3 short Stories in Turbulence and Singularities

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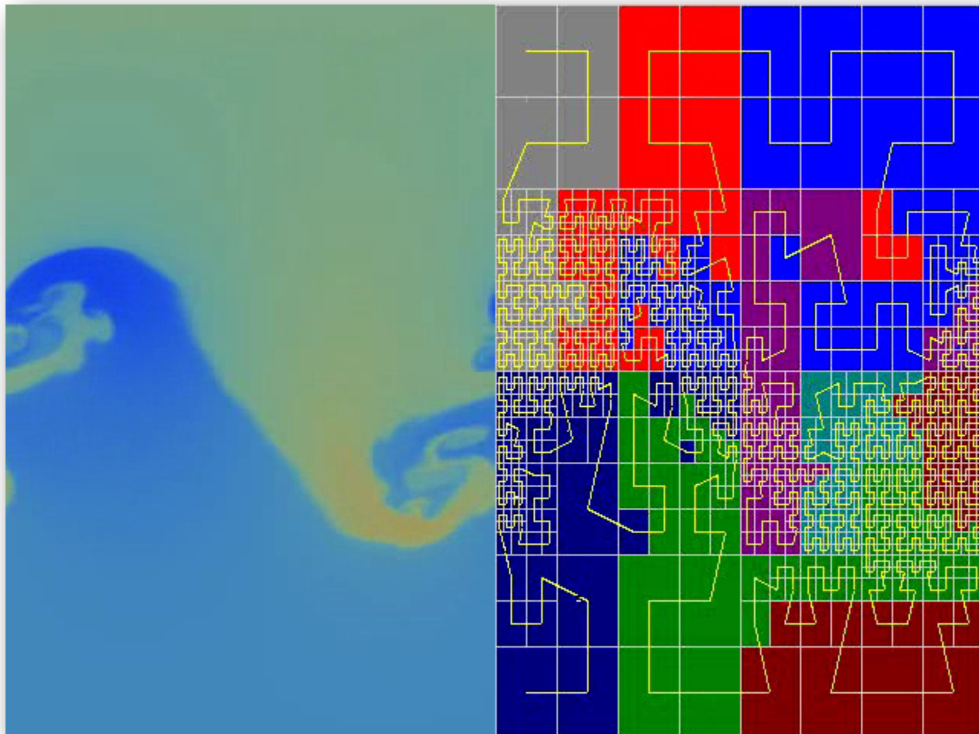
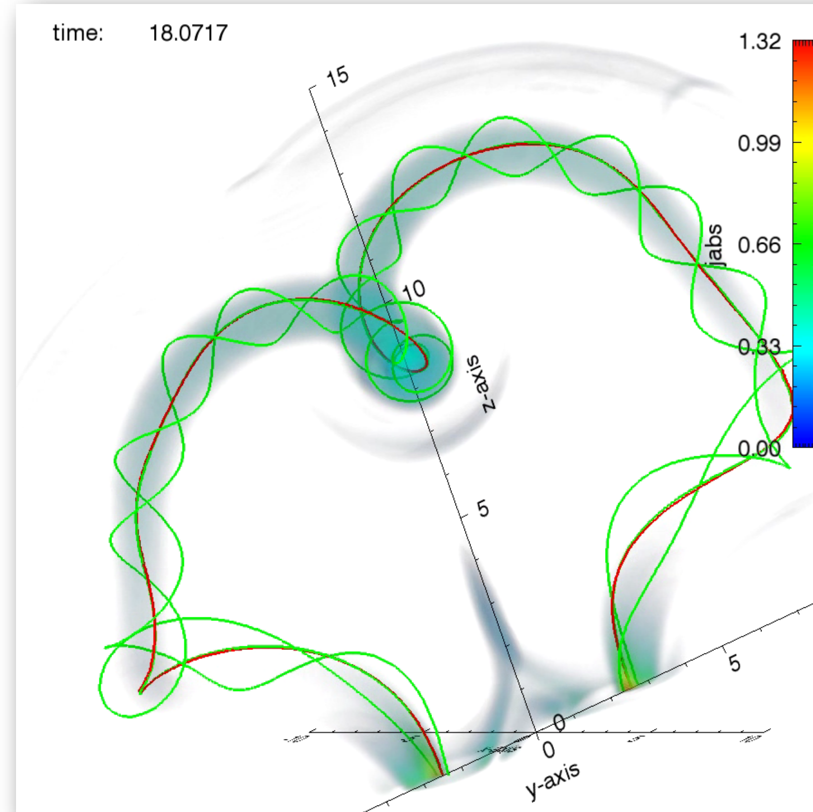
What are we doing ?

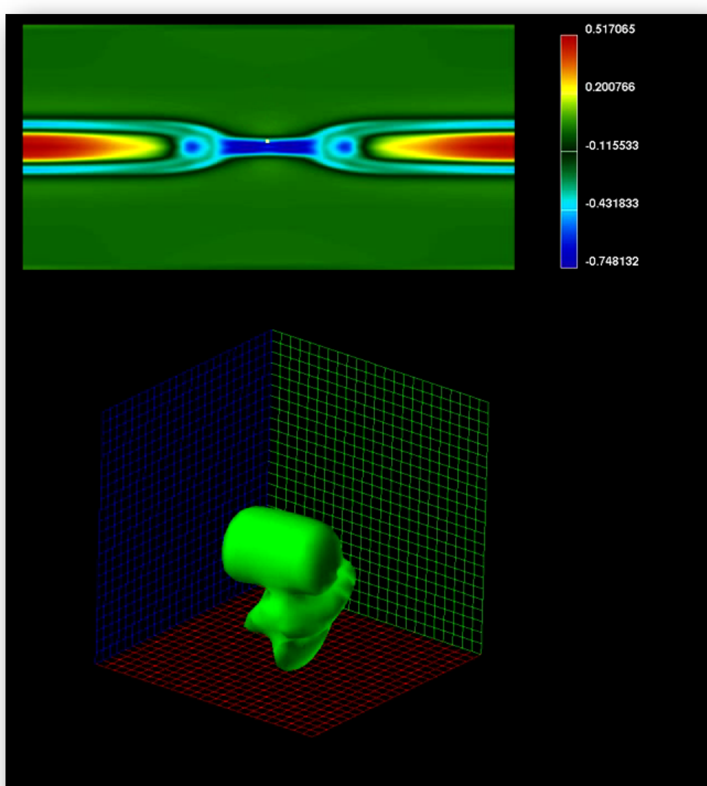
Numerical Methods



FlareLab



Vlasov Simulations



Turbulence

3 short stories

- Euler eqns.
- Exact relations between Euler and Lagrangian
- Conditional statistics

Millennium Prize Problems

P versus NP

The Hodge Conjecture

The Poincaré Conjecture

The Riemann Hypothesis

Yang–Mills Existence and Mass Gap

Navier–Stokes Existence and Smoothness

The Birch and Swinnerton–Dyer Conjecture

Announced 16:00, on Wednesday, May 24, 2000
Collège de France

Statement from the Directors and Scientific Advisory Board

Rules for the CMI Millennium Prize Problems

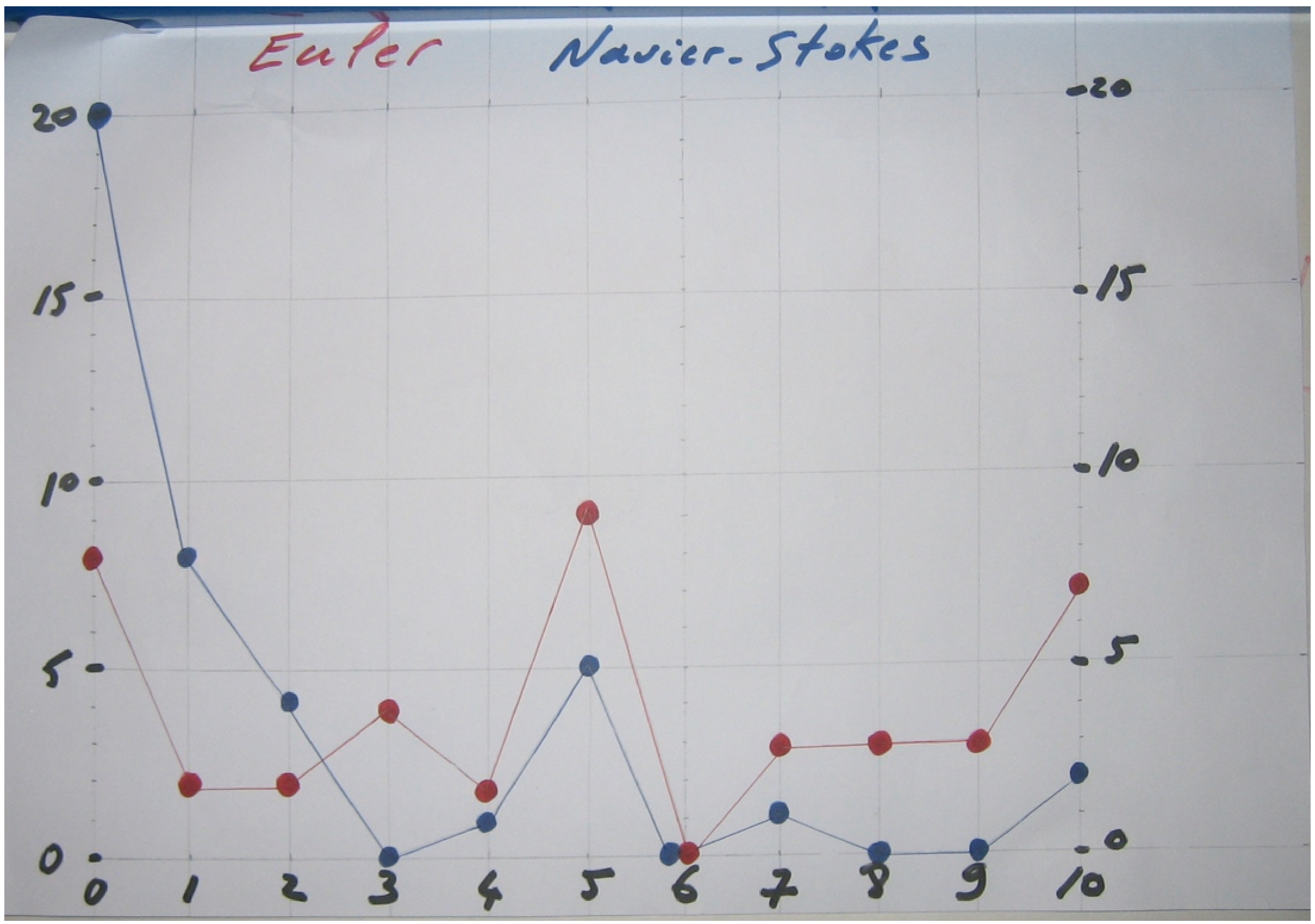
Historical Context

Press Statement

Press Reaction

Remarks

Aussois: 250 Euler equations



democracy doesn't help

no definite answer yet

Some results for Navier-Stokes

Leray (1934):

$\exists$ global solution in $d = 2$

Leray (1934):

if $\exists$ singularity, then $\|u\|_{L^\infty} \geq C[\nu/(t^* - t)]^{1/2}$

Caffareli, Kohn, Nirenberg (1982):

$$\|\omega\|_{L^\infty} \geq C/(t^* - t)$$

Scheffer (1978), Caffareli, Kohn, Nirenberg (1982):

space-time-dimension (Hausdorff)
of the singular set < 1.

Condition I: If

$$\int_0^T dt \left(\int |\omega|^2 \, d^3x \right)^2 < \infty$$

then the solution is smooth for $0 < t \leq T$.

Known: Any Leray weak solution satisfies

$$\int_0^T dt \int |\omega|^2 \, d^3x < \infty$$

Condition II: If

$$\int_0^T dt \left(\int |\mathbf{u}|^p \, d^3x \right)^{2/(p-3)} < \infty$$

for some p, $3 < p \leq \infty$ then the solution is smooth for $0 < t \leq T$.

Known:

$$\int_0^T dt \left(\int |\mathbf{u}|^p \, d^3x \right)^{4/(3(p-2))} < \infty$$

for $2 \leq p \leq 6$.

Condition III: If

$$\int_0^T dt \left(\int |\nabla \omega|^2 \, d^3x \right)^{2/3} < \infty$$

then the solution is smooth for $0 < t \leq T$.

Known:

$$\int_0^T dt \left(\int |\nabla \omega|^2 \, d^3x \right)^{1/3} < \infty$$

Navier-Stokes:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u}$$

$\Longrightarrow$

$$\partial_t \mathbf{u} + P\left[\mathbf{u} \cdot \nabla \mathbf{u}\right] = \nu \Delta \mathbf{u} \quad ,$$

where P is projection-operator on divergence free part.

$\Longrightarrow$

$$\partial_t \mathbf{v} + P\left[\mathbf{v}\right] \cdot \nabla P\left[\mathbf{v}\right] = \nu \Delta \mathbf{v} \quad ,$$

with $\mathbf{u} = P\left[\mathbf{v}\right]$

note: $\mathbf{v}$ is compressible

Navier-Stokes:

$$\partial_t \mathbf{v} + P[\mathbf{v}] \cdot \nabla P[\mathbf{v}] = \nu \Delta \mathbf{v}$$

in-between:

$$\boxed{\partial_t \mathbf{v} + P[\mathbf{v}] \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v}}$$

Grafke, Grauer, Sideris:
global existence + infinite # of conserved quantities

Burgers:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v}$$

Existence theory for the Euler equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0 \quad , \quad \mathbf{r} \in \mathbf{R}^d \quad , \quad d = 2, 3$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\mathbf{u}(0, \mathbf{r}) = \mathbf{u}^0(\mathbf{r}) \in H^s(\mathbf{R})$$

Beale–Kato–Majda (1984): $\exists$ a global solution for $d = 3, s \geq 3$

$$\mathbf{u} = C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$$

iff for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ holds

$$\boxed{\int_0^T \|\boldsymbol{\omega}(t, \cdot)\|_{L^\infty} dt < \infty}$$

for every $T > 0$.
(Similar statements for MHD and quasigeostrophic flow.)

Kato (1972): It exists a global solution for $d = 2, s \geq 3$

$$\|\boldsymbol{\omega}(t, \cdot)\|_{L^\infty} = \|\boldsymbol{\omega}(0, \cdot)\|_{L^\infty} \ .$$

Constantin, Fefferman, Majda (1996): If vorticity vector

$$\xi(\mathbf{x}, t) = \frac{\omega(\mathbf{x}, t)}{\|\omega(\mathbf{x}, t)\|}$$

is smoothly directed in an $O(1)$ region, i.e. the maximum norm of $\nabla\xi$ is L^2 integrable in time from 0 to T in that region, and the maximum norm of velocity in this region is uniformly bounded, then no blow-up up exists up to time T.

Cordoba, Fefferman (2001):
Vortex tubes with $O(1)$ length that don't twist or bend enough are ruled out if the infinity norm of velocity in a neighborhood of that region is integrable in time.

Cordoba, Fefferman (2001):
If a current sheet has a potato chip like structure and $\int_0^\infty sub_{x\in U} |\mathbf{u}(\mathbf{x}, t)| \, dt < \infty$ than no blow-up.

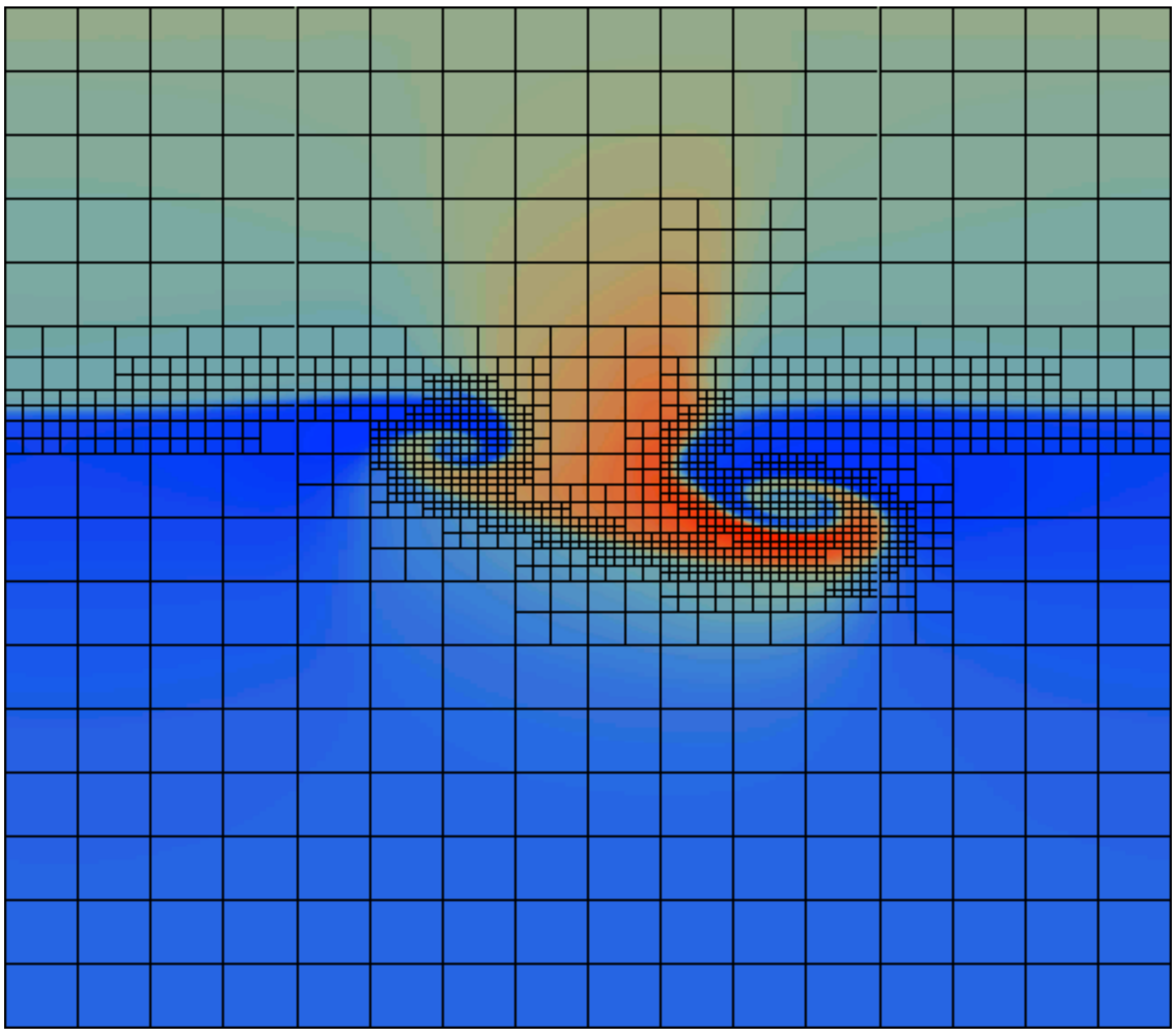
Deng, Hou, Yu (2005, 2006)

Theorem I, Theorem II will be discussed later

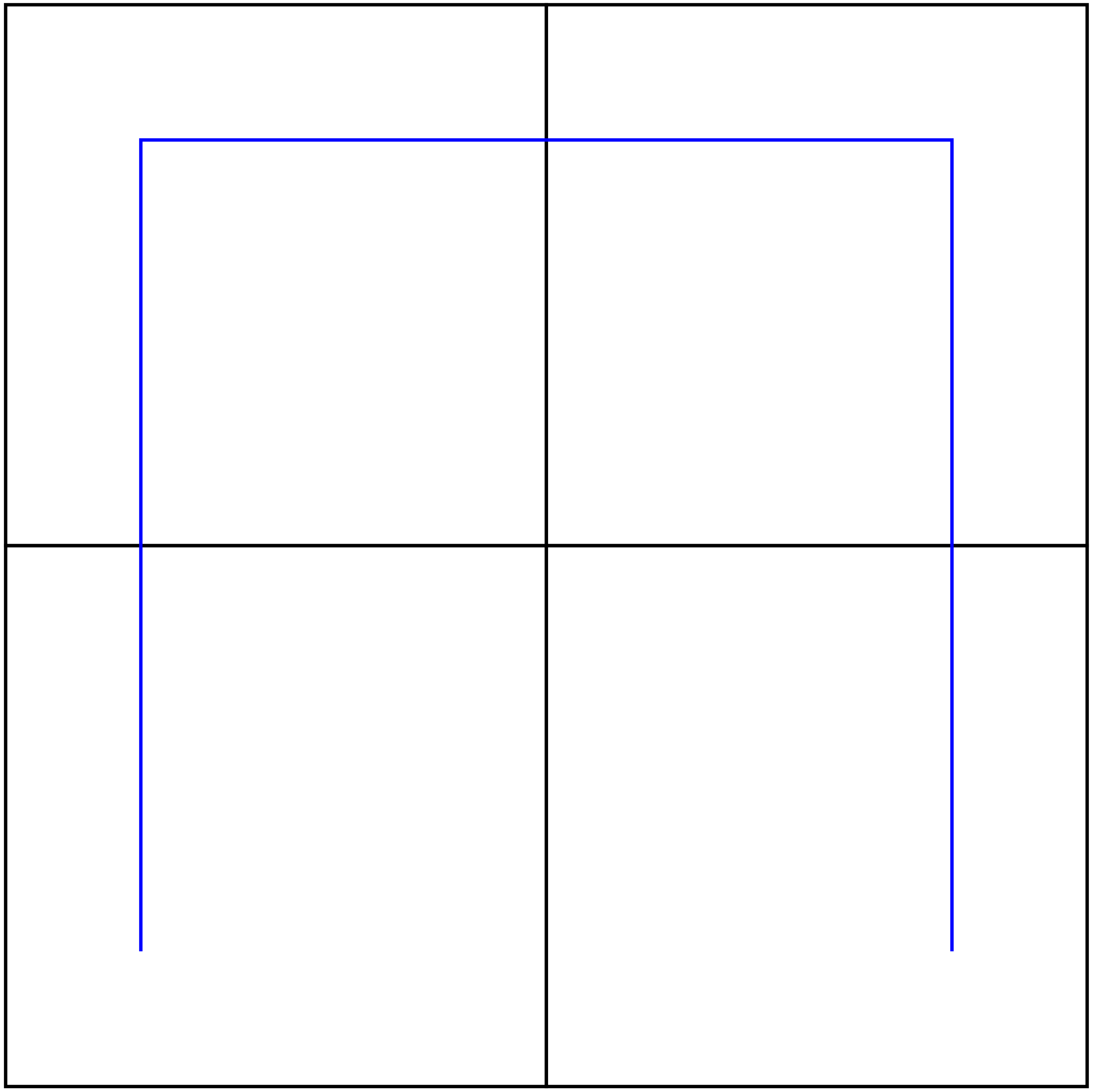
Dreher, **Grafke**, Grauer

racoon: really adaptive computations using object oriented numerics

Hilbert curves: Quadtree (2D), Octtree (3D)



distribution and load balancing: Hilbert-type space filling curve



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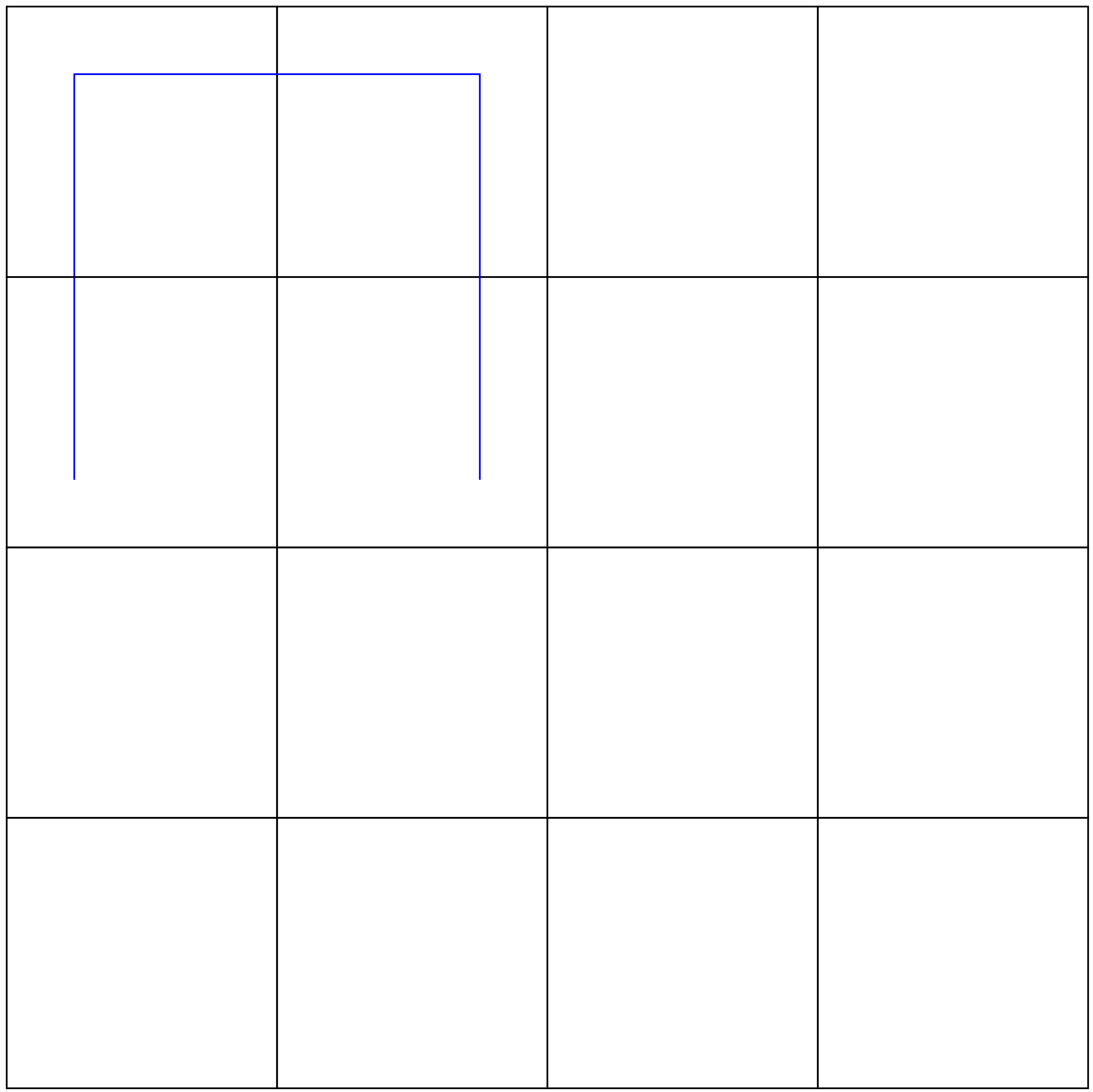

