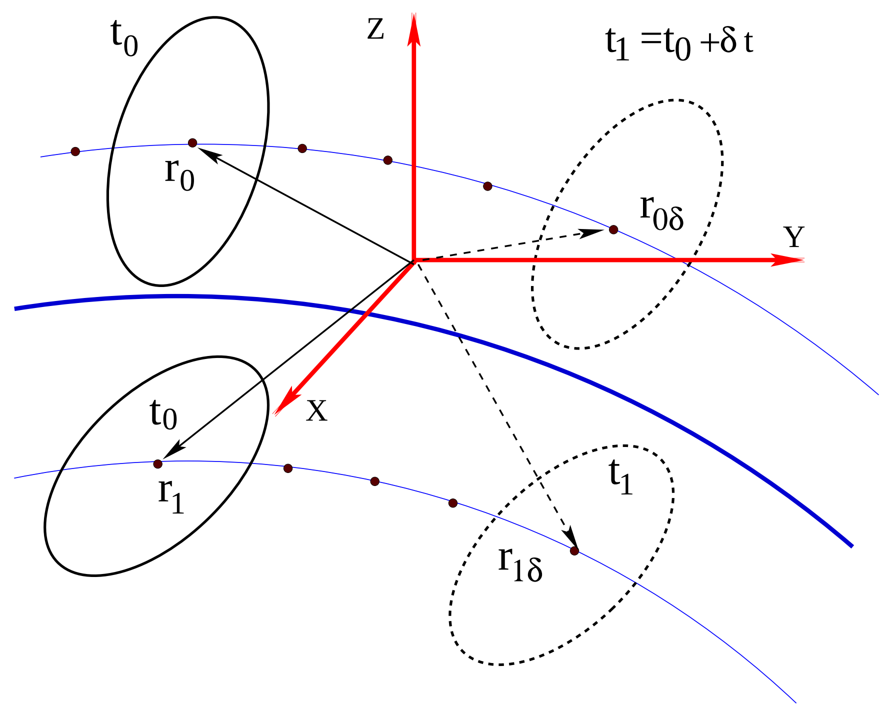
• *Problem of Turbulence*

looks amazingly simple: one has "just" to solve one-line Navier-Stokes Equation (NSE)

$$\text{Euler NSE:} \quad \frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{\nabla} p - \nu \nabla^2 \boldsymbol{u} = \boldsymbol{f}\,, \tag{1}$$

for the "Eulerian" velocity field $\boldsymbol{u}(\boldsymbol{r},t)$, in a "laboratory" reference frame with given boundary conditions, kinematic viscosity ν and forcing $\boldsymbol{f}(\boldsymbol{r},t)$ which maintains the flow. For small fluid velocities (with respect to the speed of sound) the flow can be considered as incompressible: $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, and the pressure field $p(\boldsymbol{r},t)$ can be reconstructed from $\boldsymbol{u}(\boldsymbol{r},t)$.