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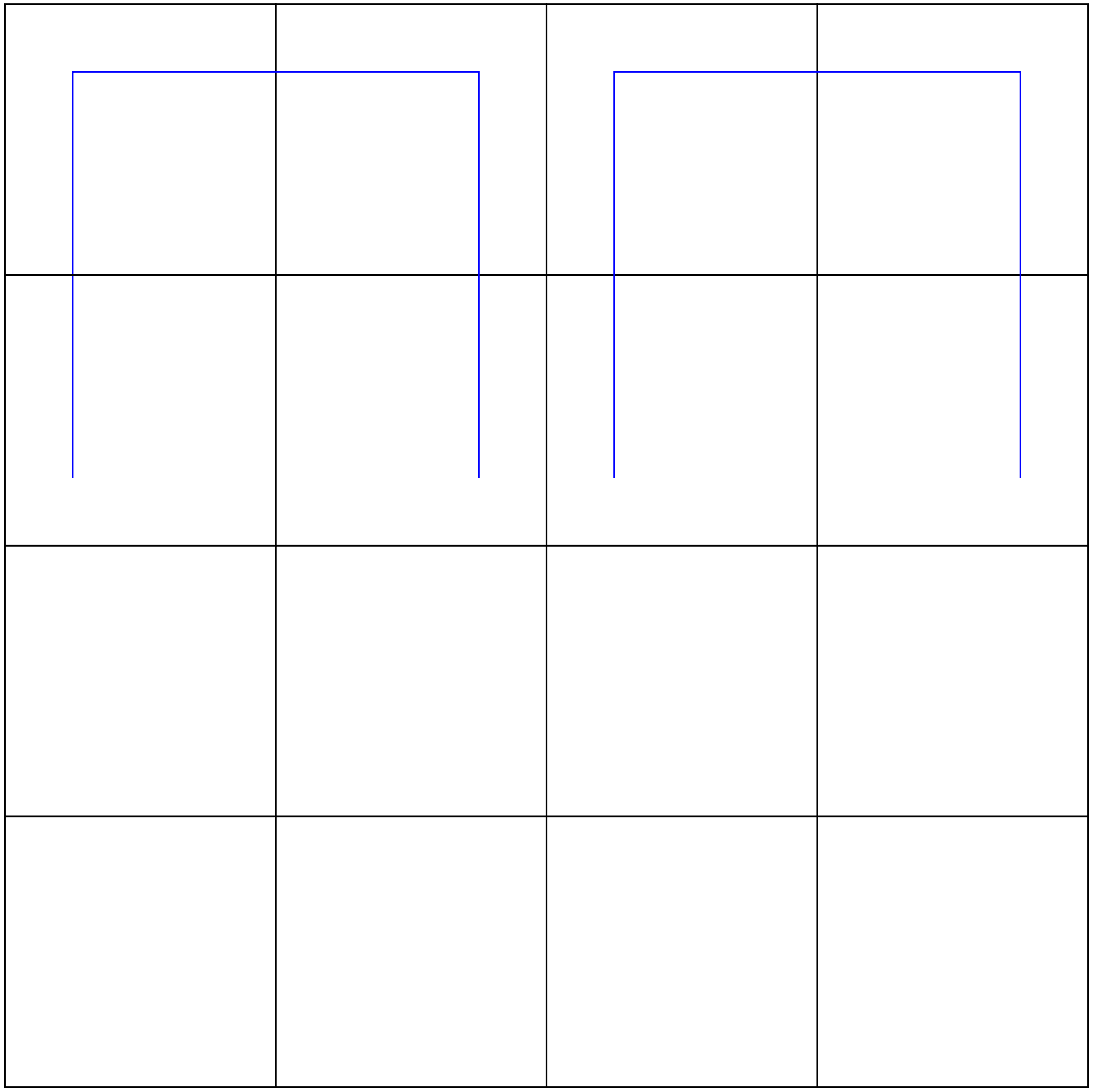

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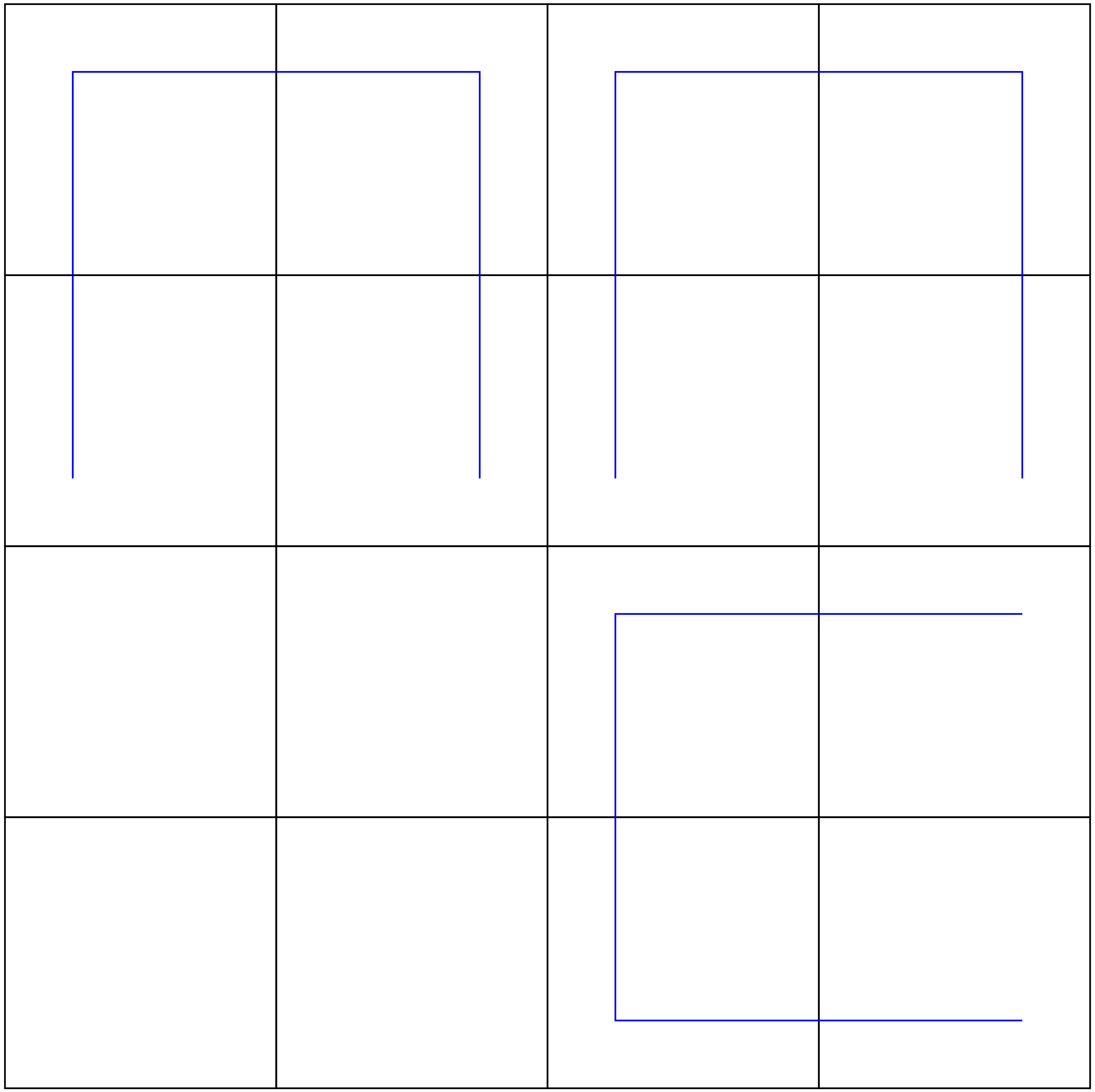
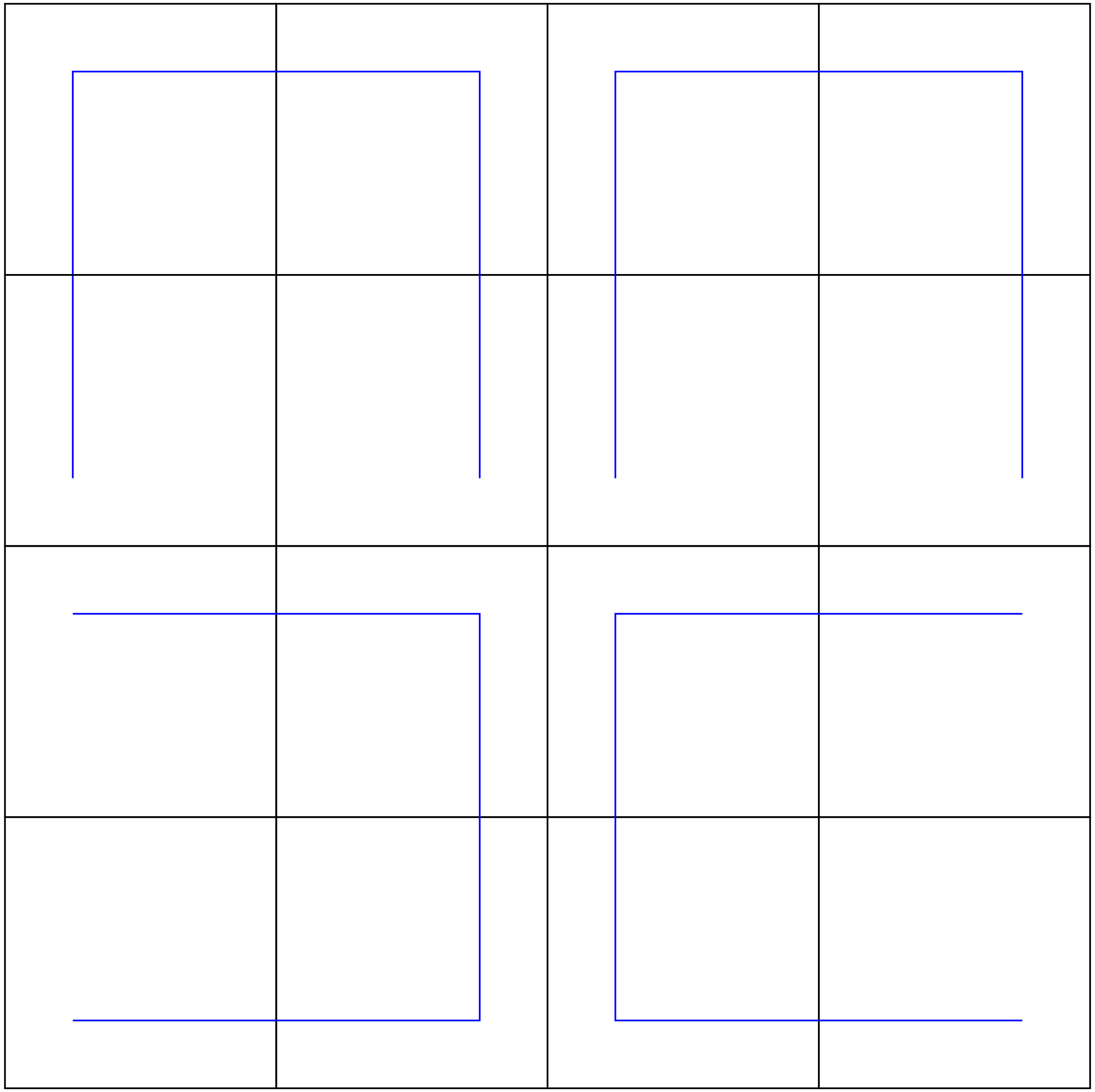
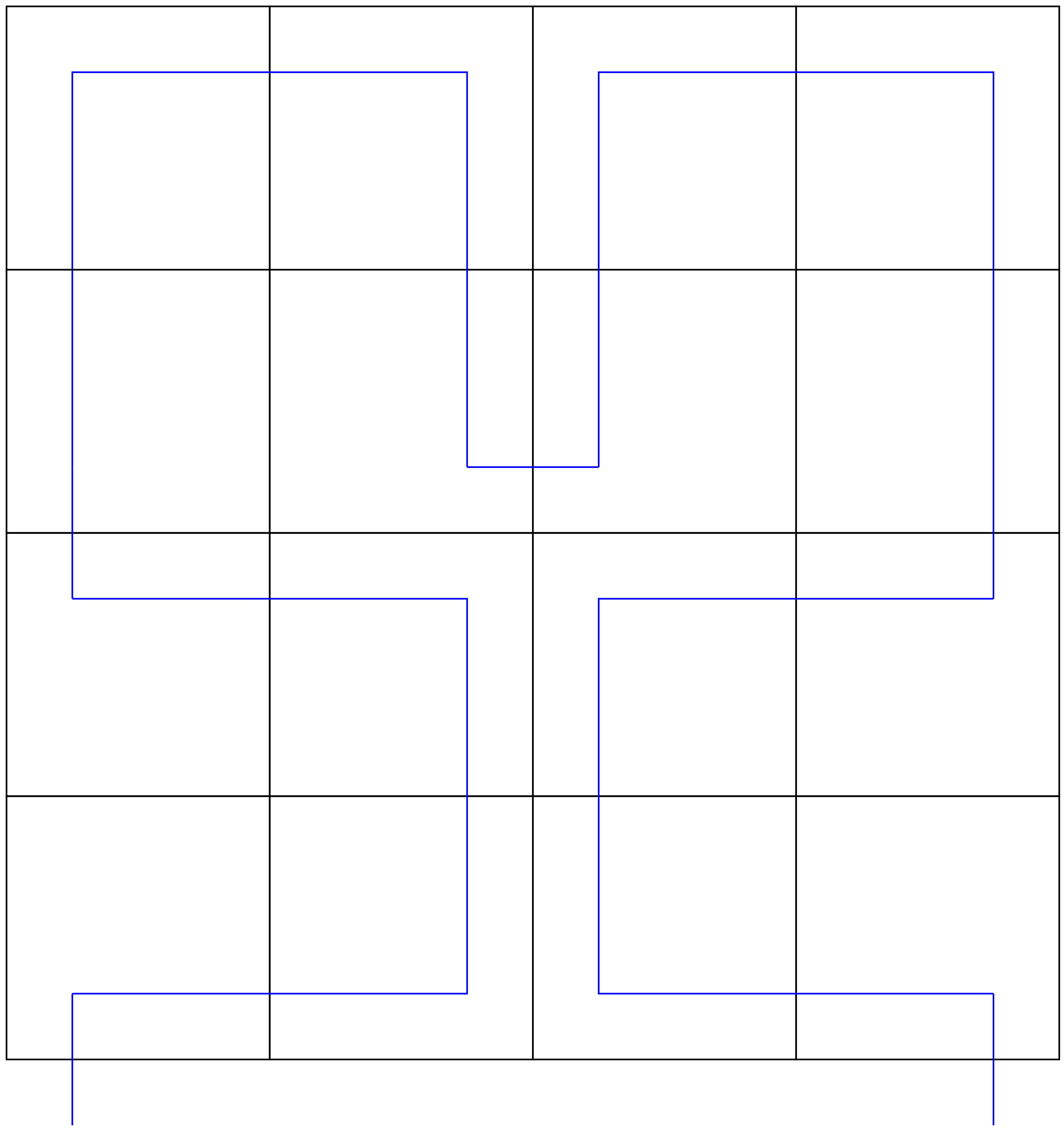


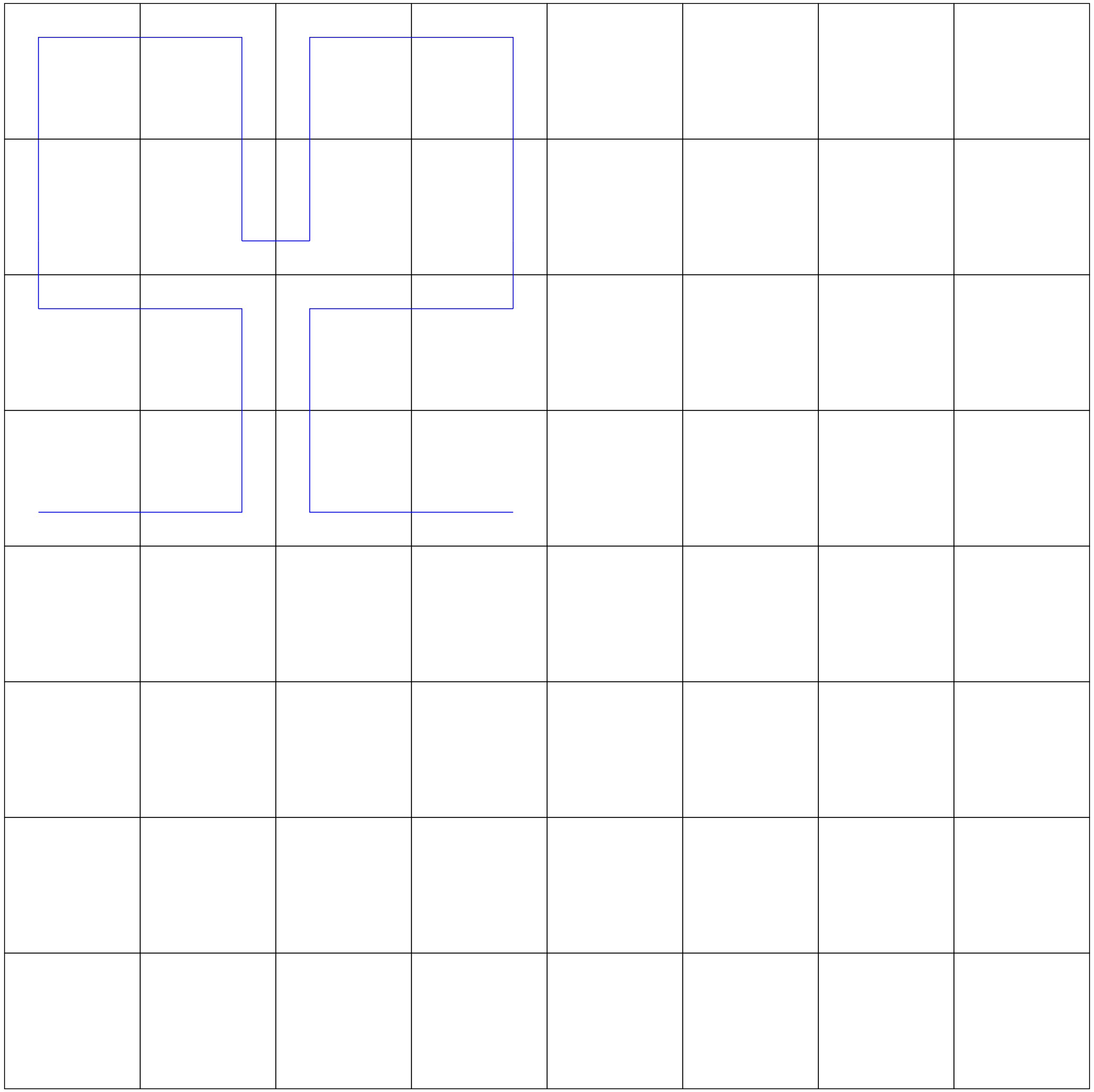
distribution and load balancing: Hilbert-type space filling curve



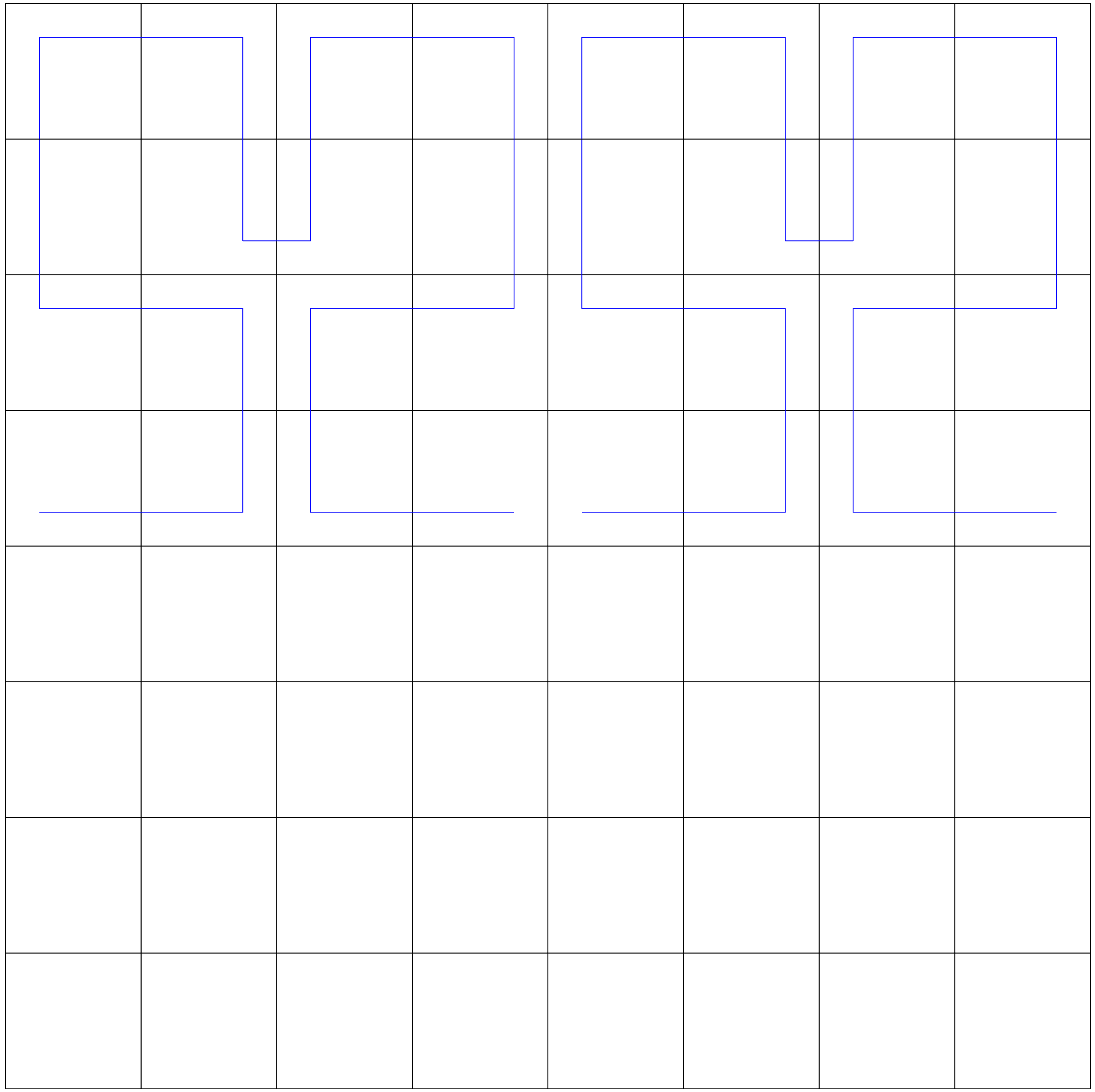
distribution and load balancing: Hilbert-type space filling curve