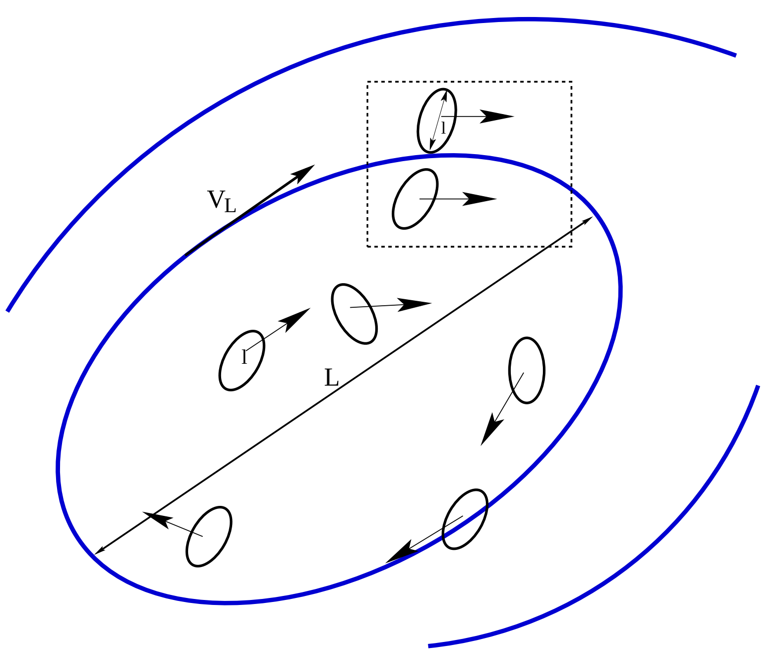


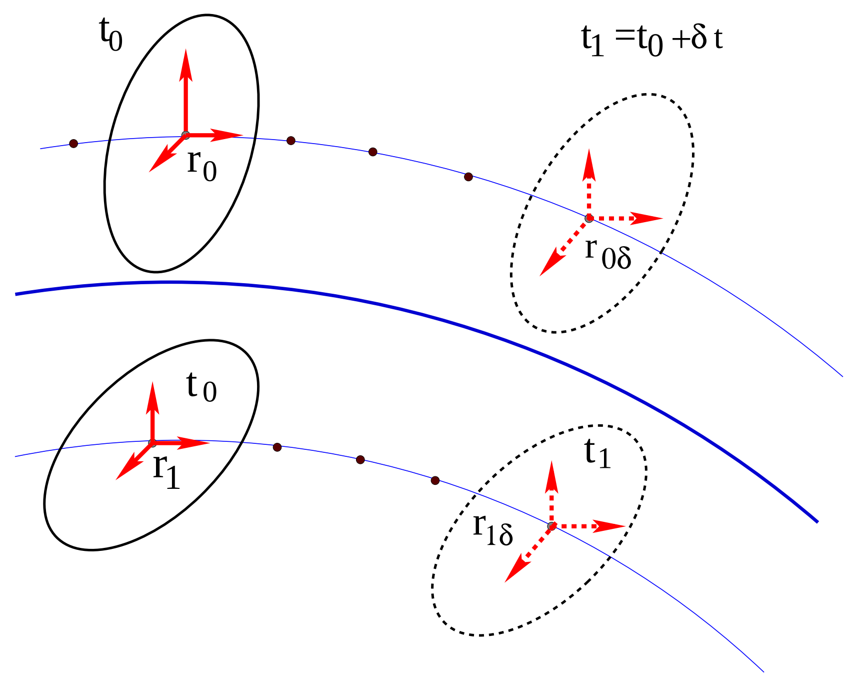
• *Lagrangian approach*

is dealing with the time-dependent Lagrangian velocity: $\boldsymbol{u}_{\rm L}(\boldsymbol{r}_0|t) \equiv \boldsymbol{u}\big(\boldsymbol{r}_{\rm L}(\boldsymbol{r}_0|t), t\big)$, in which $\boldsymbol{r}_{\rm L}(\boldsymbol{r}_0|t)$ is the trajectory of fluid point, positioned at $\boldsymbol{r}_0$ at $t = t_0$.:

$$\text{Lagrangian NSE:} \quad \frac{d\boldsymbol{u}_{\rm L}(\boldsymbol{r}_0|t)}{dt} + \left[\boldsymbol{\nabla} p - \nu \nabla^2 \boldsymbol{u}\right]_{\rm L}(\boldsymbol{r}_0|t) = \boldsymbol{f}_{\rm L}(\boldsymbol{r}_0|t)\,, \quad (2)$$

follows directly from the Eulerian NSE (1). Hereafter $\big[\Psi\big]_{\rm L}(\boldsymbol{r}_0|t)$ denotes a function $\Psi(\boldsymbol{r}, t)$, taken at $\boldsymbol{r} = \boldsymbol{r}_{\rm L}(\boldsymbol{r}_0|t)$, e.g. $\big[\boldsymbol{\nabla} p\big]_{\rm L}(\boldsymbol{r}_0|t_0) \equiv \boldsymbol{\nabla} p(\boldsymbol{r}, t)\big|_{\boldsymbol{r}=\boldsymbol{r}_{\rm L}(\boldsymbol{r}_0|t)}$. One can say that in the Eulerian approach one is looking at stormy wind through window of his lab, while in the Lagrangian case he participates simultaneously in the motions of all the air particles.





Lagrangian approach has unique physical advantages that are especially important in studies of phenomena dominated by small-scale motions or many-point correlation functions, like turbulent mixing and particle dispersion. It is natural conceptually and useful in practice for the description of turbulent transport.

However, some important aspects of small-scale statistics, transport and other related phenomena, that can be studied via the Lagrangian approach, are still unclear or poorly understood. One reason is the technical problems: the Lagrangian NSE (2) *is not local in time*: it implicitly includes, at time t, the entire history of the velocity-field for times $t_0 < t' < t$ through the Lagrangian trajectories $\boldsymbol{r}_{\mathrm{L}}(\boldsymbol{r}_0|t)$, i.e. via complicated nonlinear time evolution operators. That is why little progress has been made in the analytical description of the statistics of turbulence in the Lagrangian frame. For example, the celebrated Kraichnan's "Lagrangian History" approach, which analytically reproduced the Kolmogorov-41 energy spectrum $E(k) \propto k^{-5/3}$ from the Lagrangian NSE (2), was formulated only in simplest Direct-Interaction Approximation.