distribution and load balancing: Hilbert-type space filling curve



distribution and load balancing: Hilbert-type space filling curve



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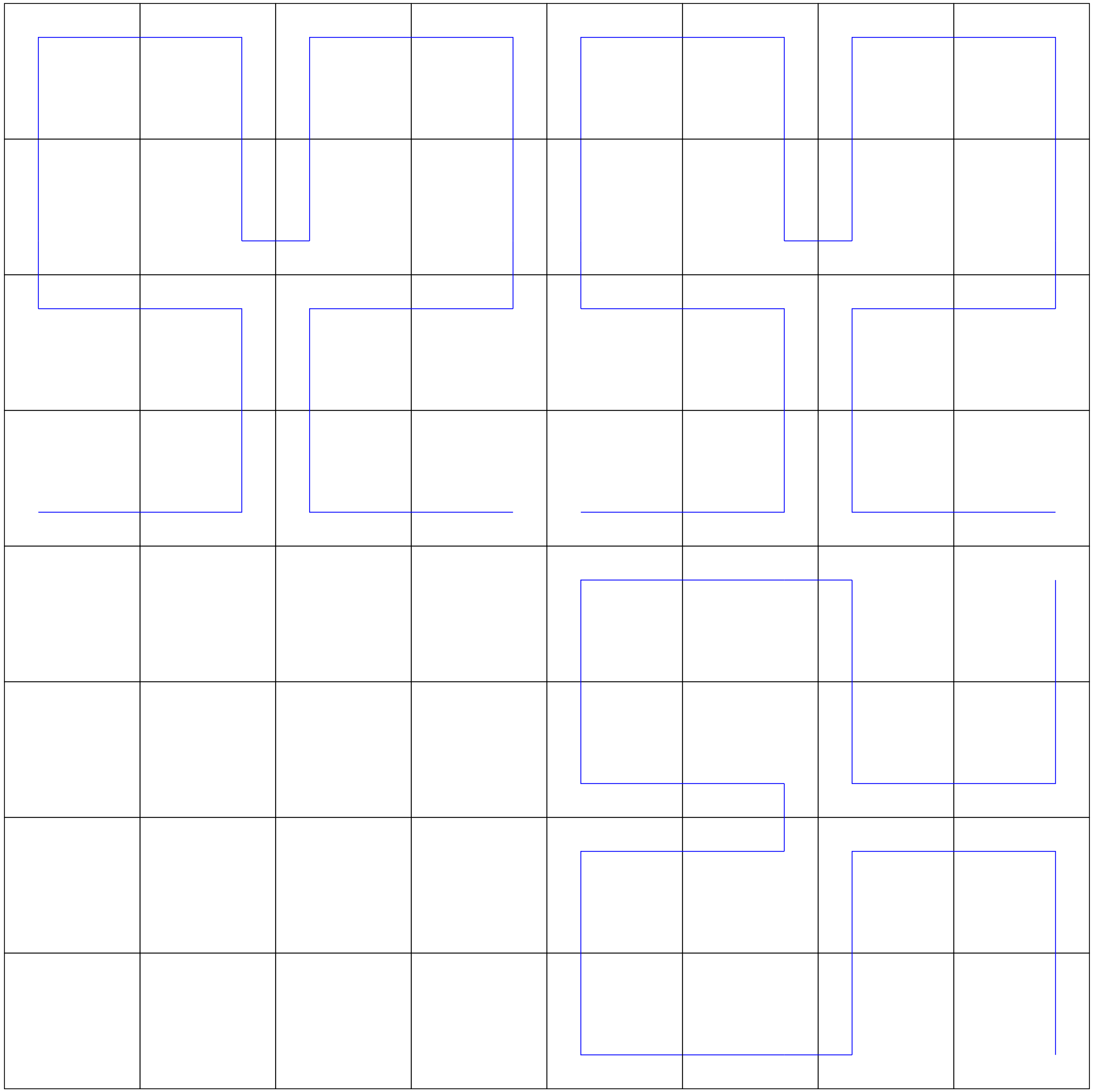
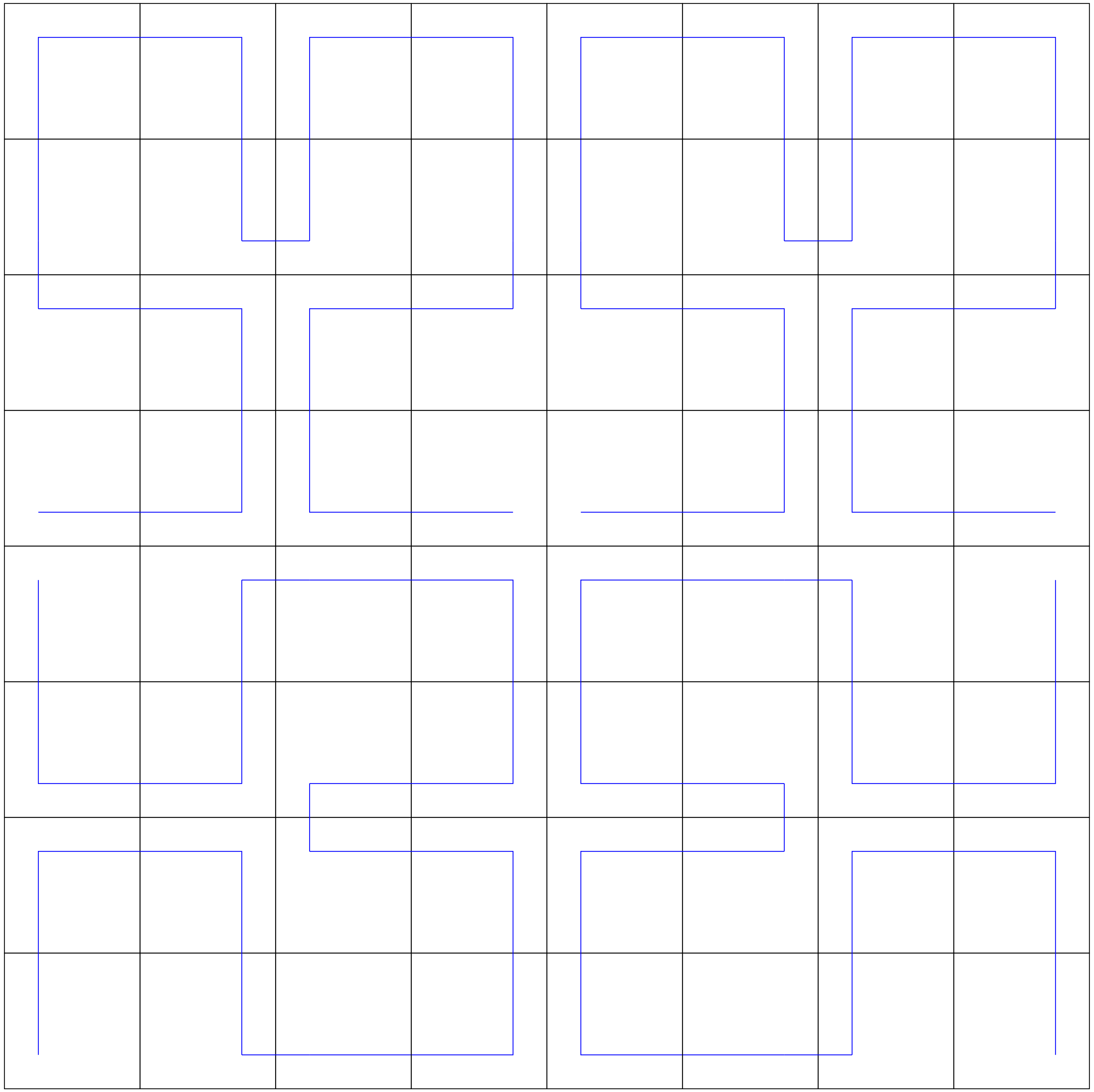
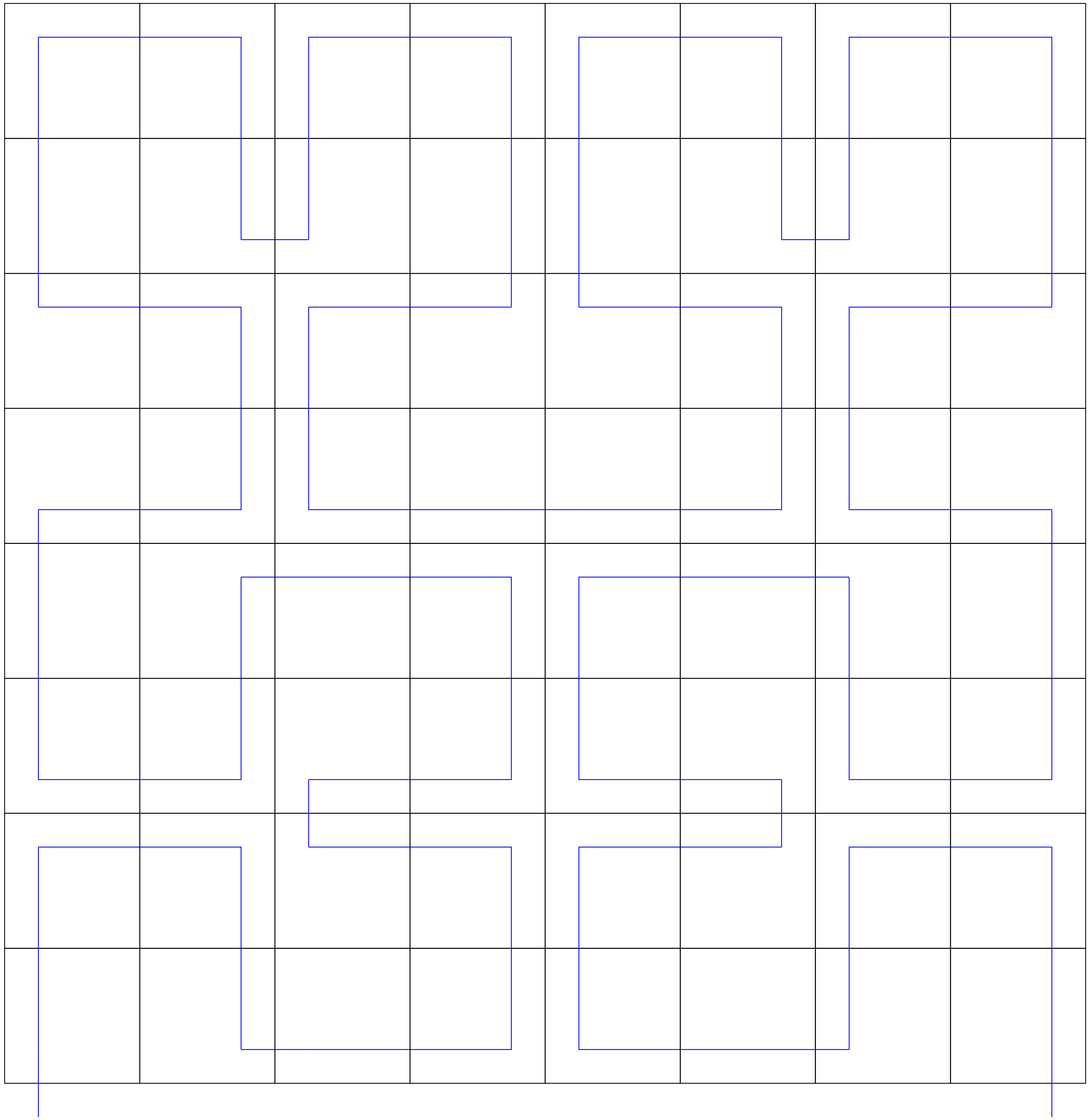


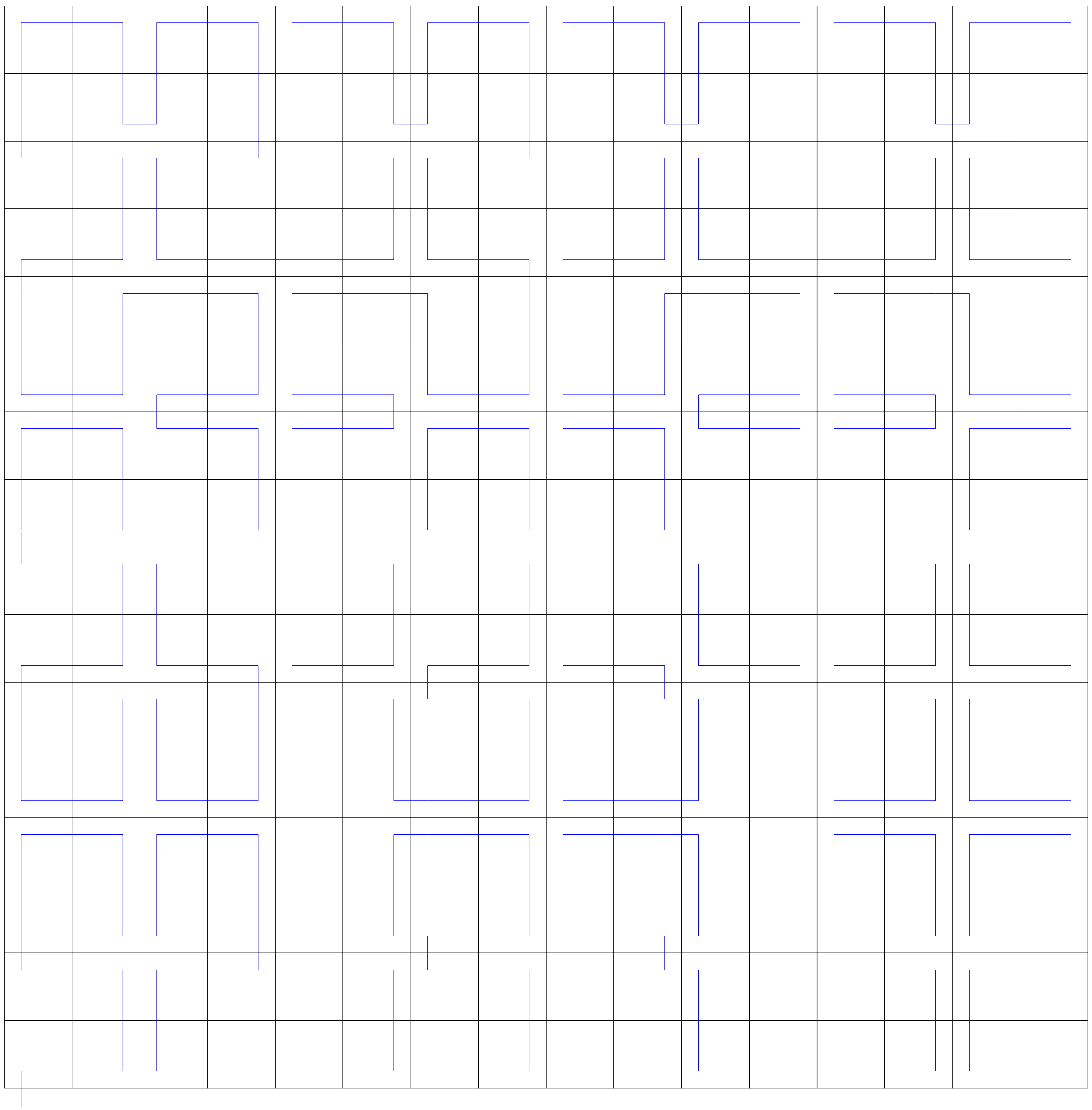
distribution and load balancing: Hilbert-type space filling curve



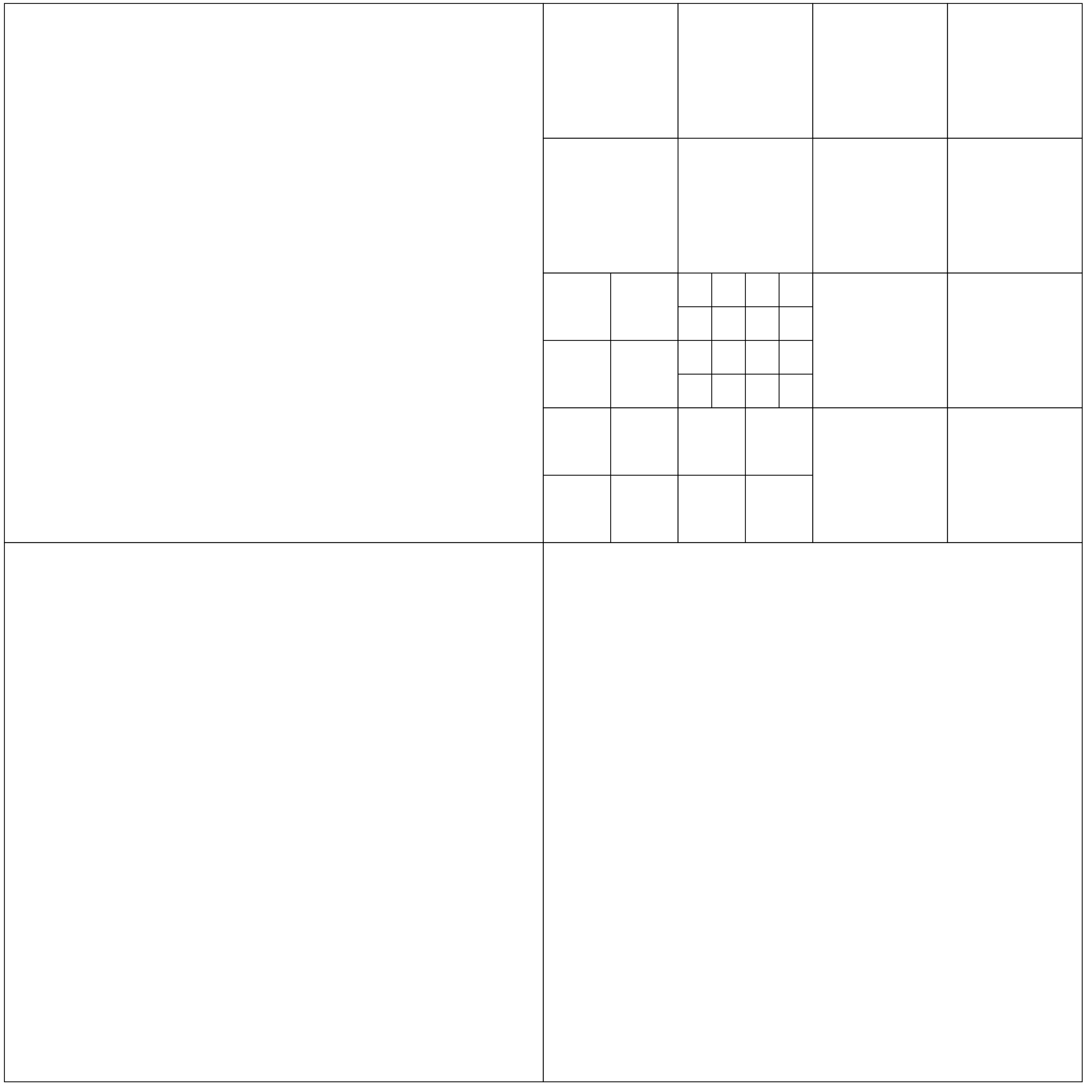
distribution and load balancing: Hilbert-type space filling curve



distribution and load balancing: Hilbert-type space filling curve



distribution and load balancing: Hilbert-type space filling curve



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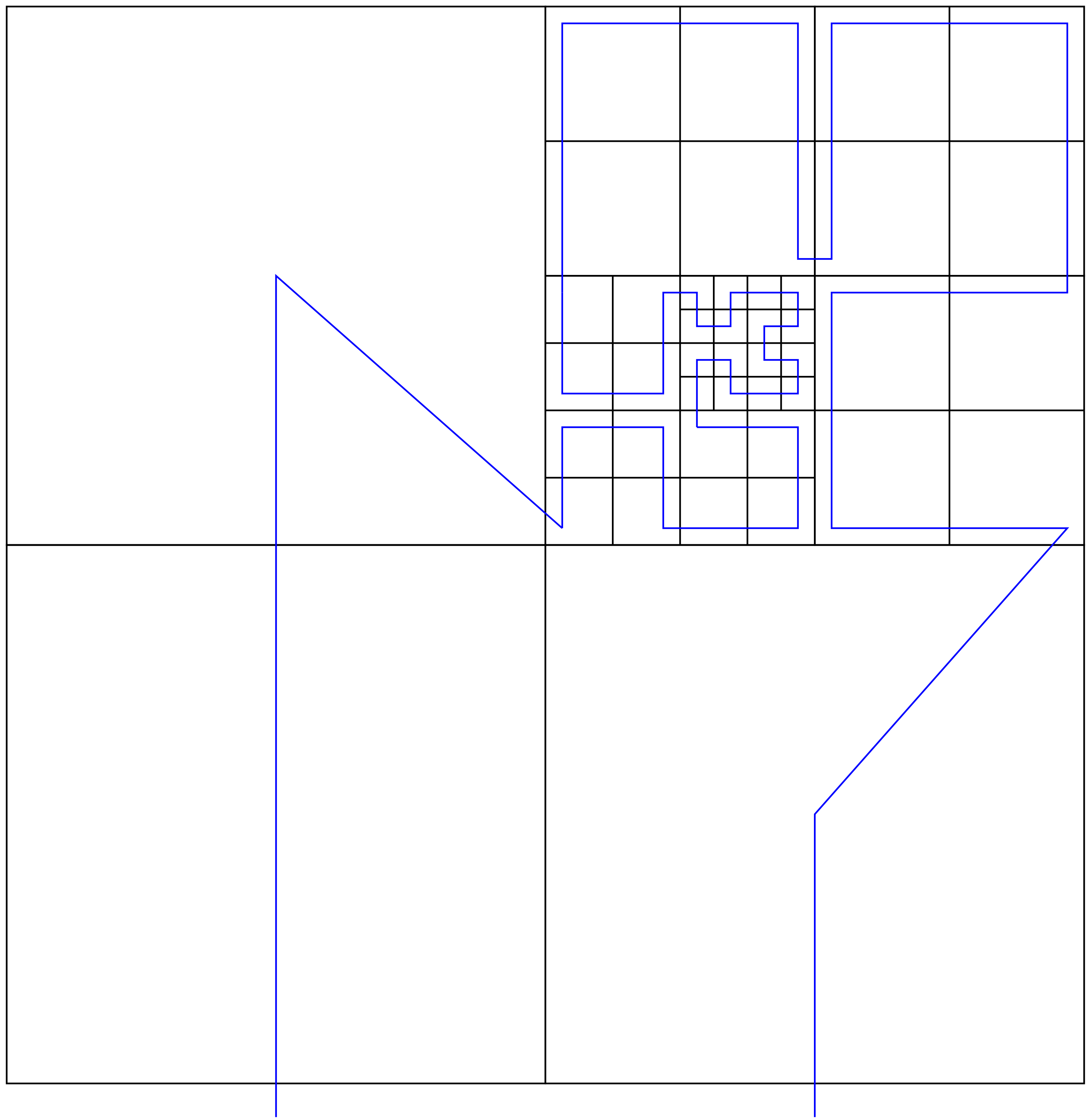
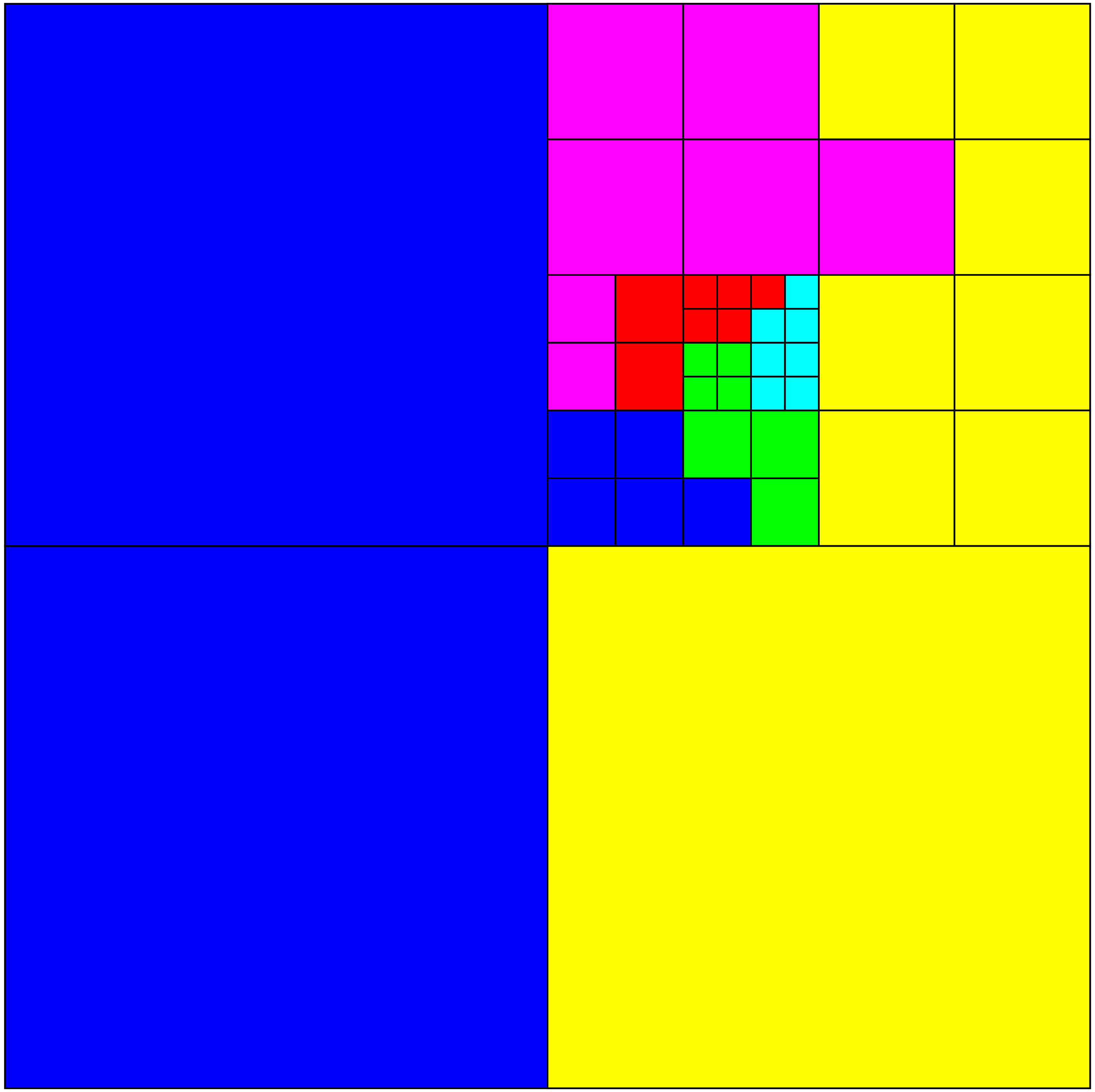
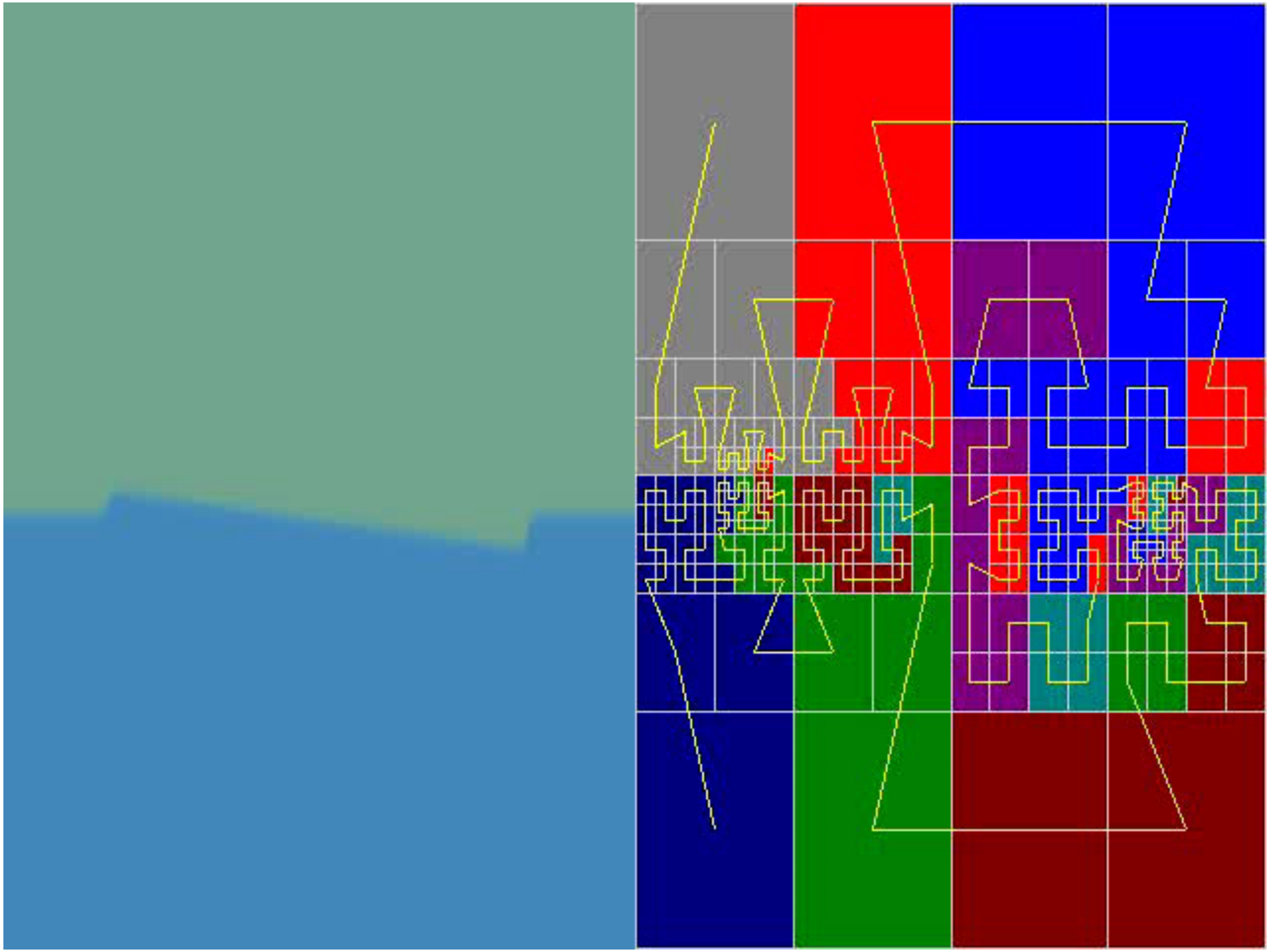
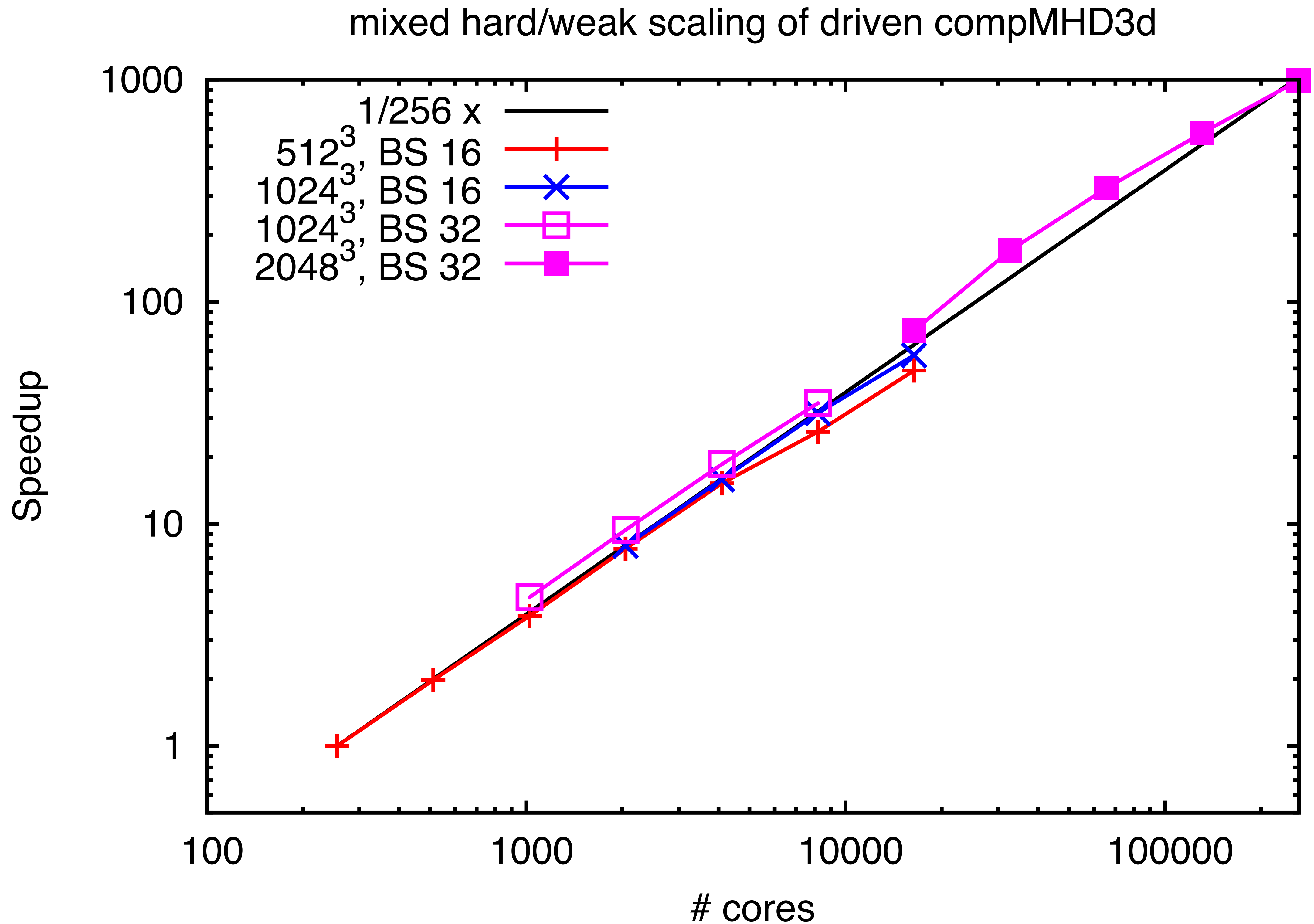
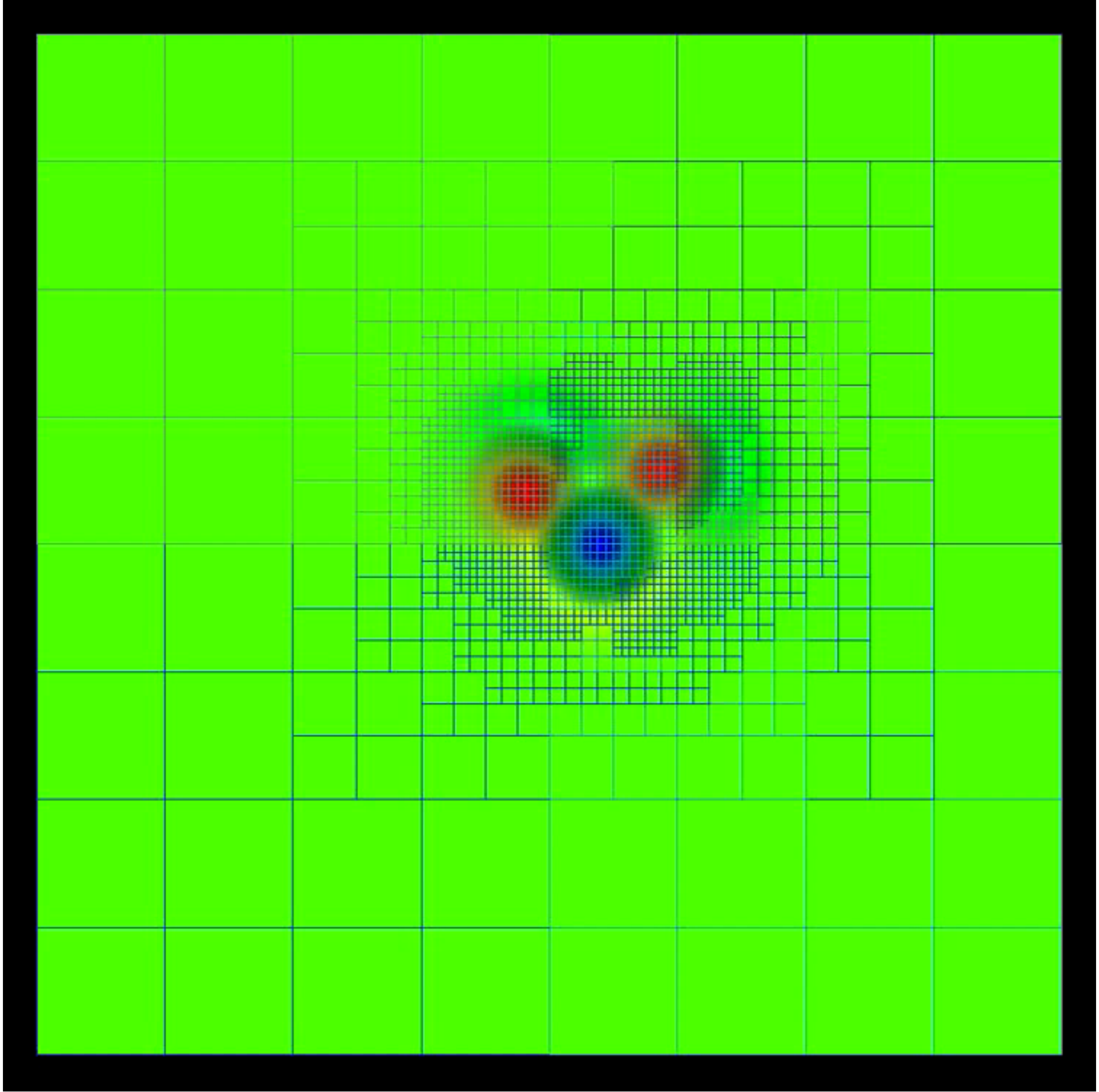


distribution and load balancing: Hilbert-type space filling curve





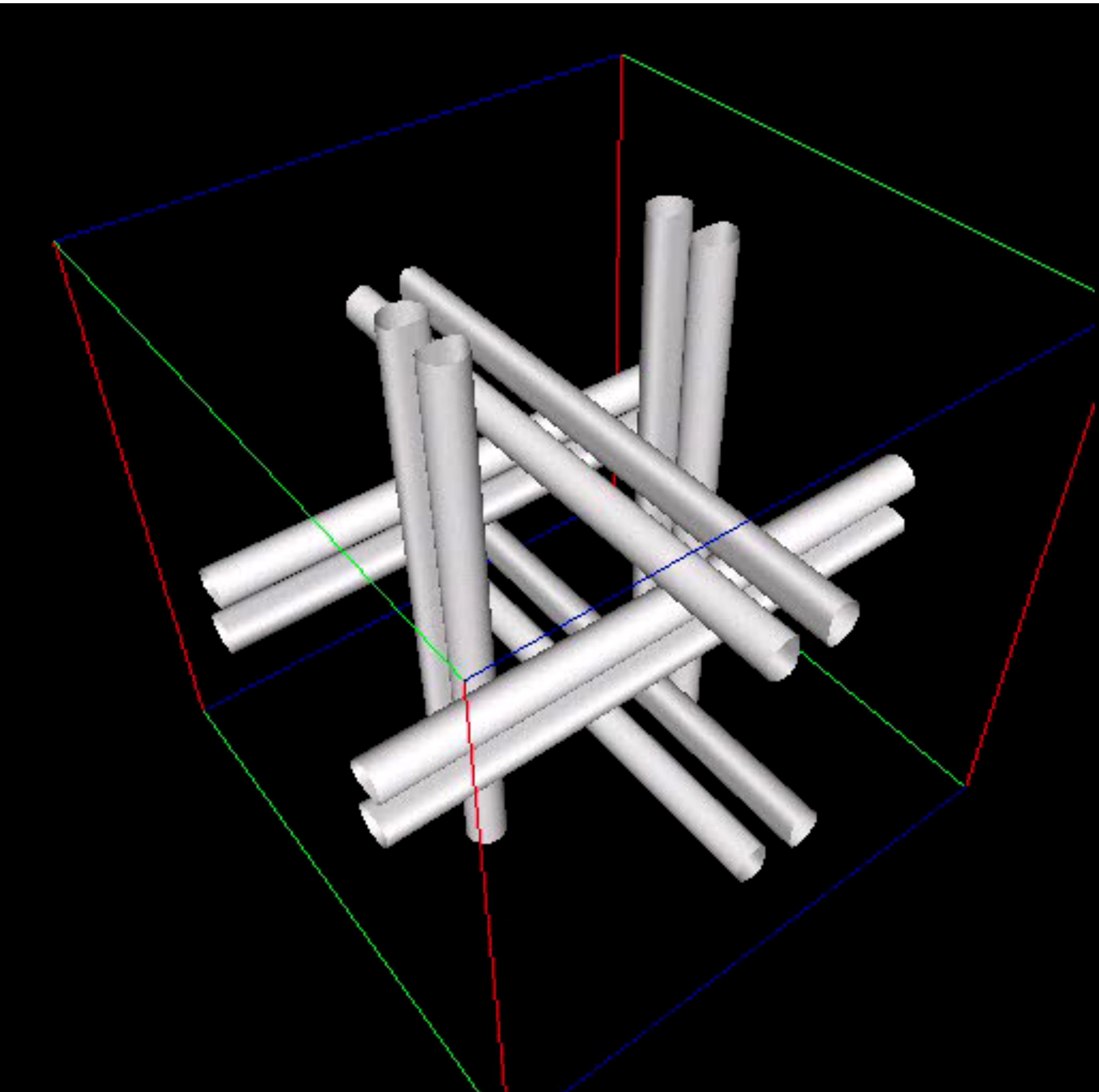

mixed hard/weak scaling of driven compMHD3d
1/256 x
512^3, BS 16
1024^3, BS 16
1024^3, BS 32
2048^3, BS 32
Speedup
cores
1000
100
10
1
100
1000
10000
100000



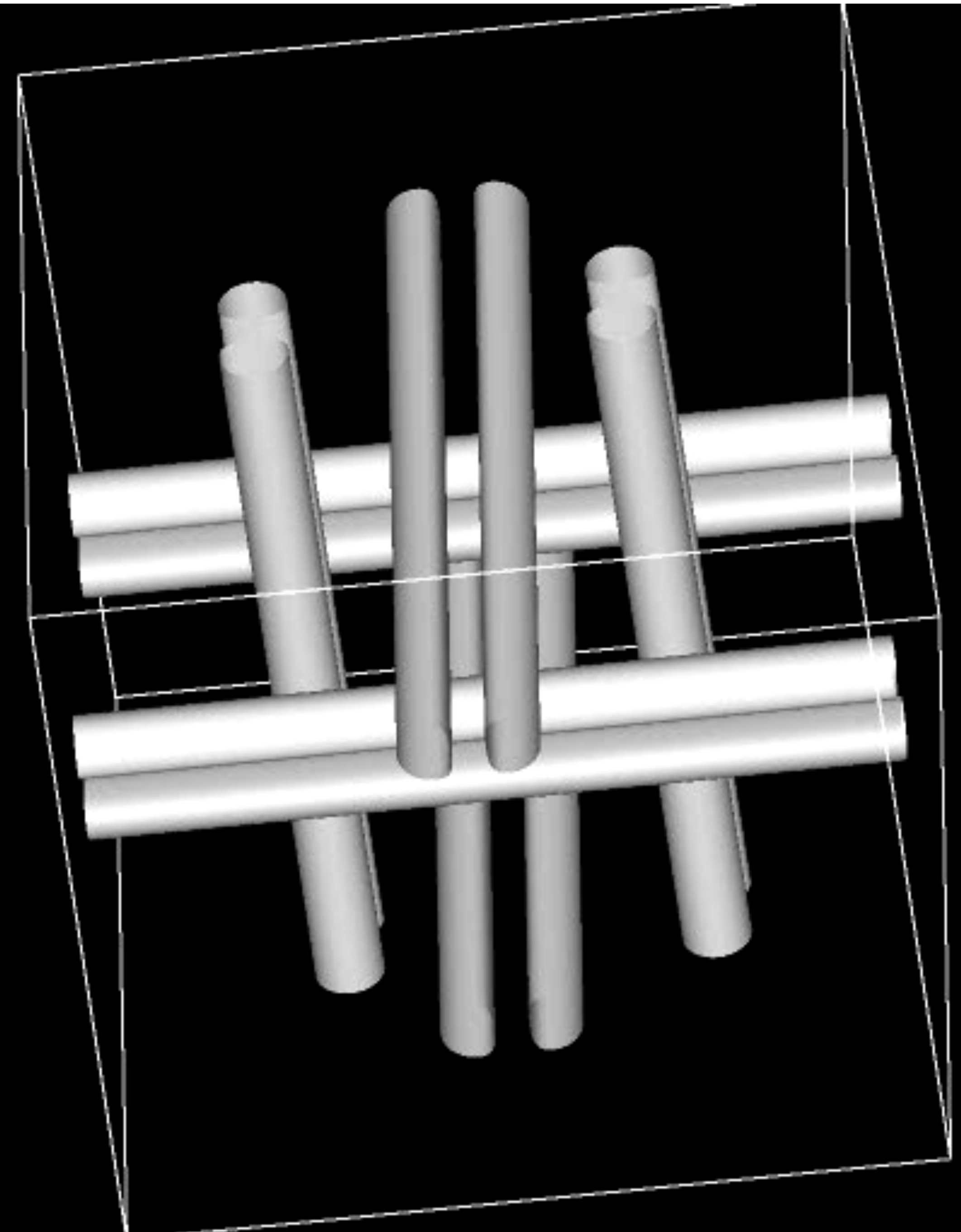
Pelz initial conditions

resolution on JUMP (FZ Jülich): 4096^3

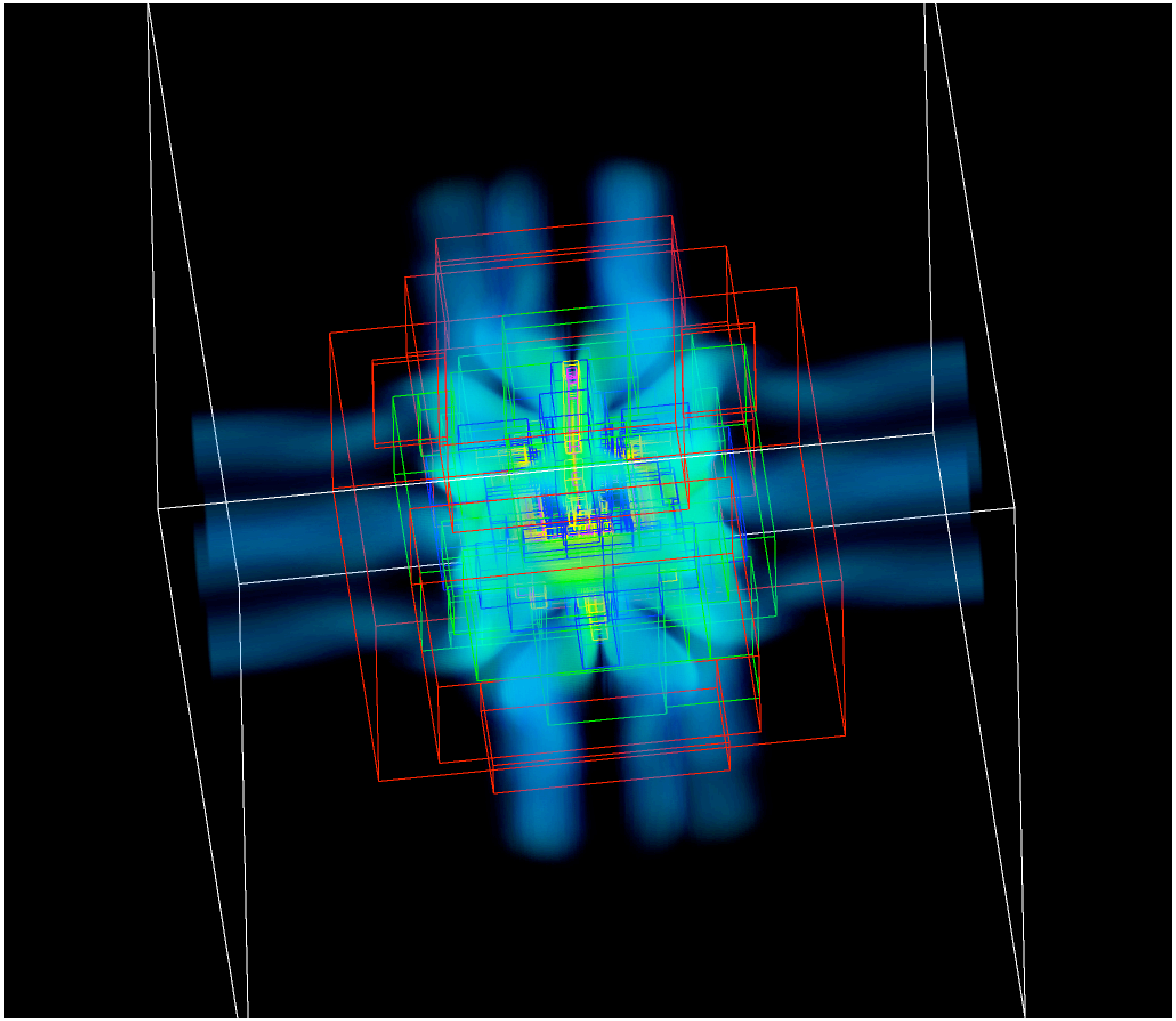
symmetry breaking



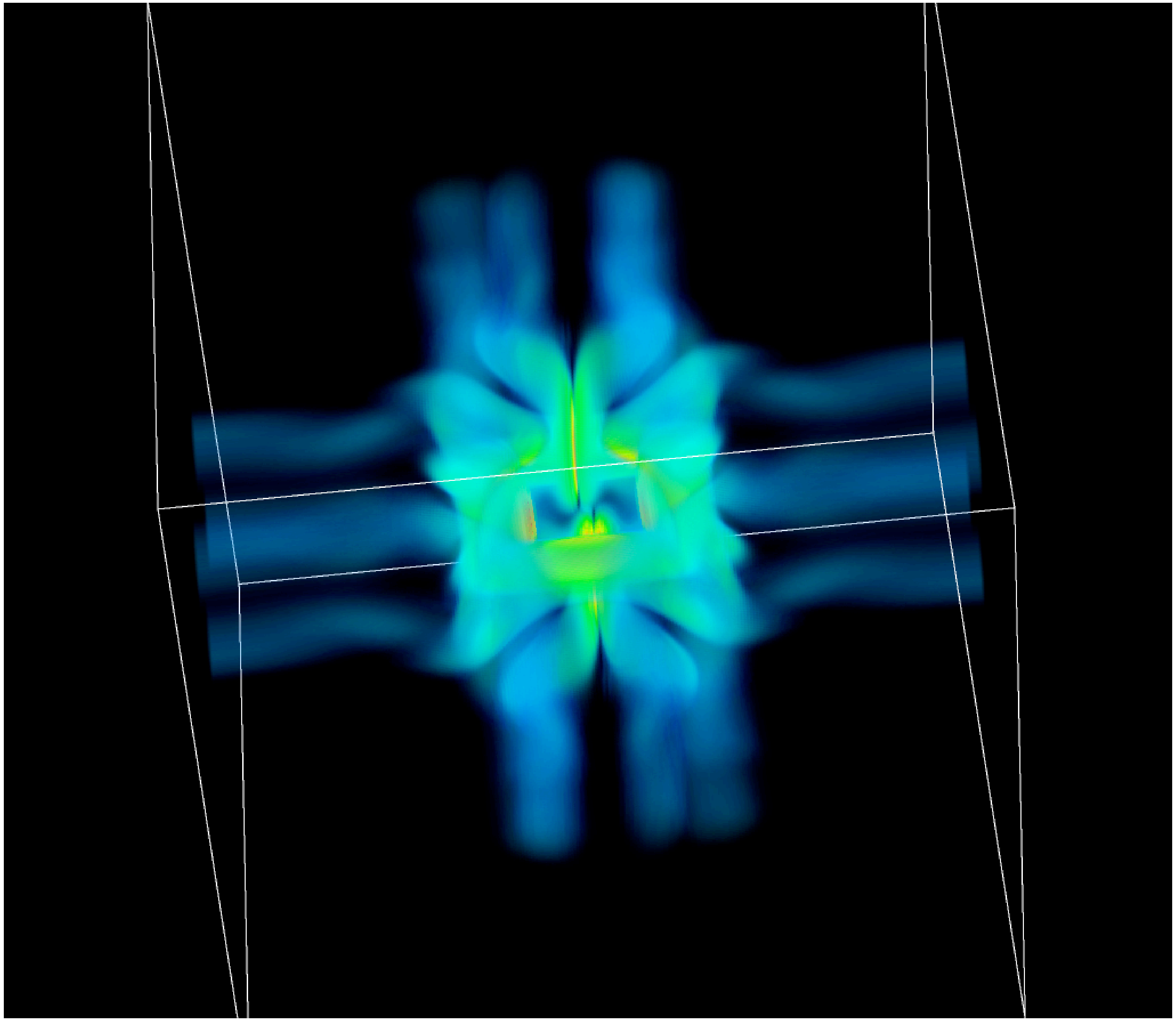
keeping symmetry artificially



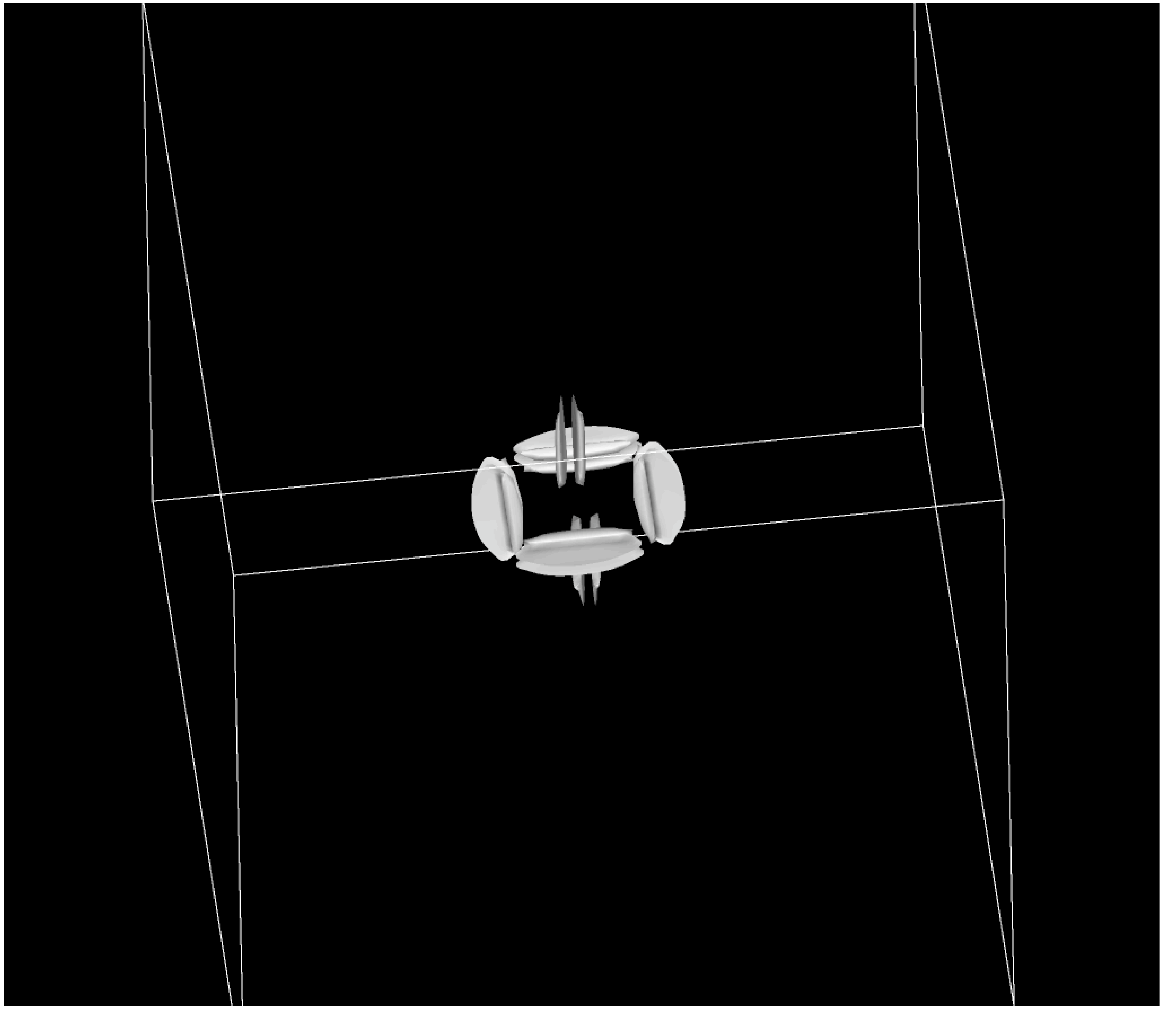
Volume rendering of vorticity at time 0.57