Even more: some fundamental (and very simple) aspects of the Lagrangian theory are still missing. For example, to the best of our knowledge, the energy balance equation for the Lagrangian spectrum, $\frac{\partial}{\partial t} E_{\mathrm{L}}(t,\omega)$ does not exist in the available literature, whereas the corresponding equation for the corresponding Eulerian spectrum $\frac{\partial}{\partial t} E_{\mathrm{E}}(t,k)$ is written in numerous papers and textbooks.

• *Evolution equation of Lagrangian power spectrum* $E_{\mathrm{L}}(t,\omega)$.

Define the Lagrangian velocity autocorrelation function:

$$R_{\mathrm{L}}(t,s) = \langle \boldsymbol{u}_{\mathrm{L}}(\boldsymbol{r}_0|t-s/2)\cdot \boldsymbol{u}_{\mathrm{L}}(\boldsymbol{r}_0|t+s/2)\rangle_{\boldsymbol{r}_0} \ ,$$

where $\langle \ldots \rangle_{\boldsymbol{r}_0}$ denotes an ensemble averaging over an infinite number of pathlines with different initial positions $\boldsymbol{r}_0$. In homogeneous turbulence $R_{\mathrm{L}}(t,s)$ is independent of the space coordinates (including $\boldsymbol{r}_0$). In stationary turbulence $R_{\mathrm{L}}(t,s)$ depends only on the lag time s. In homogeneous turbulence the same-point-Lagrangian and Eulerian correlation functions are equal. Thus,

$$E(t) = E_{\mathrm{L}}(t) \equiv \frac{1}{2}\langle |\boldsymbol{u}_{\mathrm{L}}(\boldsymbol{x}_0|t)|^2\rangle_{\boldsymbol{r}} = \frac{1}{2}R_{\mathrm{L}}(t,0)\,,$$

where $\langle \ldots \rangle_{\boldsymbol{r}}$ denotes a spacial averaging over Eulerian positions $\boldsymbol{x}$. Let $E_{\mathrm{L}}(t,\omega)$ be the Fourier transform of $\frac{1}{2}R_{\mathrm{L}}(t,s)$:

$$E_{\mathrm{L}}(t,\omega) = \frac{1}{2}\mathcal{F}\{R_{\mathrm{L}}(t,s)\}\,,$$

where $\mathcal{F}\{\varphi(s)\} \equiv \int_{\infty}^{\infty}\varphi(s)\exp(i\omega s)ds$.

Note also that $E_{\mathrm{L}}(t)$ is the integral of the Lagrangian power spectrum $E_{\mathrm{L}}(t,\omega)$ with respect to ω, and with the factor $\frac{1}{2\pi}$:

$$E_{\mathrm{L}}(t) = E_{\mathrm{E}}(t) = E(t) = \int E_{\mathrm{L}}(t,\omega)\frac{d\omega}{2\pi}\ .$$

The Lagrangian NSE (2) allows one to derive straightforwardly

$$\frac{\partial E_{\rm L}(t,\omega)}{\partial t} = \mathcal{T}_{\rm L}(t,\omega) - \varepsilon_{\rm L}(t,\omega) + \Psi_{\rm L}(t,\omega)\,, \tag{3a}$$

where the transfer rate of $E_{\rm L}(t,\omega)$ across the ω-space:

$$\mathcal{T}_{\rm L}(t,\omega) = -\frac{1}{2}\mathcal{F}\langle \boldsymbol{u}_{\rm L}\big(\boldsymbol{x}_0|t-\frac{s}{2}\big)\cdot\big[\boldsymbol{\nabla}p\big]_{\rm L}\big(\boldsymbol{r}_0|t+\frac{s}{2}\big) + \boldsymbol{u}_{\rm L}\big(\boldsymbol{r}_0|t+\frac{s}{2}\big)\cdot\big[\boldsymbol{\nabla}p\big]_{\rm L}\Big(\boldsymbol{r}_0|t-\frac{s}{2}\big)\rangle_{\boldsymbol{r}_0}\,, \tag{3b}$$

and the viscous dissipation rate of $E_{\rm L}(t,\omega)$ and the rate of energy addition (via forcing) to $E_{\rm L}(t,\omega)$ to maintain turbulence stationarity are:

$$\varepsilon_{\rm L}(t,\omega) \;=\; -\frac{1}{2}\nu\,\mathcal{F}\langle \boldsymbol{u}_{\rm L}\big(\boldsymbol{r}_0|t-\frac{s}{2}\big)\cdot\nabla^2\boldsymbol{u}_{\rm L}\Big(\boldsymbol{r}_0|t+\frac{s}{2}\big) + \boldsymbol{u}_{\rm L}\big(\boldsymbol{r}_0|t+\frac{s}{2}\big)\cdot\nabla^2\boldsymbol{u}\big]_{\rm L}\Big(\boldsymbol{r}_0|t-\frac{s}{2}\Big)\rangle_{\boldsymbol{r}_0}\,, \tag{3c}$$

$$\Psi_{\rm L}(t,\omega) \;=\; \frac{1}{2}\mathcal{F}\langle \boldsymbol{u}_{\rm L}\Big(\boldsymbol{r}_0|t-\frac{s}{2}\big)\cdot\boldsymbol{f}_{\rm L}\Big(\boldsymbol{r}_0|t+\frac{s}{2}\big) + \boldsymbol{u}_{\rm L}\big(\boldsymbol{r}_0|t+\frac{s}{2}\big)\cdot\boldsymbol{f}_{\rm L}\big(\boldsymbol{r}_0|t-\frac{s}{2}\big)\rangle_{\boldsymbol{r}_0}\,, \tag{3d}$$

Unfortunately, this equation for $\frac{\partial}{\partial t}E_{\rm L}(t,\omega)$ is not local in time as it depends on the flow history and thus can be straightforwardly evaluated only numerically from direct numerical simulations (DNS).

Next I will describe a way to circumvent this limitation by derivation of *Eurelian-Lagrangian bridges* that allow the determination of the Lagrangian (frequency) kinetic energy spectrum, $E_{\rm L}(\omega)$, from a given Eulerian energy spectrum, $E_{\rm E}(k)$, as well as the Lagrangian (frequency) dissipation spectrum, $\varepsilon_{\rm L}(\omega)$, from a given Eulerian counterpart, $\varepsilon_{\rm E}(k)$. The bridge-relationships are *not* limited by either the inertial interval of scales, or by the requirement of large Re$_\lambda$.

• *Sweeping elimination from a theory of turbulence*

The modern statistical theory of hydrodynamic turbulence goes back to the DIA-paper by Kraichnan [1] and DT-paper by Wyld [2] who suggested to simulate excitation of stationary spatially homogeneous developed hydrodynamic turbulence with the help of a spatially distributed variable force in the NSE.

IR divergence in Kraichnans DIA led to the following erroneous energy spectrum in the inertial interval

$$E(k) \simeq \sqrt{\varepsilon V_{\rm T}} k^{-3/2} \simeq (\varepsilon/\rho)^{3/2} k^{-5/3} (kL)^{-1/6}\,, \tag{4}$$

and to the de-correlation frequency of the different-time Eulerian velocity correlator: $\Omega_{\rm D}(k) \simeq kV_{\rm T}$, which should be considered as the Doppler shift of the frequency of k-eddies in a random velocity field of energy contained eddies of the scale L. The reason for this difficulty is that DIA does not correctly separate the sweeping [with the frequency $\Omega_{\rm D}(k)$] and dynamic interactions with the turnover frequency

$$\gamma(k) \simeq \varepsilon^{1/3} k^{2/3} \ll \Omega_{\rm D}(k) \simeq \gamma(k)(kL)^{1/3}\,. \tag{5}$$

Therefore, the problem consists in distinguishing and studying a relatively weak dynamic-interaction that determines the turbulence spectrum in the formal technique of the theory on the background of the effect of $kV_{\rm D}$ -sweeping, masking the interaction. The natural way to solve this problem is to use, for the description of the dynamic interaction of eddies, variables without the kinematic effect of sweeping. Kraichnan in his LHDIA papers [3] used the Lagrangian description of fluid flows for this purpose, but this led to serious technical difficulties, which did not allow him to go further than the DIA.

[1] R.H. Kraichnan, I. Fluid Mech. **6** (1959) 497

[2] H.W. Wyld, Ann. Phys. 14 (1961)143.

[3] RH. Kraichnan, Phys. Fluids **8** (1965) 575-598; **9** (1966) 1728].

The authors of a number of papers [4], [5], [6], [7], [8], [9] tried to solve this problem by the explicit introduction of a separation scale, k', into the theory, $L < k' < k$. Unfortunately, in so doing sweeping of k-eddies by significantly larger k-eddies is always kept and the difficulties in the theory remain.

In 1977 [10] I suggested a method of the sweeping elimination without using a separation scale, by developing some "intrinsic" perturbational approach, in which random sweeping was eliminated from the statistical theory AFTER averaging over small scale statistics. This approach is much better the the Kraichnan's DIA because it accounts the sweeping in every order of perturbation theory, while the rather weak effect of the dynamic interaction of eddies is considered approximately, by the first diagram of the internal diagram technique. Unfortunately, a simple analysis of some diagrams of fourth order in the vertices in the internal diagram technique in Euler variables reveals logarithmic divergences.