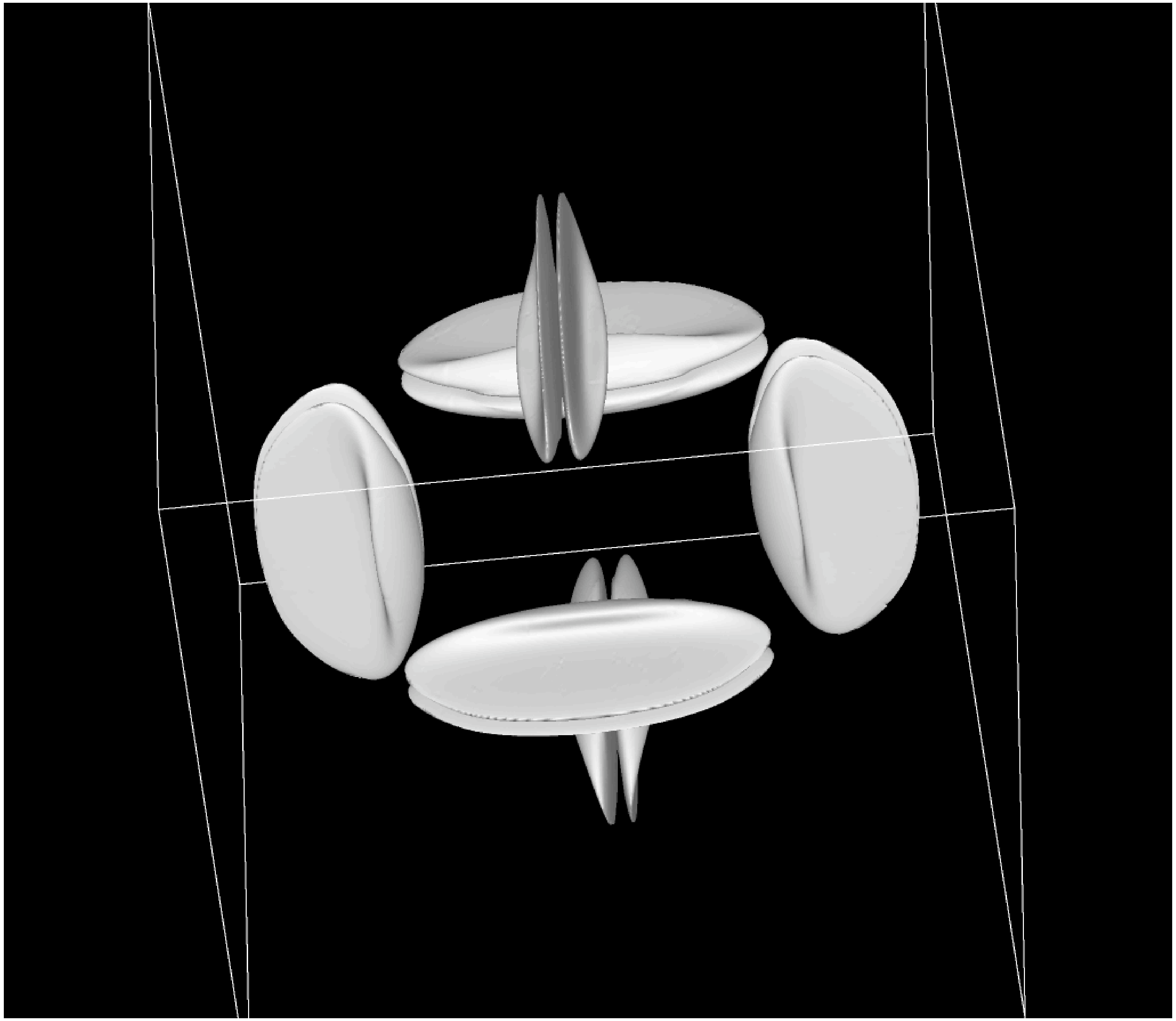
without grids

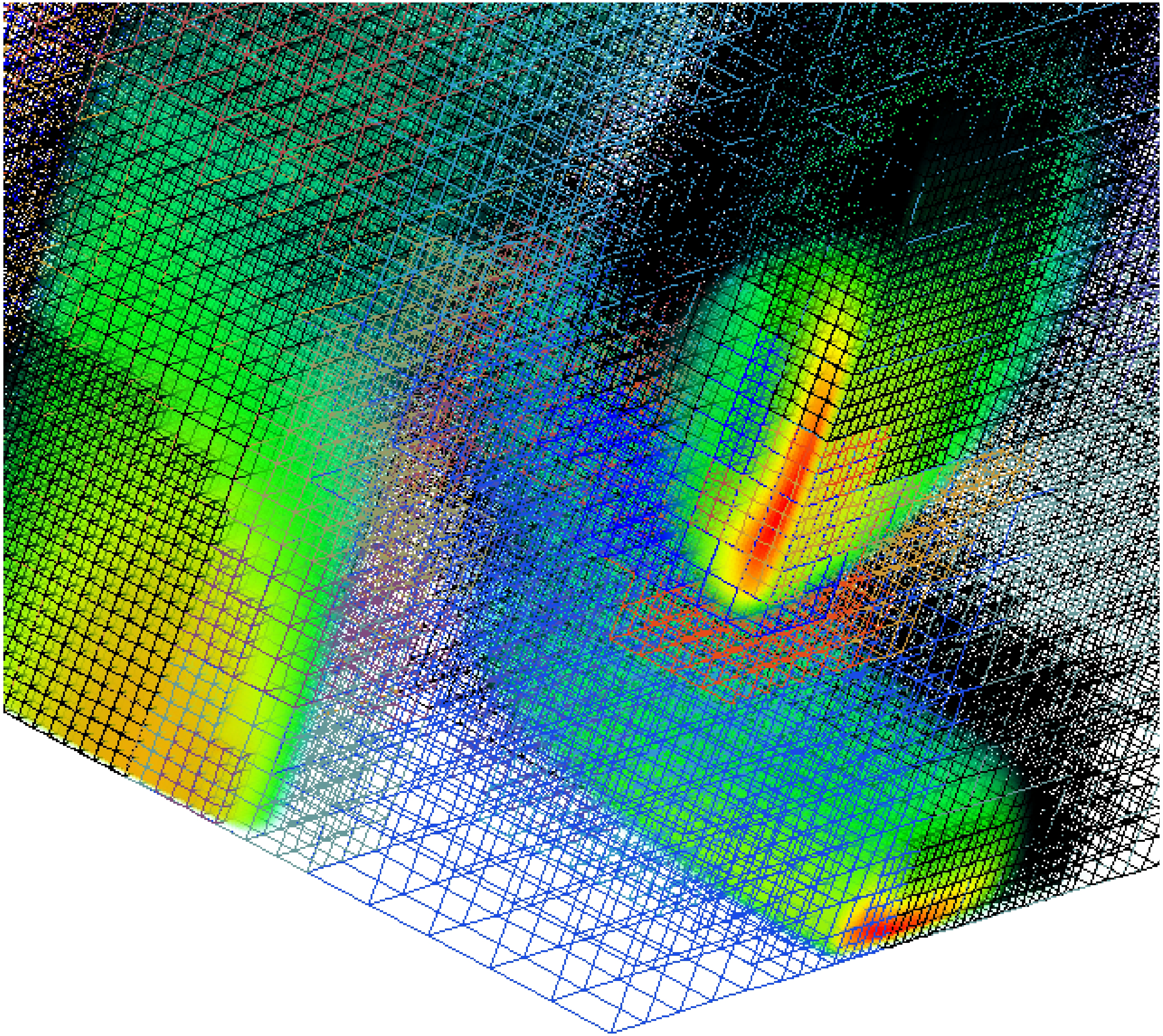


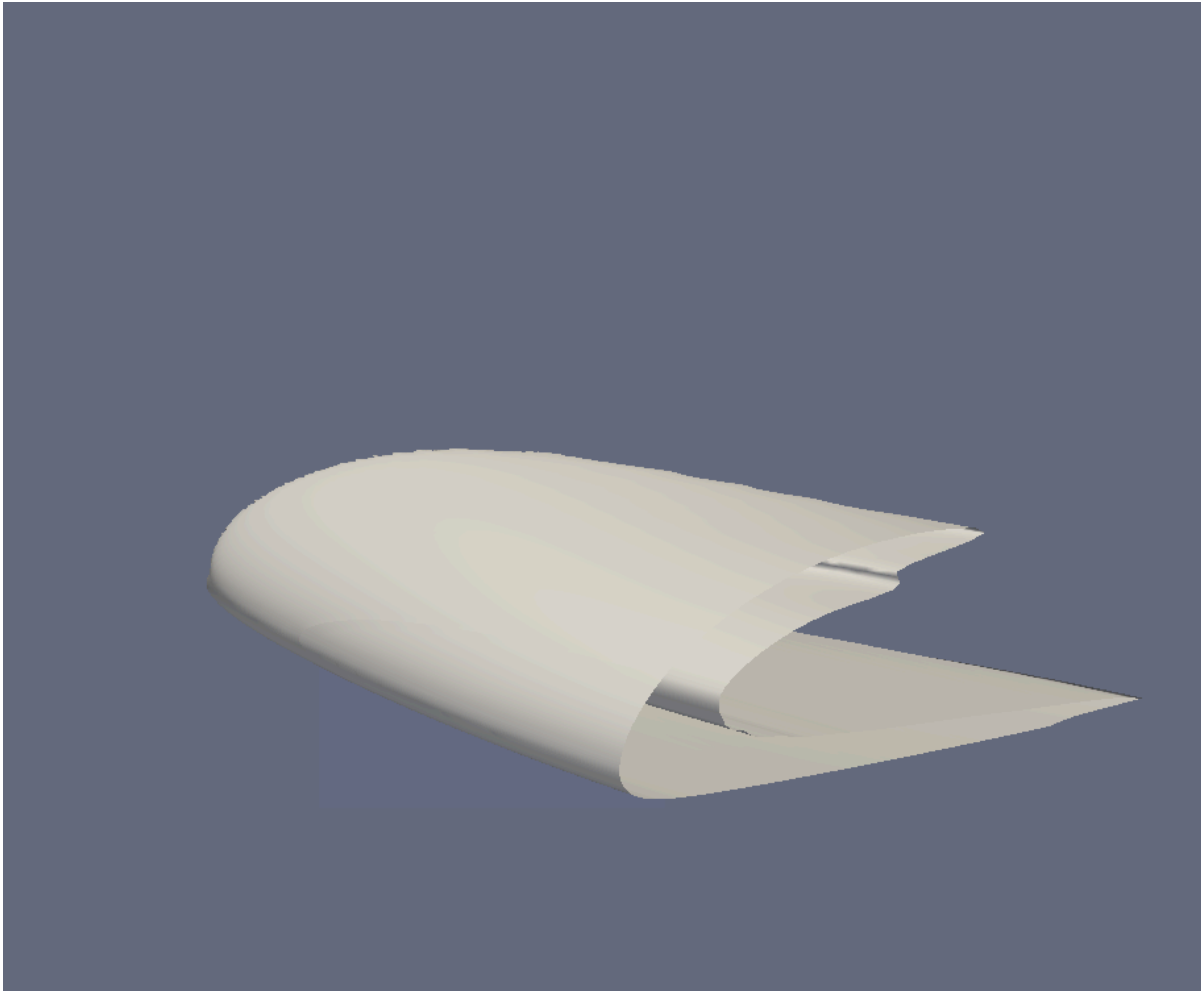
isosurface 70%

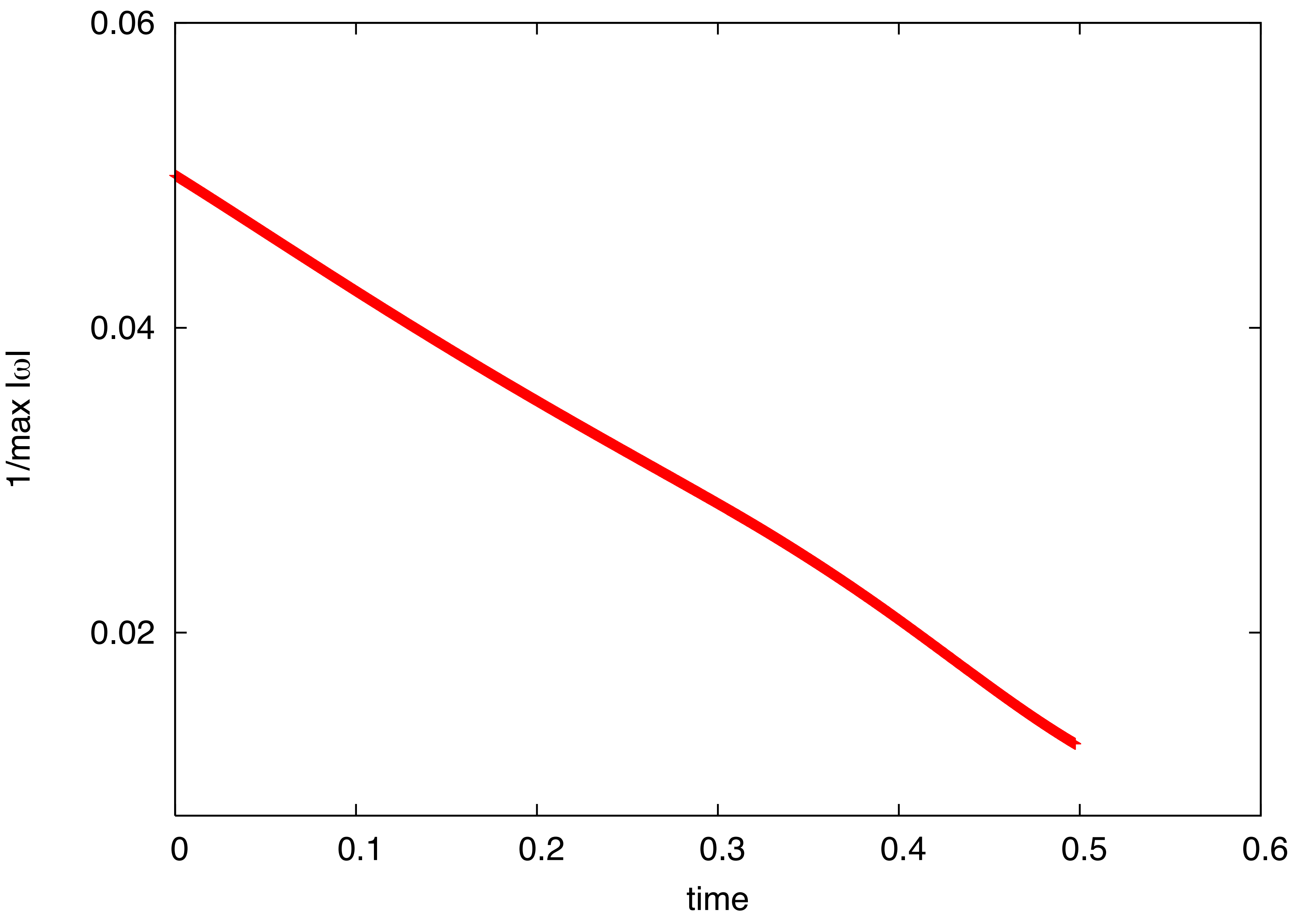


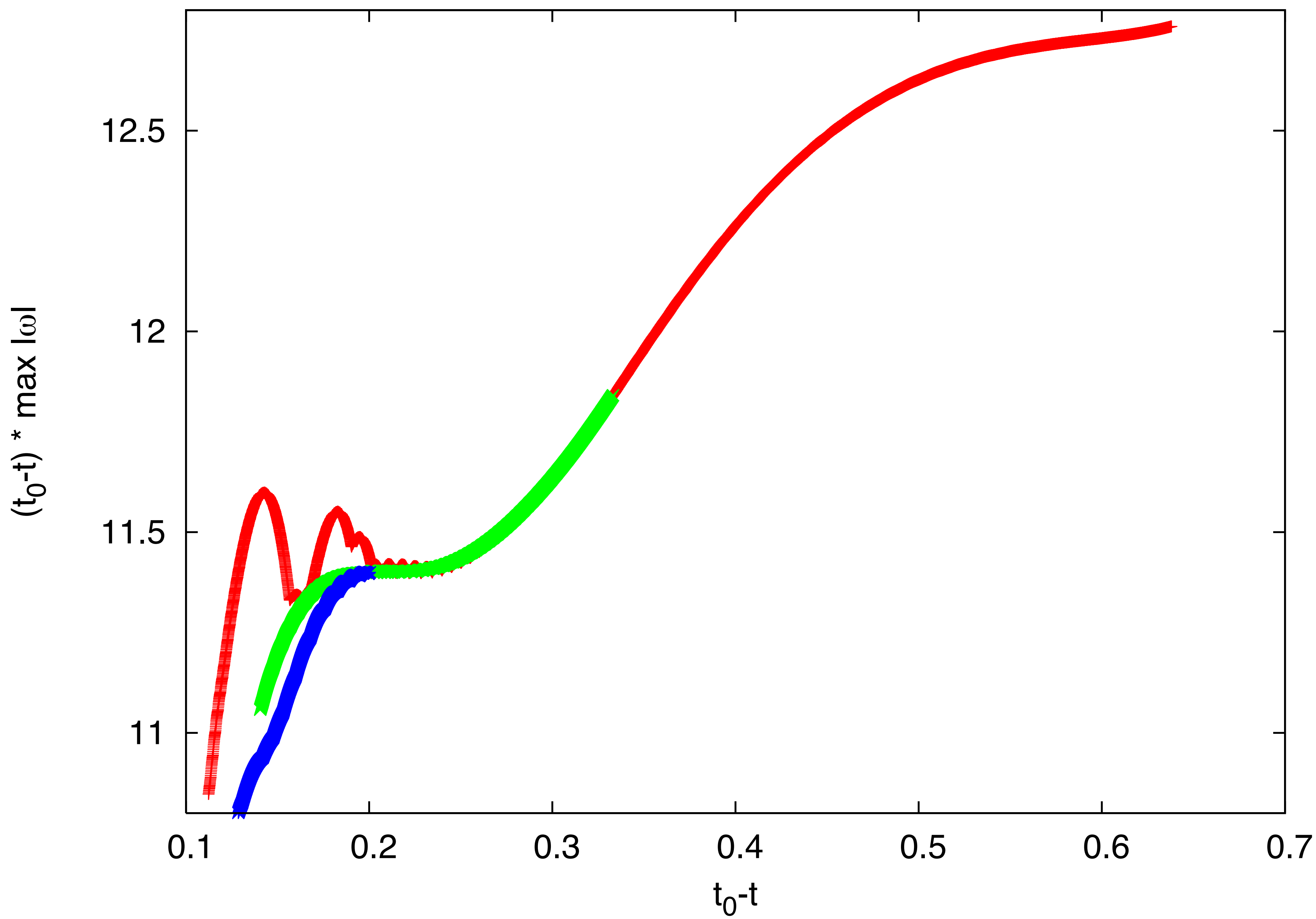
isosurface 70%, zoom



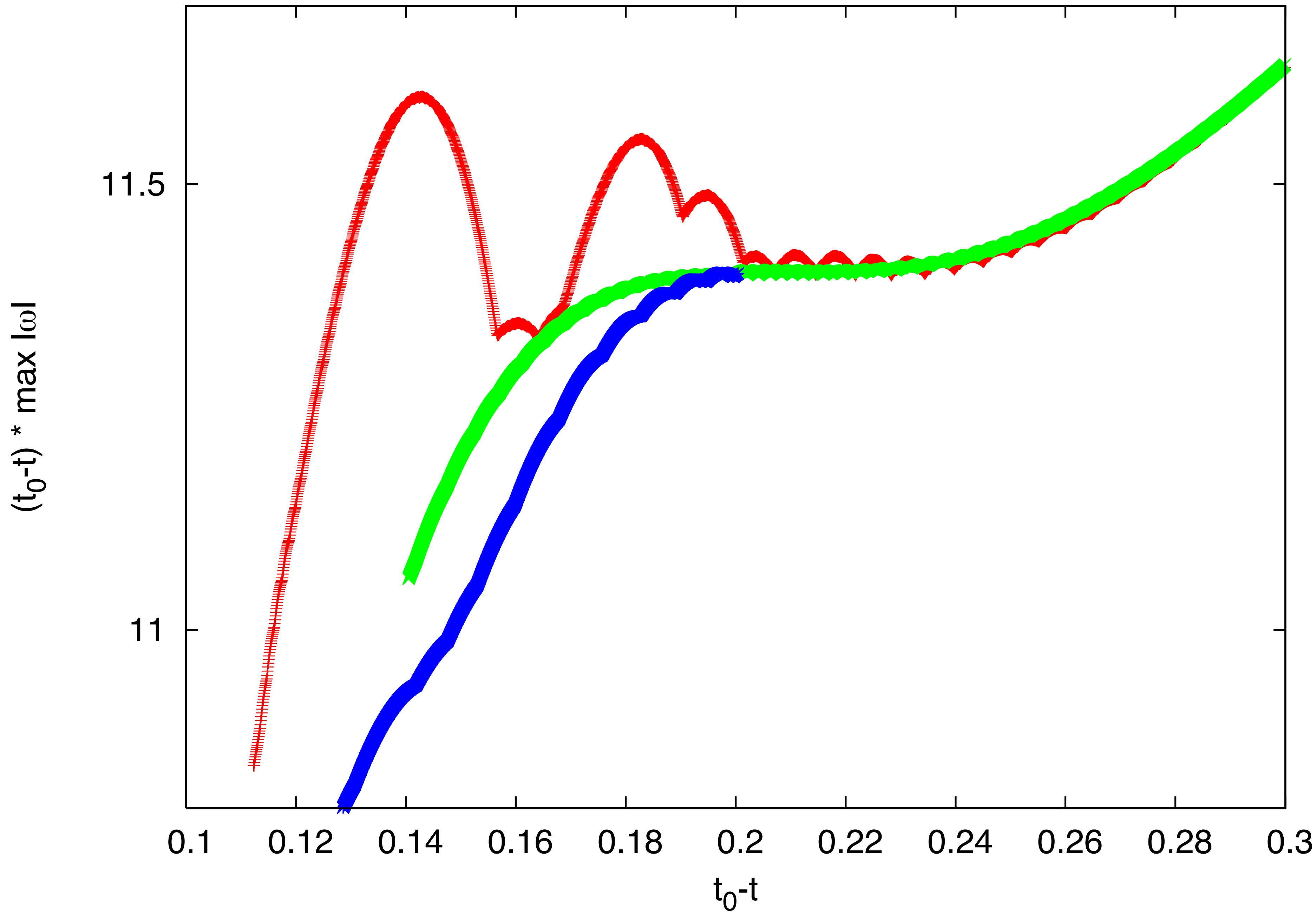





0.06
0.04
0.02
1/max |ω|
0
0.1
0.2
0.3
0.4
0.5
0.6
time



12.5
12
11.5
11
0.1
0.2
0.3
0.4
0.5
0.6
0.7
(t_0-t) * max |ω|
t_0-t



message: need extreme resolution

DHY Theorem 1

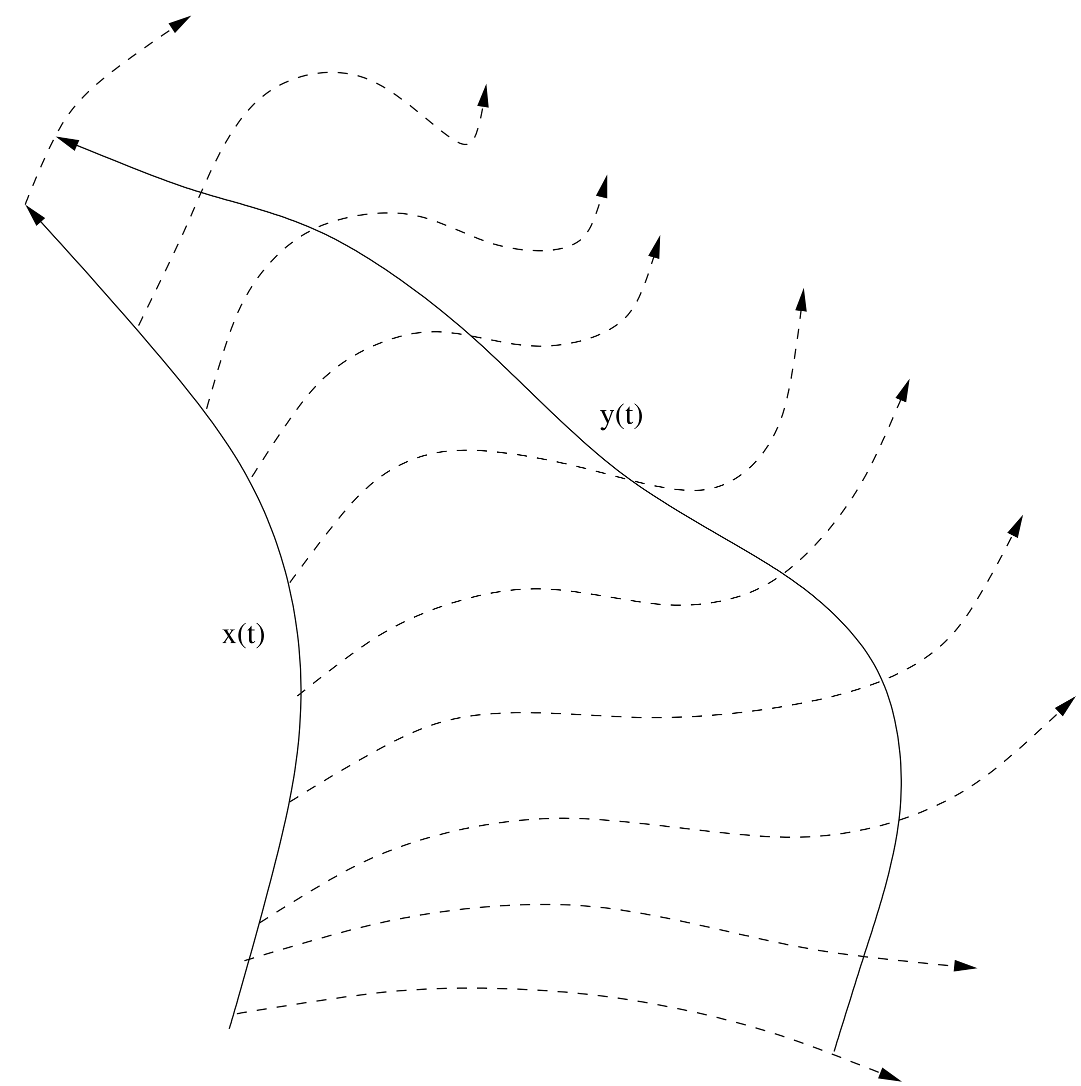
Let $x(t)$ be a family of points such that $|\omega(x(t), t)|$ is comparable to $\Omega(t)$. Assume that for all $t \in [0, T)$ there is another point $y(t)$ on the same vortex line as $x(t)$, such that the direction of vorticity $\xi(x, t) = \omega(x, t)/|\omega(x, t)|$ along the vortex line between $x(t)$ and $y(t)$ is well-defined. If we further assume that

$$\left| \int_{x(t)}^{y(t)} (\nabla \cdot \xi)(s, t) \, ds \right| \leq C$$

for some absolute constant C, and

$$\int_0^T |\omega(y(t), t)| \, dt < \infty; ,$$

then there will be no blowup up to time T.