The problem of eliminating sweeping in all orders of perturbation theory was solved in [11] by using quasi-Lagrangian reference system in which the origin is moving with the velocity of ONE Lagrangian tracer. The main technical achievement is a proof that the in this approach the sweeping is eliminated in ALL orders of the perturbation approach, because the sweeping was eliminated BEFORE statistical averaging. Thus it can be called *sweeping-free quasi-Lagrangian approach*.

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[4] B.B. Kadomtsev, Plasma Turbulence (Academic Press, London, 1965)

[5] GA. Kuzmin and A.Z. Patashinskii, J.AppI. Mech. Tech. Phys. **19** (1978) 50

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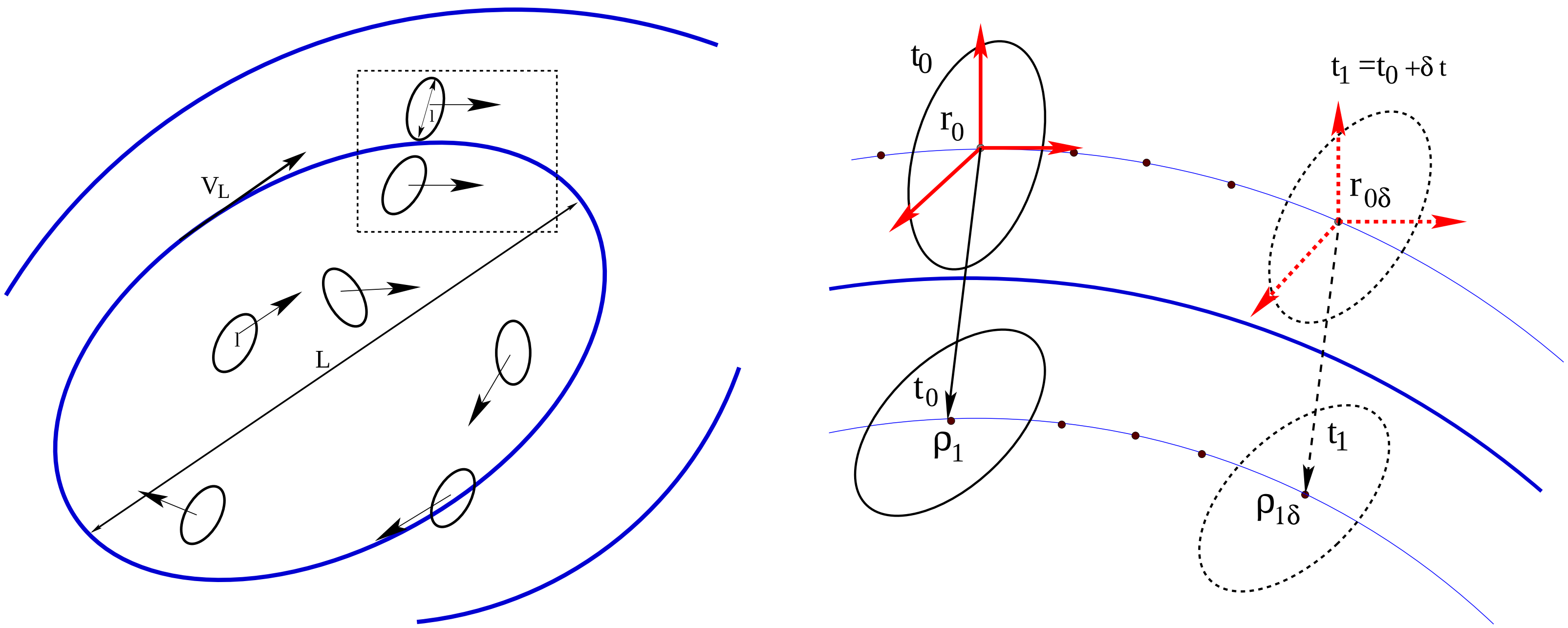
[10] V.S. Lvov, preprint No. 53, Institute of Automation and Electrometry, Novosibirsk (1977).

[11] V.1. Belinicher and VS. Lvov, Zh. Eksp. Teor. Fiz. **93** (1987) 533 [Sov.Phys. JETP **66** (1987) 303]

• *Sweeping-free quasi-Lagrangian approach*

Derivation of the Eulerian-Lagrangian Bridges requires elimination from the theory of *the kinematic effect of sweeping* of the small scale motion by the energy-contained (large scale) motions. In principle this can be done in the Lagrangian approach. However, due to time non-locality of the Lagrangian NSE (2) this is extremely difficult and hardly possible.

A way out, suggested by Belinicher-L'vov (hereafter BL) in a 1987-JETP paper is to *eliminate the kinematic sweeping effect* by using a reference frame shared by *all* fluid points inside a large eddy which was named by BL in the 1987-JETP quasi-Lagrangian (qL) frame [12]. Its origin moves along the Lagrangian trajectory $\boldsymbol{x}_{\rm L}(\boldsymbol{x}_0|t)$ of a particular fluid point, denoted as O-point, with position $\boldsymbol{x}_0$ at time $t = t_0$



[12] Monin in 1959 used the same coordinate system (without giving it a name) to extend the region of validity of the well known Kolmogorov's relation between the second-order and third-order structure functions.

The BL-1987 JETP paper introduces

$$\text{sweeping-free qL-velocity:} \qquad \boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t) \equiv \boldsymbol{u}\big(\boldsymbol{r}-\boldsymbol{r}_0+\boldsymbol{r}_{\rm L}(\boldsymbol{r}_0,t),t\big). \tag{6a}$$

Quasi-Lagrangian NSE follows directly from Eq. (6a) and the Eulerian NSE (1):

$$\text{qL-NSE:} \qquad \Big\{\frac{\partial}{\partial t} + \big[\boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t) - \boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r}_0,t)\big]\cdot\boldsymbol{\nabla}\Big\}\boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t)$$
$$+\ \boldsymbol{\nabla}p_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t) = \nu\nabla^2\boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t) + \boldsymbol{f}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t)\ . \tag{6b}$$

Note that both the quasi-Lagrangian NSE (6b) and Eulerian NSE (1) do not involve the history of the fluid points trajectories, whereas Lagrangian NSE (2) does.

However, the kinematic sweeping effect, inherent in the Eulerian NSE (1), is eliminated from the qL-NSE (6b) as it was proven by BL in the 1987-JETP paper at all orders of the Wyld perturbation technique. The reason is that the large scale eddies do not contribute to the qL-velocity difference $\big[\boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r},t) - \boldsymbol{u}_{\rm qL}(\boldsymbol{r}_0|\boldsymbol{r}_0,t)\big]$, in contrast to the Eulerian NSE.

In the framework of the qL-NSE (6b) L'vov-Procaccia in a set of 1995-2000 PRE papers

- invented fusion rules for the many-point velocity correlation functions,
- proved the K41 hypothesis of the cascade picture of the energy transfer over scales
- demonstrated that anomalous scaling is consistent with NSE, being a non-perturbation effect
- used the fusion rules for analytic calculation of anomalous scaling exponents in turbulence.

Based on the results above I will use qL-NSE (6b) to derive the Eulerian-Lagrangian bridges.

• *Derivation of the Eulerian-Lagrangian Bridges*

Consider the two-point, different-time cross-velocity correlation of the qL-velocities

$$W(\boldsymbol{r},s) \equiv \langle \boldsymbol{u}^{\mathrm{qL}}(\boldsymbol{r}_0|\boldsymbol{r}',t')\cdot \boldsymbol{u}^{\mathrm{qL}}(\boldsymbol{r}_0|\boldsymbol{r}'',t'')\rangle\,, \quad \text{where } \boldsymbol{r}=\boldsymbol{r}'-\boldsymbol{r}''\,,\ s=t'-t''\,.$$

Denote the $(\boldsymbol{k},\omega)$-Fourier transform of $W(\boldsymbol{r},s)$ as $\mathcal{W}(\boldsymbol{k},\omega)$, i.e. $\mathcal{W}(\boldsymbol{k},\omega)\equiv \mathcal{F}\{W(\boldsymbol{r},s)\}$.

$\mathcal{W}(\boldsymbol{k},\omega)$ is the turbulence kinetic energy density (per unit mass) in the $\boldsymbol{k}$- and ω-spaces simultaneously.

In isotropic turbulence $\mathcal{W}(\boldsymbol{k},\omega)\Rightarrow \mathcal{W}(k,\omega)$. Then, instead of $\mathcal{W}(k,\omega)$ it is convenient to introduce $E_{\mathrm{qL}}(k,\omega)\equiv 2\pi k^2\mathcal{W}(k,\omega)$.