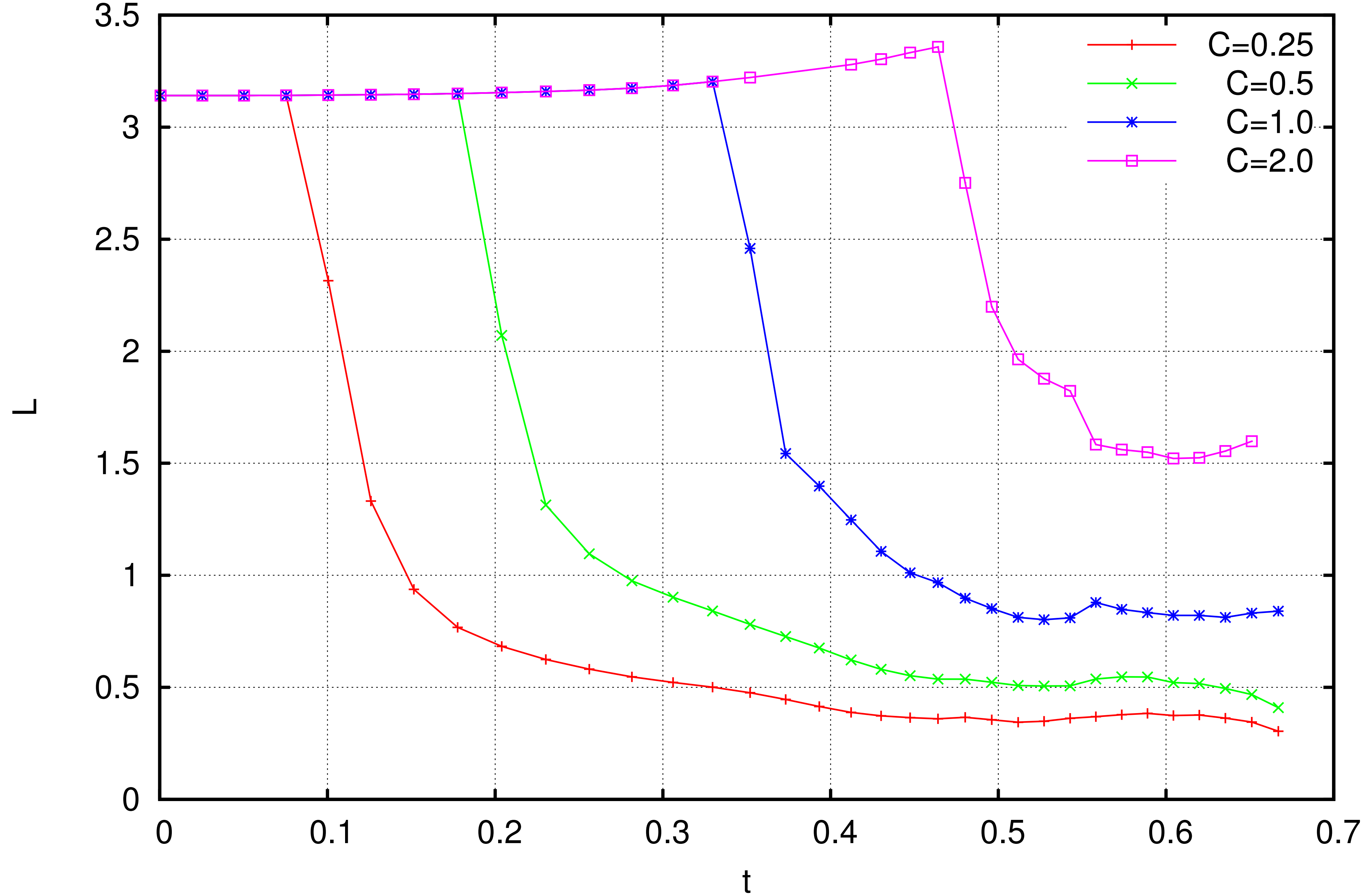
What to take as $x(t)$?
Answer: Location of $\Omega(t)$

What to take as $y(t)$?
Answer: Choose $y(t)$ such that C is constant in time.

This leads to the question: Does $y(t)$ approach $x(t)$?

This leads to the question: Does $y(t)$ approach $x(t)$?

Answer (from numerics) for Pelz-like initial conditions: No



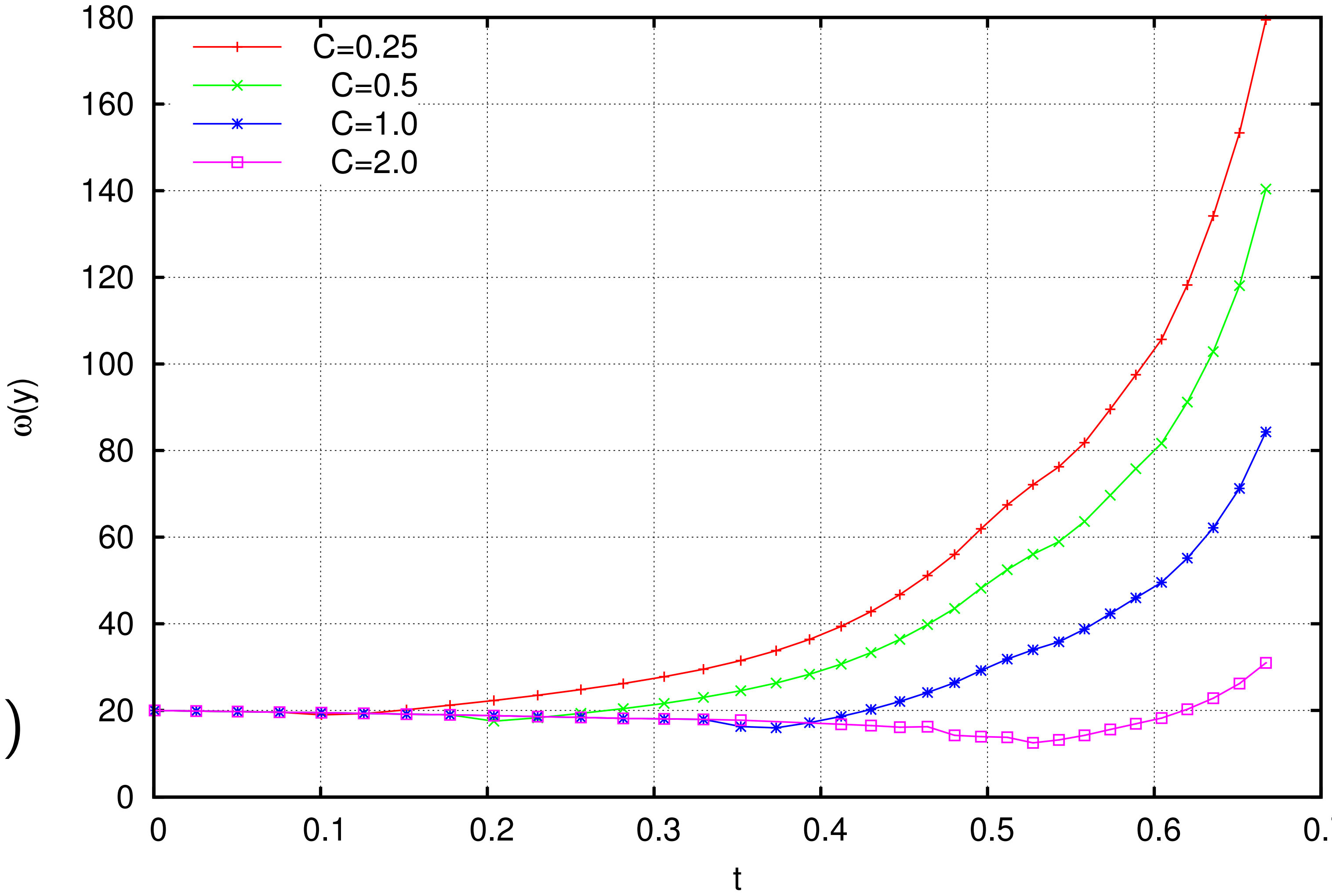
Citation from Deng, Hou, Yu (2006):

"Let us take the point $x(t)$ to be the point insinde one tube where the maximum vorticity is attained, and y(t) to be a point on the same vortex line, but outside the tube. It is easy to check that within this inner region, condition (2.1) is satisfied. By Theorem 1 we see that if the maximum vorticity *outside* these small tubes is integrable in time, then there is no blowup inside the tubes. It is likely that the maximum vorticity outside these small tubes has a growth rate smaller than that inside these small regions. This casts doubt on the validity of Pelz's claim.."

It's not that easy:

in the outer region vorticity looks harmless but it isn't

Conclusion:
Theorem 1 does not rule out Pelz singularity
(but whole vortex line must blow up)



Deng, Hou, Yu (2006) Theorem 2

Assume there is a family of vortex line segments L_t and $T_0 \in [0, T)$, such that $L_{t_2} \subseteq X(L_{t_1}, t_1, t_2)$ for all $T_0 < t_1 < t_2 < T$. We also assume that $\Omega(t)$ is monotonically increasing and $\|\omega(t)\|_{L^\infty(L_t)} \geq c_0 \Omega(t)$ for some $c_0 > 0$ when t is sufficiently close to T. Furthermore, we assume that

- $U_\xi(t) + U_n(t) M(t) L(t) \lesssim (T - t)^{-A}$ for some $A \in (0, 1)$
- $M(t) L(t) \leq C_0$,
- $L(t) \gtrsim (T - t)^B$ for some $B < 1 - A$.

Then there will be no blowup in the 3D incompressible Euler flow up to time T.

Here $L(t)$ is the arc length of L_t and

$$U_\xi(t) = \max_{x,y \in L_t} |(u \cdot \xi)(x, t) - (u \cdot \xi)(y, t)|$$

$$U_n(t) = \max_{L_t} |u \cdot n|$$

$$M(t) = \max(\|\nabla \cdot \xi\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)}) .$$

where κ and n are the curvature and the unit normal vector of L_t, respectively.

Which vortex line to choose ?

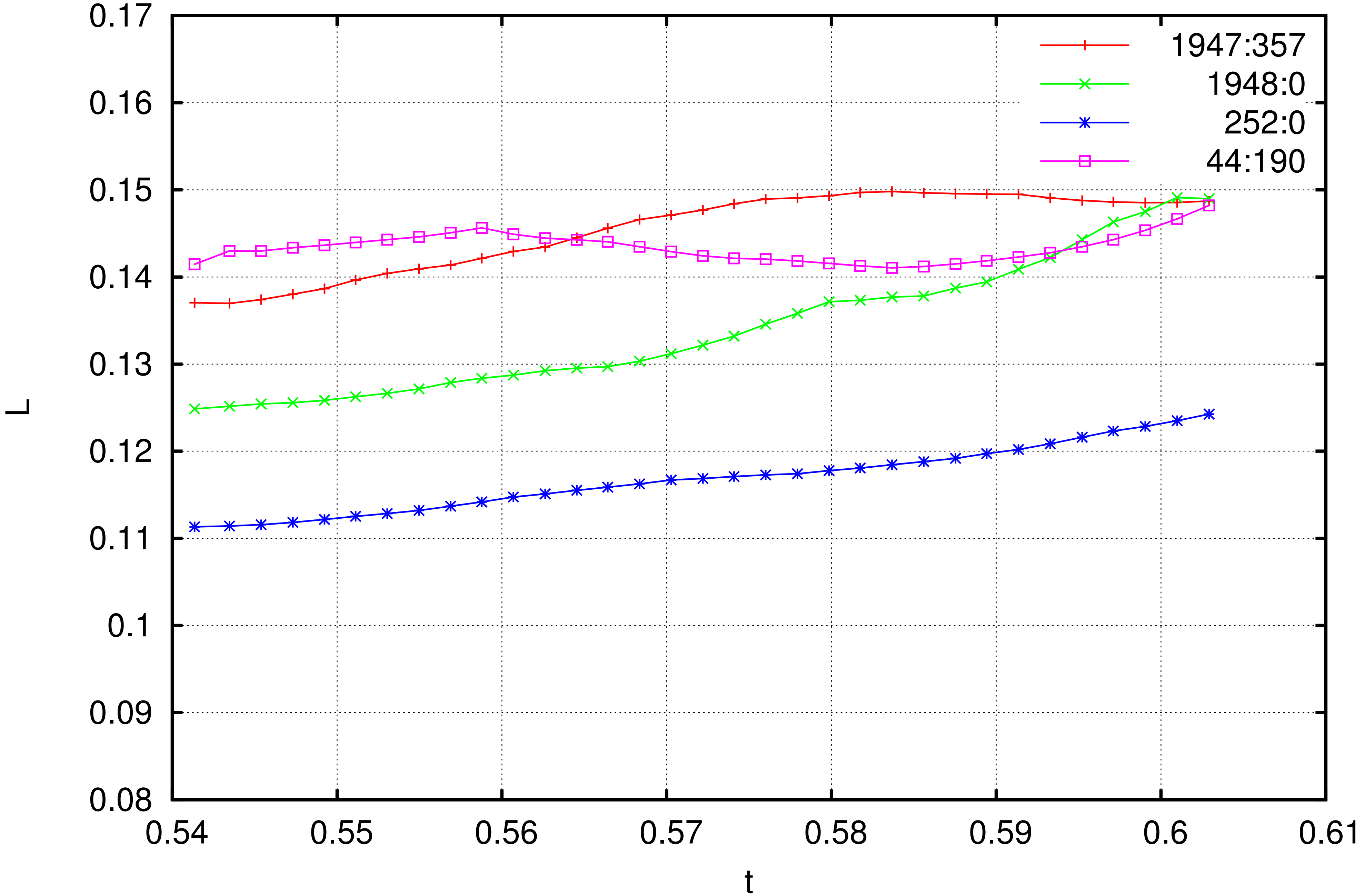
Answer:
Using the “back-to-labels” map obtained via the tracers it is possible to follow the vortex line that will contain the maximum $\Omega(x, t)$ at a late time.

How to choose the length ?

Answer:
The critical length is such that $M(t)L(t)$ is constant.

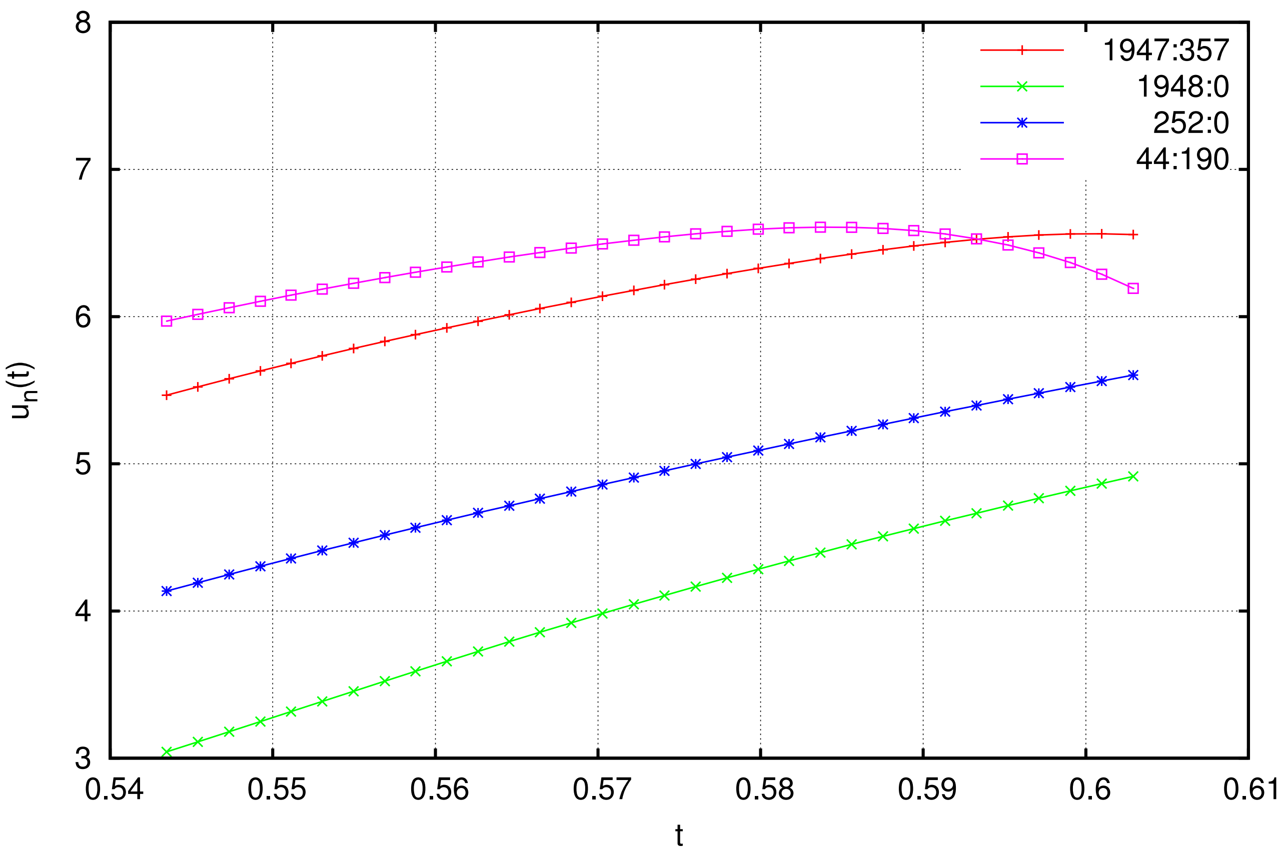
Observation (from numerics):

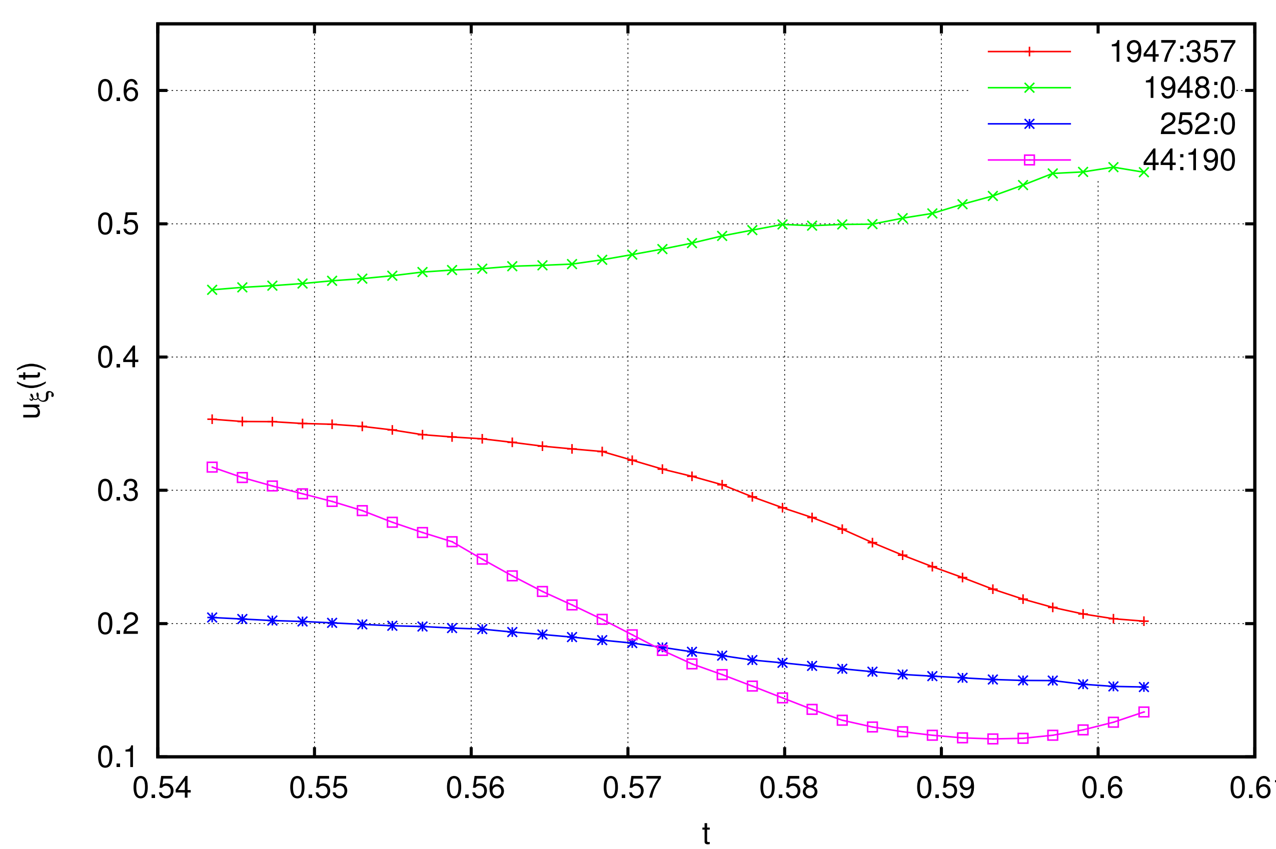
Our observations confirm uncritical scaling of length because curvature and $\nabla \cdot \xi$ are well behaved on vortex line segment.



Only chance for finite time singularity:
velocity blows up like $1/(t^* - t)$

but it doesn't !!!





Conclusion: no finite time blowup in Pelz like initial condition

Questions and wish list:

i) What is the distance to the finite time singularity?
If $t - t^* = O(1)$ you are in trouble?

ii) Check, whether resolution is sufficient?
a) analyticity strip method
b) redo with different resolutions

iii) Locations of max. vorticity and max. strain must converge !

iv) Do not plot $1/||\omega||_\infty$!

v) Agree on initial conditions !

vi) Talk to the mathematicians !

vii) Try to understand the flow:
what is the local (self-induction) and what is background strain?
(see P. E. Hamlington, J. Schumacher, and W. J. A. Dahm, Phys. Fluids 20 (2008) 111703)

Eulerian versus Lagrangian description

Lagrangian description: pressure and dissipation

Viellefosse (1984), Chertkov, Pumir & Shraiman (1999), Chevillard & Meneveau (2006)
Gibbon (2002), Gibbon, Holm, Kerr & Roulstone (2006), Gibbon & Holm (2007)

Velocity gradients: taking the gradient of the Navier-Stokes equations yields

$$\frac{d\mathcal{A}_{ij}}{dt} + \mathcal{A}_{ik}\mathcal{A}_{kj} = -\partial_{ij} p + \nu \Delta \mathcal{A}_{ij}$$

with $\frac{d\mathcal{A}_{ij}}{dt} = \frac{\partial \mathcal{A}_{ij}}{\partial t} + u_k \partial_k \mathcal{A}_{ij}$ and $\mathcal{A}_{ij} = \frac{\partial u_i}{\partial x_j}$.