$E_{\mathrm{qL}}(k,\omega)$ is the qL kinetic energy density in *one-dimensional* k- and frequency ω-space simultaneously.

$$E_{\mathrm{E}}(k)=\int E_{\mathrm{qL}}(k,\omega)d\omega/2\pi\,,\quad E_{\mathrm{L}}(\omega)=\int E_{\mathrm{qL}}(k,\omega)dk,\quad \varepsilon_{\mathrm{L}}(\omega)=2\nu\int k^2E_{\mathrm{qL}}(k,\omega)dk\,. \tag{7a}$$

Here we have used the BL result that the dissipative term in the balance equation for $\mathcal{W}(\boldsymbol{k},\omega)$ has the simple form $2\nu k^2\mathcal{W}(\boldsymbol{k},\omega)$. It was shown by BL that, in the qL-reference frame, the kinematic sweeping of small eddies by the velocity field of the larger eddies is absent, and thus the characteristic frequency for the k-scale motions is $\gamma(k)\approx k\sqrt{kE_{\mathrm{E}}(k)}$. Therefore, one can write:

$$E_{\mathrm{qL}}(k,\omega)\approx 2\pi\Phi[\,\omega/\gamma(k)\,]E(k)/\gamma(k)\,, \tag{7b}$$

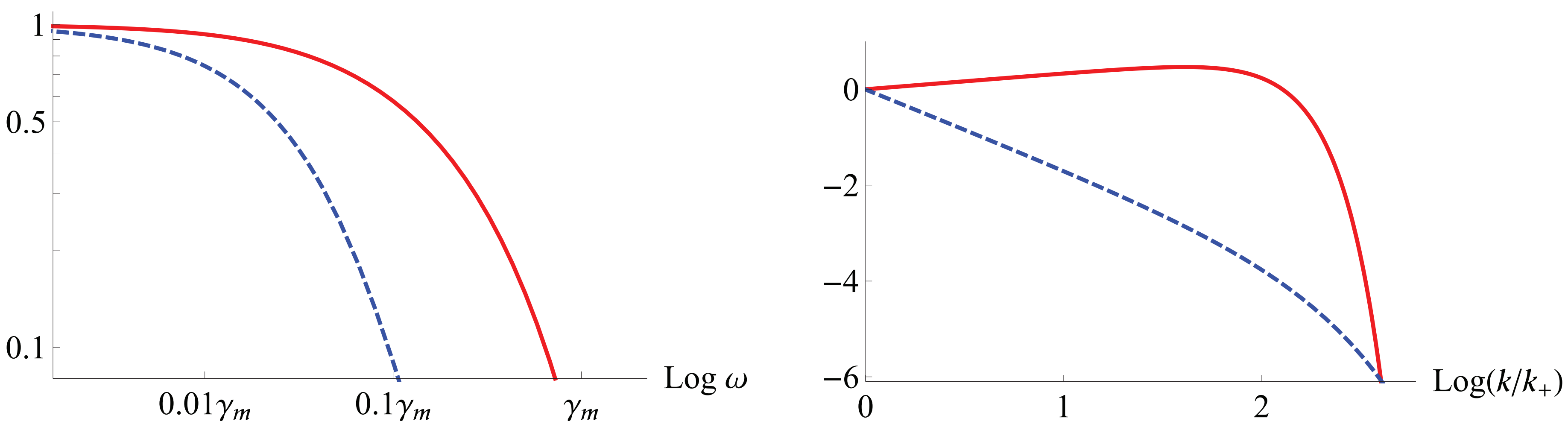
where $\int\Phi(x)\,dx=1$. The exact form of $\Phi(x)$ is not known. However, BL indicated that $\mathcal{W}(\boldsymbol{k},\omega)$ decays much faster than $1/x^2$ for $\lim_{x\to\infty}$. For simplicity we chose: $\Phi(x)=\exp(-x)$.

Next: Eqs.(7) give the Eulerian-Lagrangian bridges for the energy $(8\mathrm{a})$ and for the dissipation rate $(8\mathrm{a})$:

$$E_{\rm L}(\omega) \simeq 2\pi \int \frac{E_{\rm E}(k)}{\gamma(k)} \exp\Big[-\frac{\omega}{\gamma(k)}\Big] dk \qquad \Rightarrow \omega^{-2} - \omega_\eta^{-2}\,, \tag{8a}$$

$$\varepsilon_{\rm L}(\omega) \simeq 4\pi\nu \int \frac{k^2 E_{\rm E}(k)}{\gamma(k)} \exp\Big[-\frac{\omega}{\gamma(k)}\Big] dk \;\Rightarrow \nu\big(\,\omega_\eta \,-\, \omega\,\big). \tag{8b}$$

The Eulerian-Lagrangian bridges (written to the left of $\Rightarrow$) (8) are *not* limited by either the inertial interval of scales, or by the requirement of large Re_λ. For $\mathrm{Re}_\lambda \gg 1$ Eqs. (8) can be simplified using that $\exp[-\omega/\gamma(k)] \ll 1$ if $\gamma(k) < \omega$. Introducing $k(\omega) \equiv \sqrt{\omega^3/\varepsilon}$ we can replace the zero lower limit by $k(\omega)$ and the upper limit by $(1/\eta)$ (η-the Kolmogorov micro-scale) of the integrals in (8). This gives the red-marked results in Eqs. (8) in which $\omega_\eta \,= \pi\sqrt{\varepsilon/\nu}$ is the turnover frequency of η-scale eddies.