Recent fluid deformation closure (RFD):

$$d\mathcal{A}_{ij} = \left(C_{ij}^{-1} \frac{\mathcal{A}_{nm}\mathcal{A}_{mn}}{C_{kk}^{-1}} - \mathcal{A}_{ik}\mathcal{A}_{kj} - \mathcal{A}_{ij} \frac{C_{mm}^{-1}}{3T} \right) dt + dW_{ij}$$

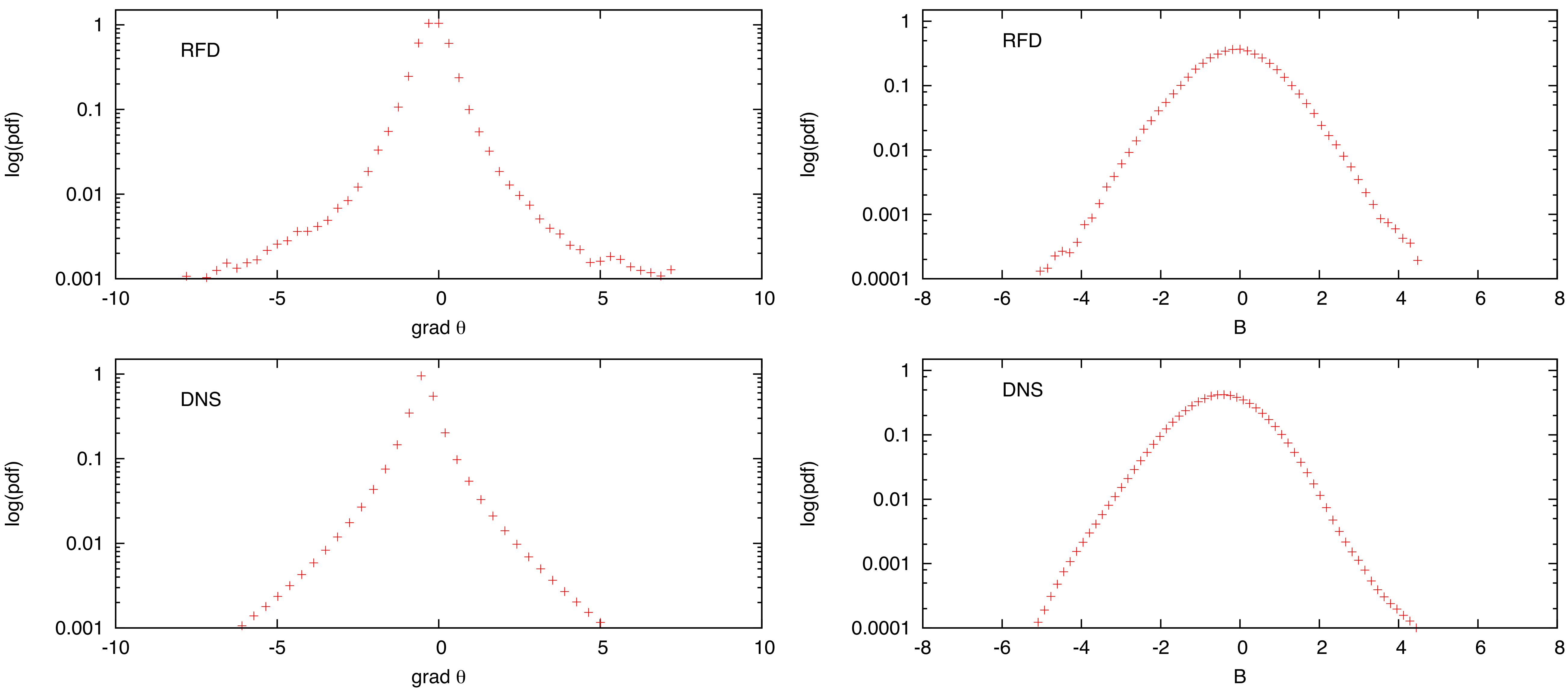
with $D_{ij}^{-1} = \frac{\partial X_i}{\partial x_j}$ and Cauchy-Green tensor $C_{ij} = D_{ik} D_{kj} \approx \exp(\tau \mathcal{A}) \exp(\tau \mathcal{A}^T)$,

Kolmogorov time $\tau \sim \sqrt{\frac{\nu}{\epsilon}}$, Gaussian stochastic forcing $d\mathbf{W}$.

Eulerian versus Lagrangian description

Extension to scalar gradient $\partial_k \theta$ and magnetic field fluctuations B_k:

$$d\partial_k\theta = -\left(\mathcal{A}_{kj}\partial_j\theta + \partial_k\theta\frac{C_{mm}^{-1}}{3T_\theta}\right)dt + dV_{\theta\,k} \quad , \quad dB_k = +\left(\mathcal{A}_{kj}B_j + B_k\frac{C_{mm}^{-1}}{3T_B}\right)dt + dV_{B\,k}$$



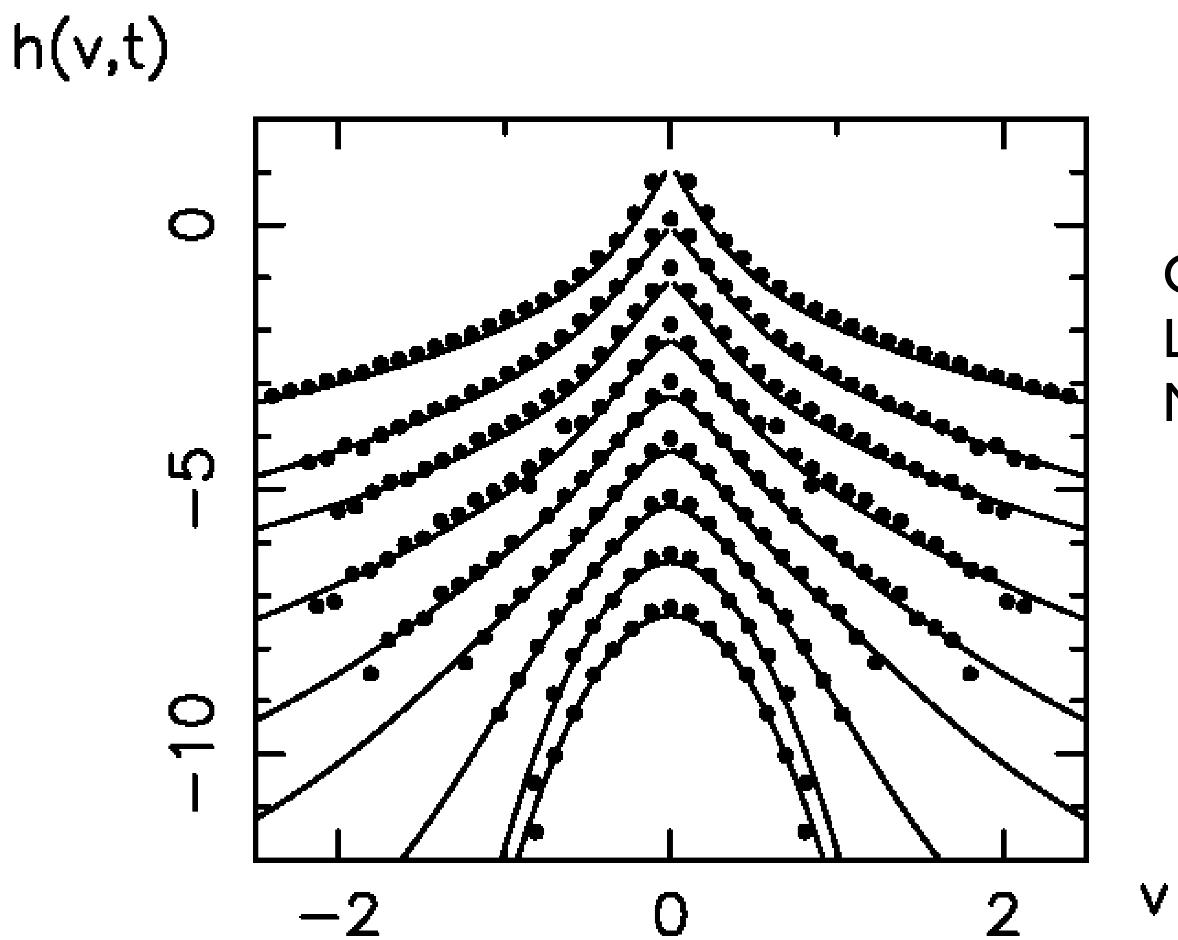
Lagrangian turbulence

Eulerian description

$$\partial_t \mathbf{u}(\mathbf{x}, t) = \ldots$$

Lagrangian description

$$\partial_t \mathbf{X}(t, \mathbf{y}) = \mathbf{u}(\mathbf{X}(t, \mathbf{y}), t) \qquad \mathbf{X}(0, \mathbf{y}) = \mathbf{y}$$
$$\partial_t \mathbf{u}(\mathbf{X}(t, \mathbf{y}), t) = \ldots$$



Ott, Mann (2000)
La Porta, Voth, Crawford, Alexander, Bodenschatz (2001)
Mordant, Metz, Michel, Pinton (2001)

Lagrangian multifractals

- Chevillard, Roux, Leveque, Mordant, Pinton, Arneodo (2003)
- Biferale, Bofetta, Celani, Devinish, Lanotte, Toschi (2004)
- Yakhot (2008)

Lagrangian multifractals

Biferale et al (2004): Take She-Leveque model

$$\zeta_E(p) = \frac{p}{9} + 2\left[1 - \left(\frac{2}{3}\right)^{p/3}\right]$$

Legrende transformation to obtain singularity spectrum:

$$\begin{aligned} D(h) &= \inf_p \left(ph + 3 - \zeta_E(p)\right) \\ &= 1 + p^*(h)\left(h - \frac{1}{9}\right) + 2\left(\frac{2}{3}\right)^{p^*(h)/3} \end{aligned}$$

with

$$p^*(h) = \frac{3}{\ln(2/3)} \ln\left[\frac{(1-9h)}{6\ln(2/3)}\right]$$

Assumptions:

$$\boxed{\delta_\tau v \sim \delta_l u \;\text{ with }\; \tau_l \sim l/\delta_l u}$$

Borgas 1993

$$\Longrightarrow \tau \sim \frac{L_0^h}{u_0} l^{1-h}$$

Frisch-Parisi for Lagrangian structure functions:

$$S_p(\tau) \sim u_0^p \int_{h\in I} d\mu(h) \left(\frac{\tau}{T_L}\right)^{\frac{hp+3-D(h)}{1-h}}$$

Saddle point approximation for $\tau \ll T_L$:

$$\zeta_L(p) = \inf_h \left(\frac{hp+3-D(h)}{1-h}\right)$$

MHD works similar:

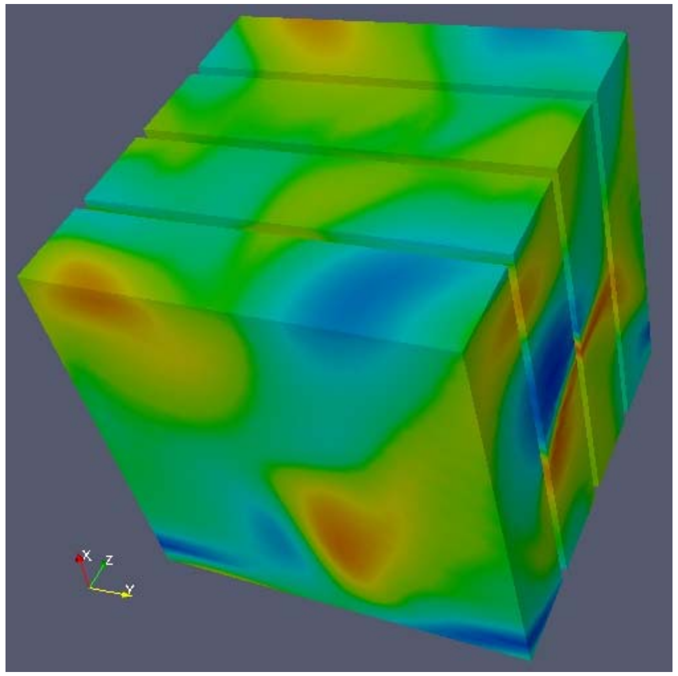
just start with the MHD Eulerian ζ_p

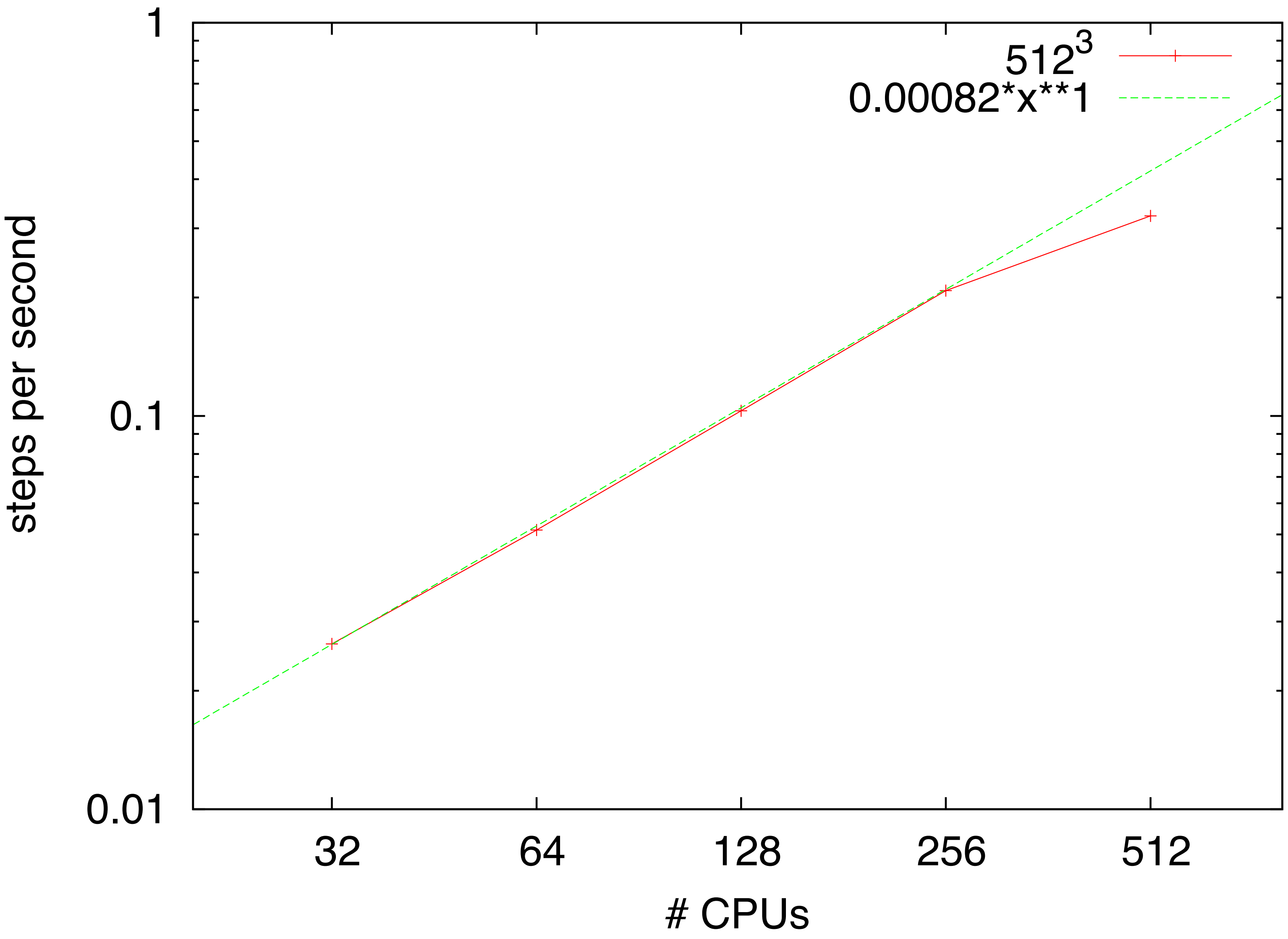
How does theory and numerics compare ?

Numerics

- pseudo spectral code
- parallel treatment of tracer particles, 2-150 million particles
- written in C++

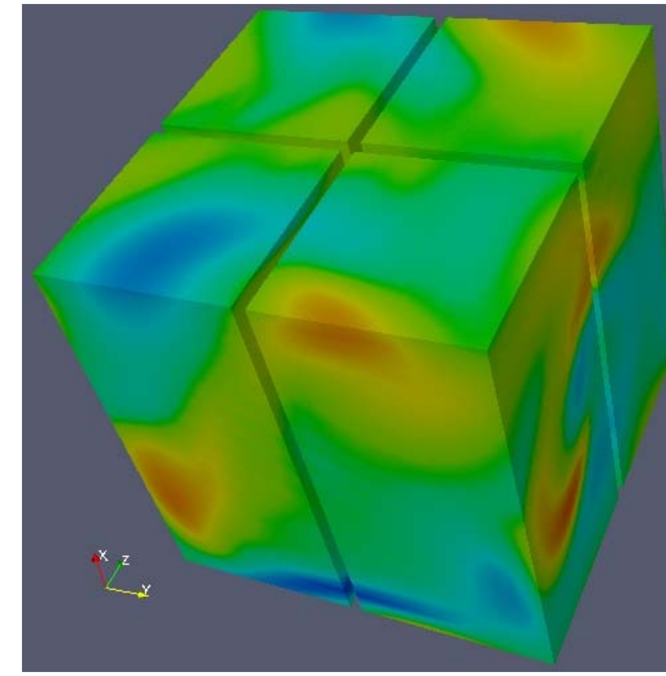
JUMP: slice based





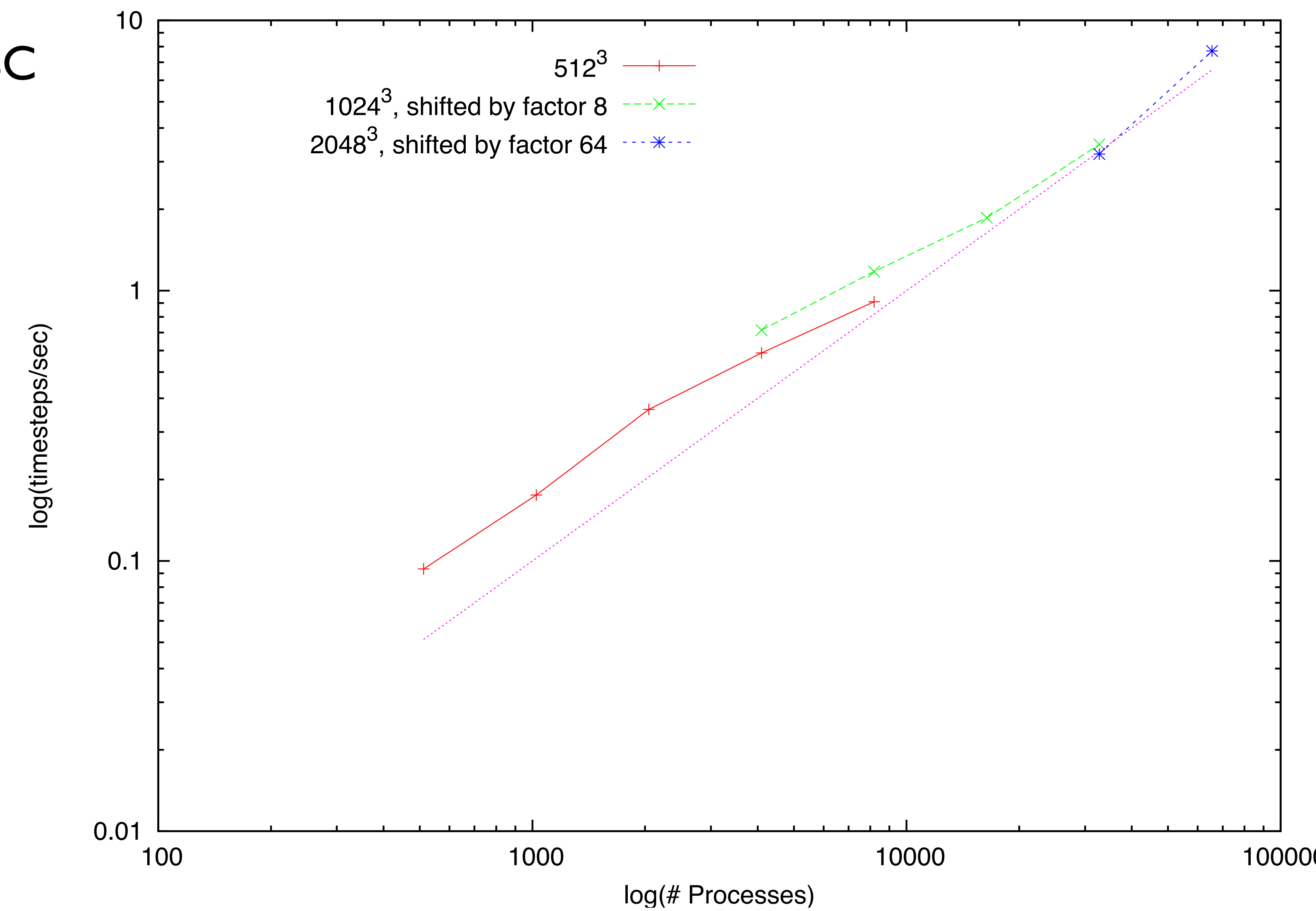
Numerics

- pseudo spectral code, Navier-Stokes:
- parallel treatment of tracer particles, 2-150 million particles
- written in C++



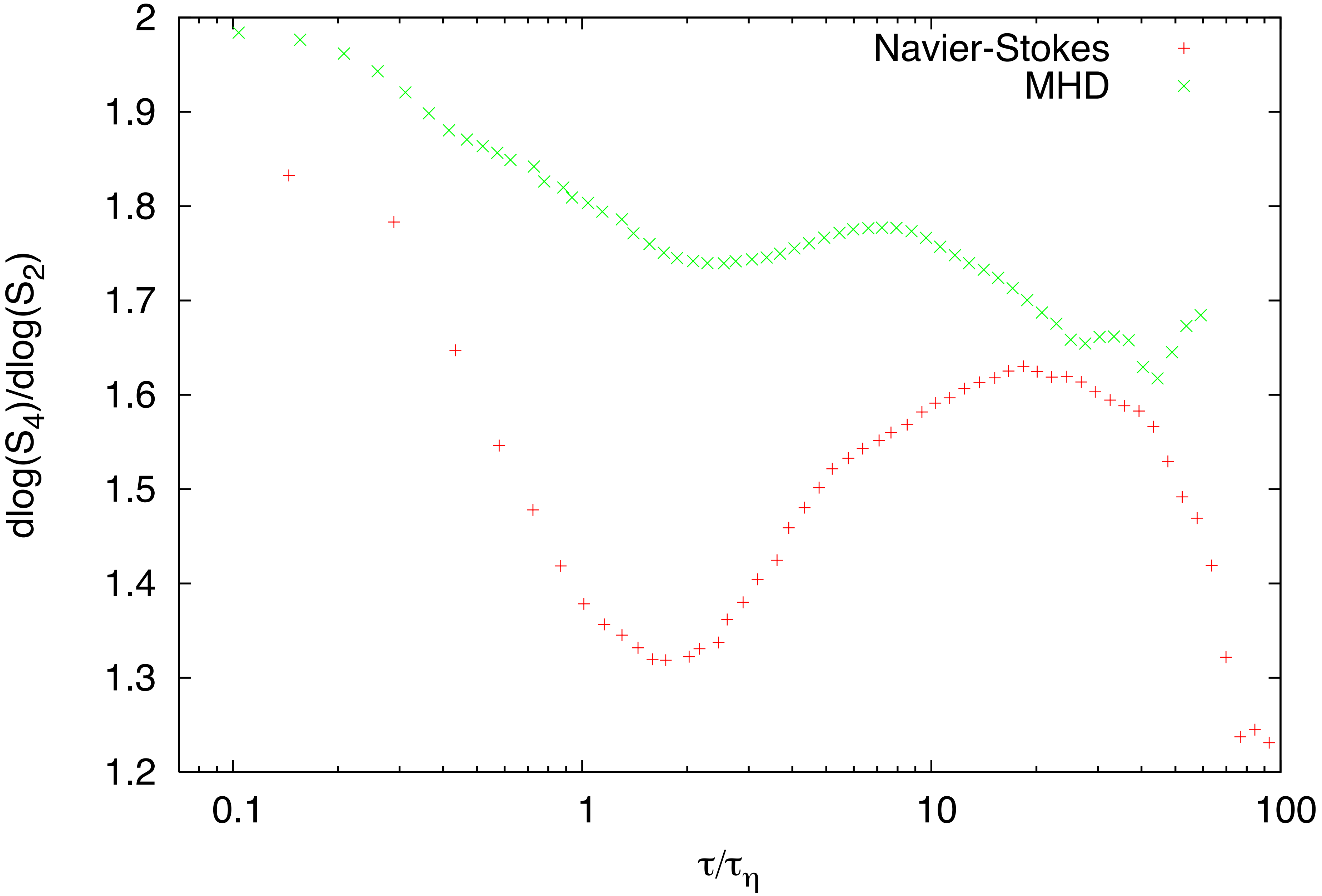
JUGENE: column based

Advertisement BlueGene: look at P3DFFT from SDSC



$\Re_\lambda$	$u_{\rm rms}$	$\epsilon_{\rm k}$	ν	dx	η	τ_η	L	T_L	N^3	N_p
460	0.189	$3.6 \cdot 10^{-3}$	$2.5 \cdot 10^{-5}$	$3.07 \cdot 10^{-3}$	$1.45 \cdot 10^{-3}$	0.083	1.85	9.9	2048^3	10^7