Left:Lagrangian spectra for energy, $E_{\rm BL}(\omega)$, and dissipation $\varepsilon_{\rm BL}(k,\omega)$ found from Eqs. (8), using examples of the Eulerian spectra for energy $E_{\rm BL}(k)$ and dissipation, $\varepsilon_{\rm BL}(k,\omega)$, shown on the Right.

• *Eulerian and Lagrangian DNS study of turbulence*

We used 1024^3 grid points in a cubical domain with periodic boundary conditions to simulate isotropic turbulent flow at $\mathrm{Re}_\lambda = 240$ for about $20\,\tau_L$, energy-contained eddy turnover times. In order to produce a stationary turbulence with total (time-independent) energy density E we used Lundgren's linear forcing method by applying a force $f_i(\boldsymbol{x}, t) = 2\varepsilon(t)\boldsymbol{u}(\boldsymbol{r}, t)/E$, in which $\varepsilon(t)$ is the current rate of energy dissipation. To obtain the Lagrangian objects we computed the trajectories of 5×10^5 fluid points, randomly released in the computational domain. The instantaneous Lagrangian velocities of the fluid points, are computed using the 4-order 3-dimensional Hermite interpolation.

We obtained preliminary DNS results of numerical simulations for for stationary isotropic turbulence Left: Lagrangian frequency spectra of turbulence energy, $E_{\mathrm{L}}(\omega)$, and dissipation rate, $\varepsilon_{\mathrm{L}}(\omega)$ and Eulerian spectra of the energy $E_{\mathrm{E}}(k)$ and the dissipation rate $\varepsilon_{\mathrm{E}}(k)$. Also we got Frequency dependence of the three terms on the RHS of the evolution Eq.$(3\mathrm{a})$ and demonstrated that the sum of the three contributions, $\frac{\partial}{\partial t}E_{\mathrm{L}}(t,\omega)$, is zero as expected in stationary turbulence. All three terms have their maximum magnitude at the lowest frequency. The energy transfer term $\mathcal{T}_{\mathrm{L}}(t,\omega)$ is negative at low frequencies and becomes positive for $\omega/\omega_\eta > 0.03$ indicating the transfer of TKE from the small to the large frequency eddies. Finally we demonstrated that

The agreement between the analytical bridges (8), and the DNS results is very good for both $E_{\mathrm{L}}(\omega)$ and $\varepsilon_{\mathrm{L}}(\omega)$.

The DNS study now is in progress and our final results will be available soon.

• *Concluding remarks*

– We presented DNS results (1024^3-cube, Re$_\lambda = 240$, 4×10^5 tracers) of the Lagrangian energy and dissipation rate spectra, $E_{_\mathrm{L}}(t,\omega)$ and $\varepsilon_{_\mathrm{L}}(t,\omega)$, in stationary homogeneous isotropic turbulence.

– Our DNS results show that not only $E_{_\mathrm{L}}(t,\omega)$ but also $\varepsilon_{_\mathrm{L}}(t,\omega)$ has its maximum at low frequencies ω (about the turnover frequency of energy containing eddies) and vanishes at large ω (about a half of the Kolmogorov microscale frequency).

– We derived Eulerian-Lagrangian bridge relationships that allow the determination $E_{_\mathrm{L}}(t,\omega)$ and $\varepsilon_{_\mathrm{L}}(t,\omega)$ in terms of the Eulerian energy and dissipation spectra $E_{_\mathrm{E}}(t,k)$ and $\varepsilon_{_\mathrm{E}}(t,k)$.

– We used NSE in the sweeping-free qL-representation (intermediate between Eulerian and Lagrangian frameworks) and the combined (k,ω) energy spectrum $E_{_\mathrm{BL}}(t,k,\omega)$.

– The approach can be generalized to account for the intermittency effects and can be used to find bridges for more complicated objects, like energy transfer term, that involves 3-rd order correlations.

– We consider the *Eurelian-Lagrangian bridges* and their agreement with our DNS as our central result and expect that they will shed light on the connections between the Eulerian and Lagrangian frameworks in turbulent flows.