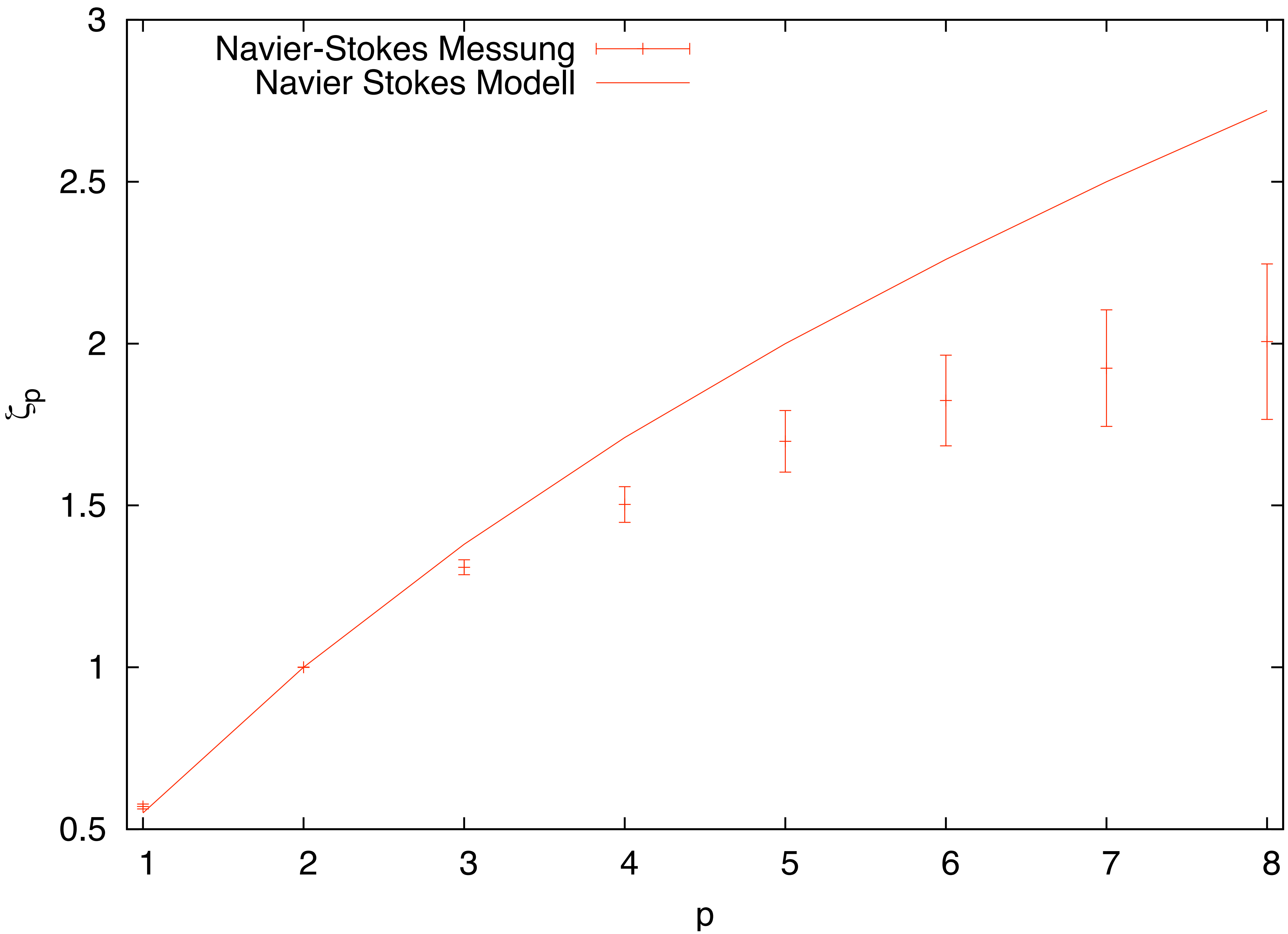
TABLE I. Parameters of the numerical simulations. $\Re_\lambda = \sqrt{15VL/\nu}$: Taylor-Reynolds number, $u_{\rm rms}$: root-mean-square velocity, $\epsilon_{\rm k}$: mean kinetic energy dissipation rate, ν: kinematic viscosity, dx: grid-spacing, $\eta = (\nu^3/\epsilon_{\rm k})^{1/4}$: Kolmogorov dissipation length scale, $\tau_\eta = (\nu/\epsilon_{\rm k})^{1/2}$: Kolmogorov time scale, $L = (2/3E)^{3/2}/\epsilon_{\rm k}$: integral scale, $T_L = L/u_{\rm rms}$: large-eddy turnover time, N^3: number of collocation points, N_p: number of tracer particles.



Navier-Stokes
MHD
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
0.1
1
10
100
dlog(S4)/dlog(S2)
τ/τη

Lagrangian scaling exponents (Navier-Stokes)

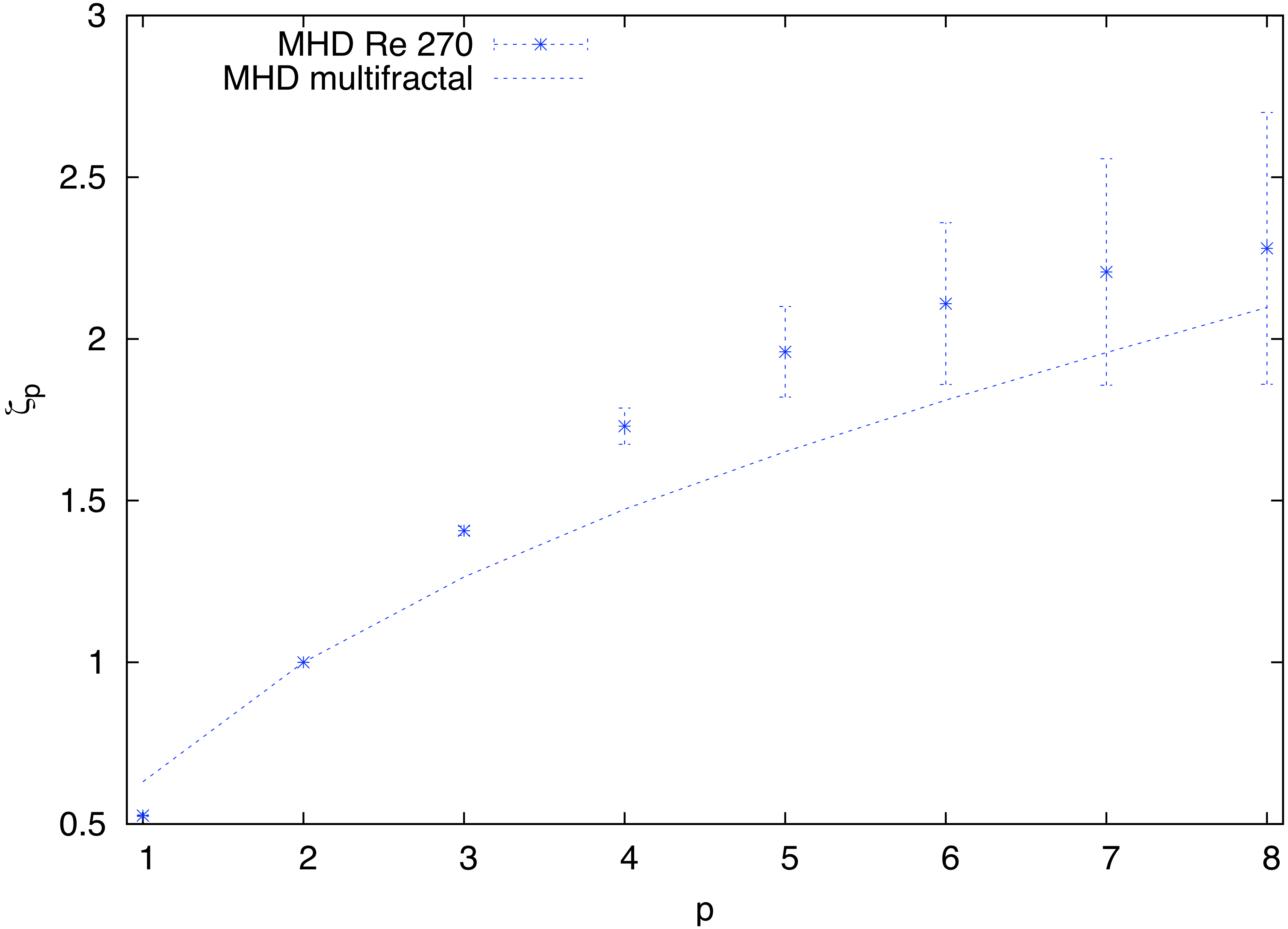
Multifractal model: Biferale et al. (2004)



Navier-Stokes exponents **differ** from model

Lagrangian scaling exponents (MHD)

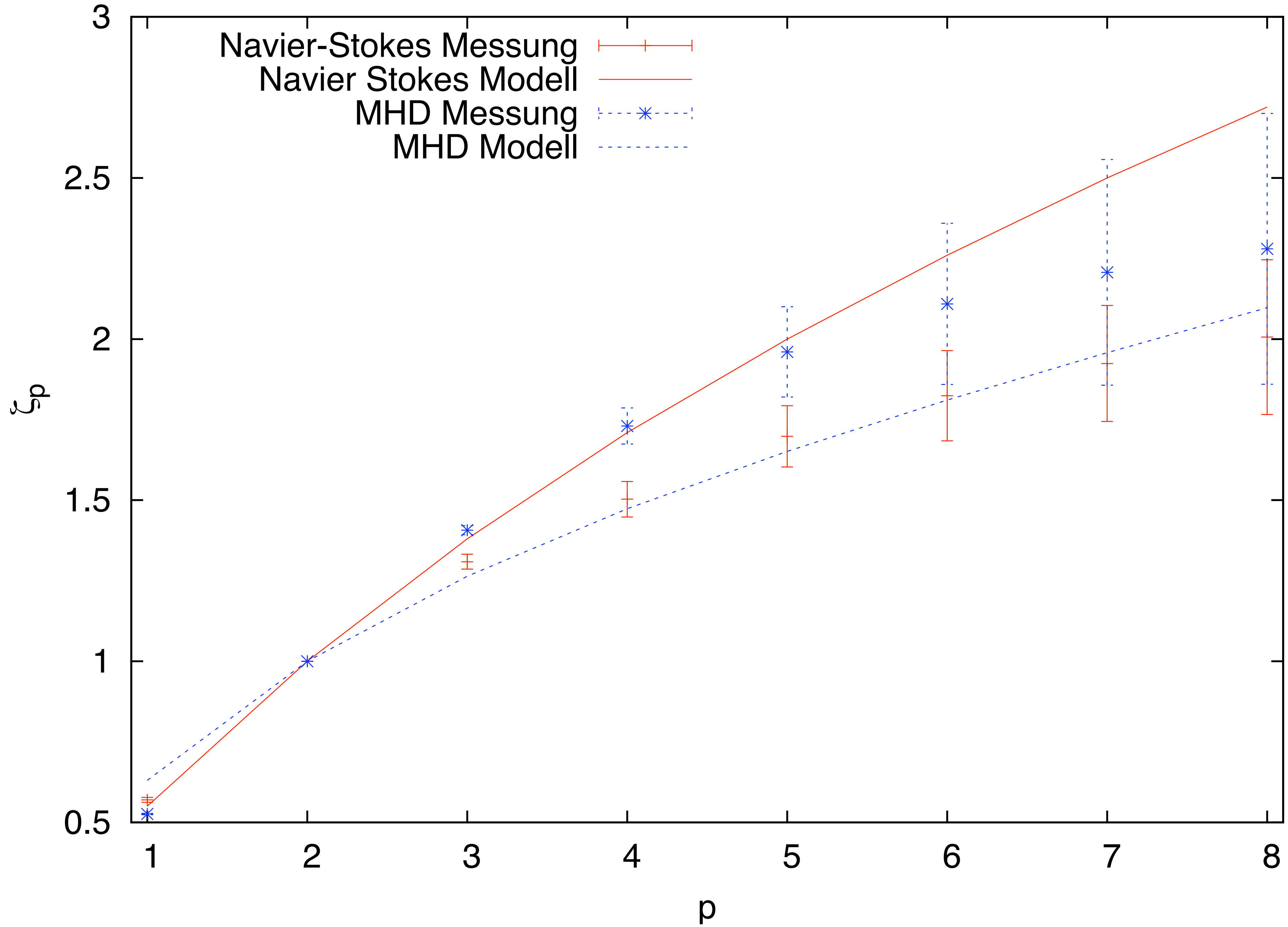
Multifractal model: Biferale et al. (2004)



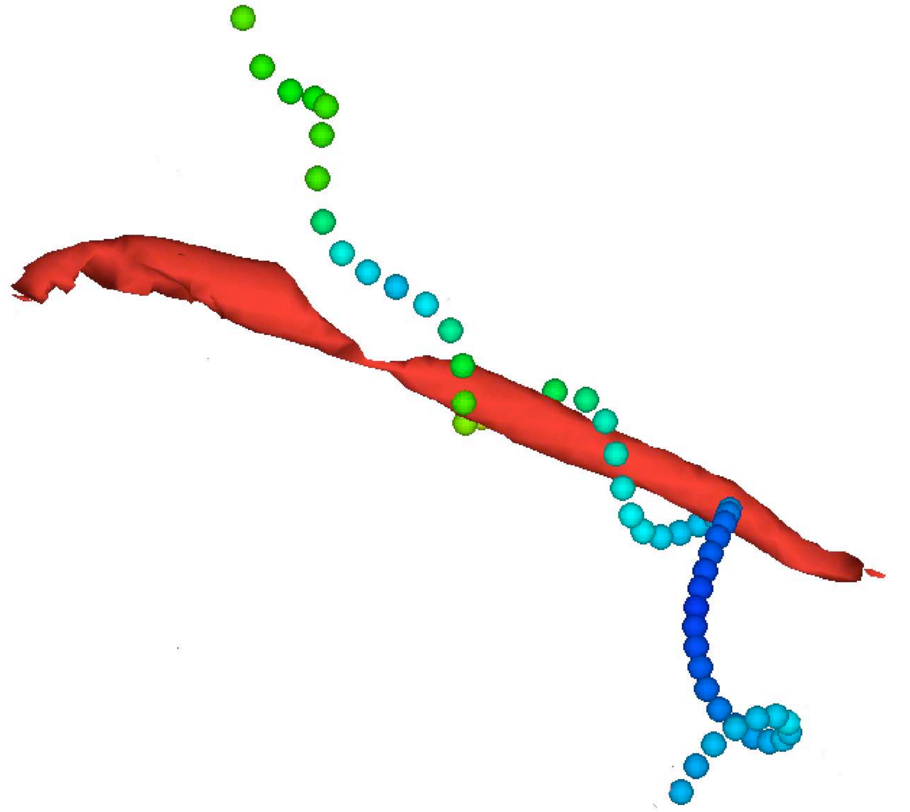
MHD exponents **differ** from model

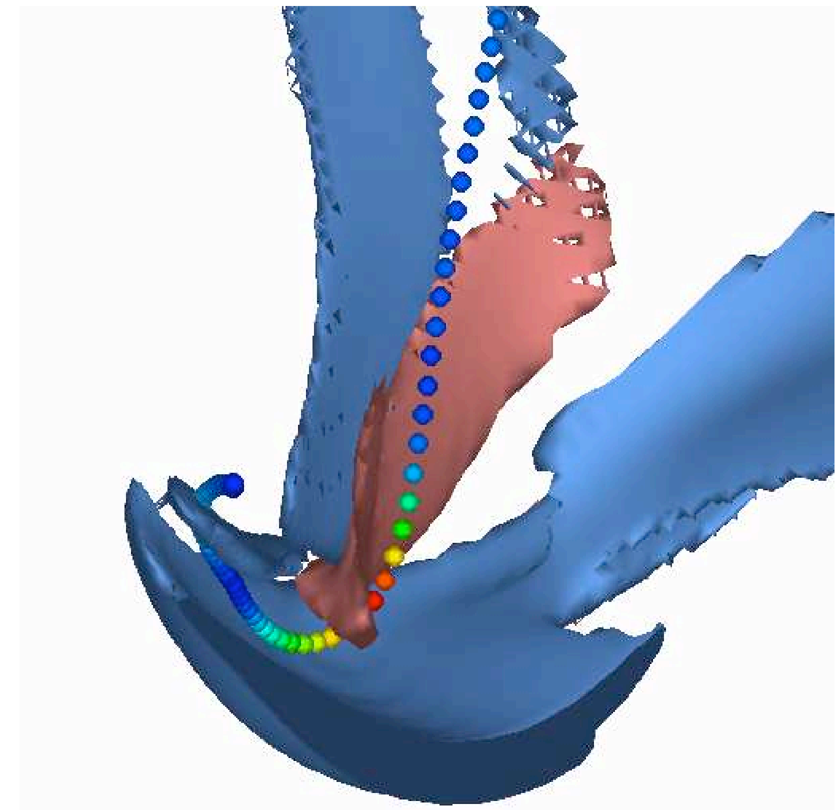
Lagrangian scaling exponents (NS & MHD)

Multifractal model: Biferale et al. (2004)



exponents **differ** from model





Navier-Stokes

MHD

Eulerian view:

$C_0 = 2$

$C_0 = 1$

Lagrangian view:

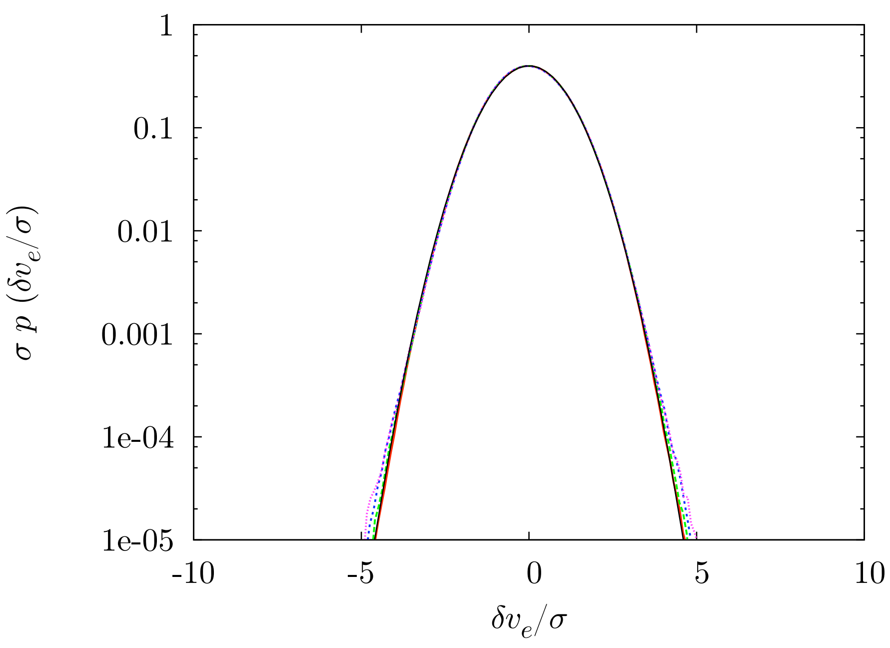
$C_0 = 1$

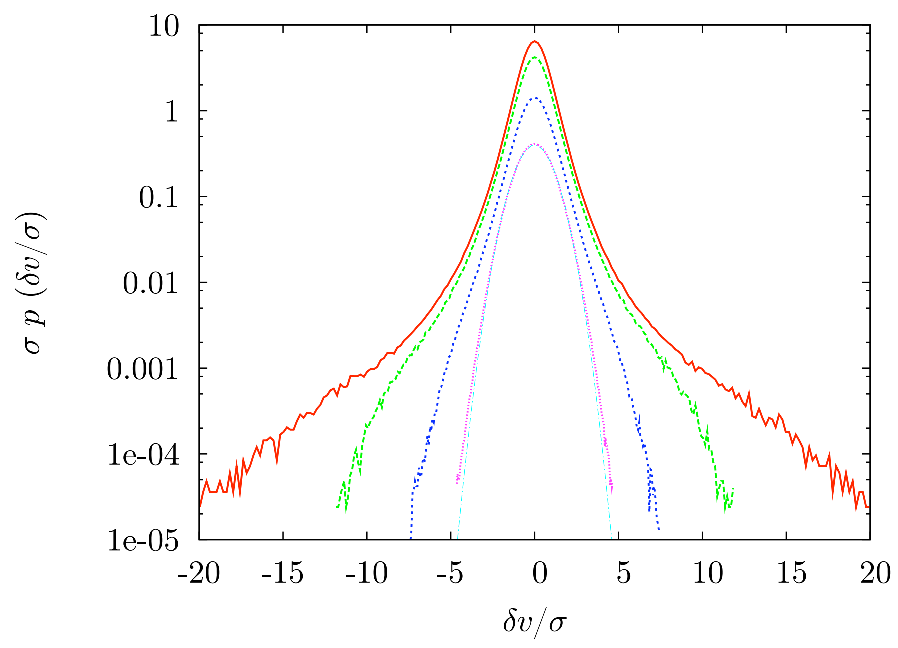
$C_0 = 2$

Two open problems:

- missing monotonicity in relating Eulerian to Lagrangian turbulence
- 2D Navier-Stokes: no Eulerian but Lagrangian intermittency

2D numerics (Kamps, Friedrich 2007)





p	1	3	4	5
ζ_p^a	0.557 ± 0.002	1.267 ± 0.007	1.35 ± 0.018	1.313 ± 0.033
ζ_p^b	0.557 ± 0.003	1.313 ± 0.008	1.45 ± 0.019	1.588 ± 0.029

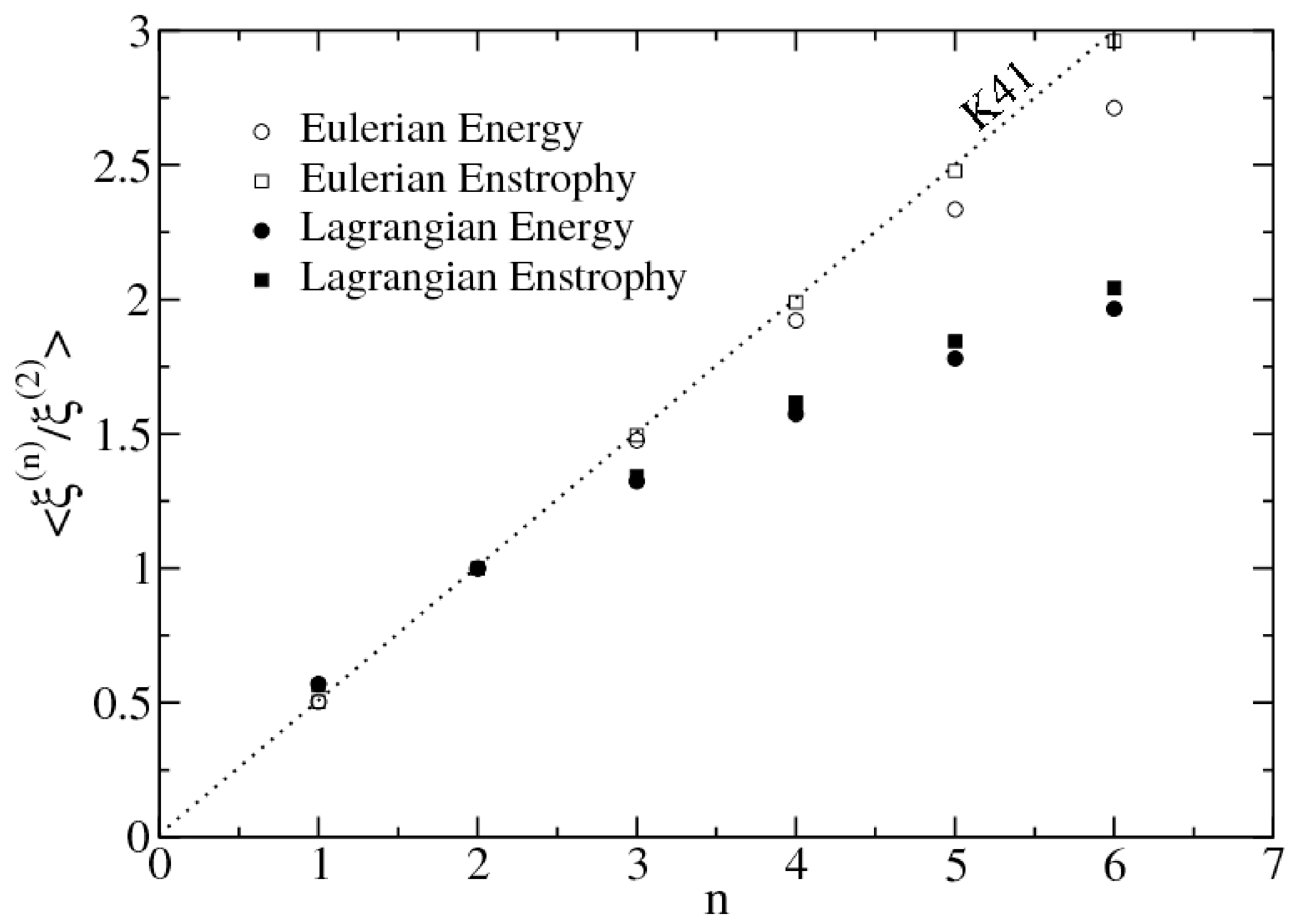
▸ 2D experiment (Rivera, Ecke)

Eulerian and Lagrangian velocity statistics in weakly forced two-dimensional turbulence

Michael K. Rivera and Robert E. Ecke
The Condensed Matter and Thermal Physics Group (MPA-10) and The Center for NonLinear Studies (T-CNLS), Los Alamos National Laboratory, Los Alamos, NM, 87545
(Dated: November 21, 2007)

We present statistics of velocity fluctuations in both the Lagrangian and Eulerian frame for weakly driven two-dimensional turbulence. We find that simultaneous inverse energy and enstrophy ranges present in the Lagrangian and Eulerian Fourier spectra are not directly echoed in real-space moments of velocity difference. The spectral ranges, however, do line up very well with ratios of the real-space moments *local* exponents, indicating that though the real-space moments are not scaling "nicely", the relative behavior of the velocity difference probability distribution functions is changing over very short ranges of length scales. Utilizing this technique we show that the ratios of the local exponents for Eulerian moments in weak two-dimensional turbulence behave in agreement with Kolmogorov predictions over the spectrally identified ranges. The Lagrangian local exponent ratios, however, behave in a different manner compared to their Eulerian counterparts, and deviate significantly from what would be expected from Kolmogorov predictions.

PACS numbers: abc.123



Transition PDFs:

<u>Kamps</u>, Friedrich, Grauer (2008)

Eulerian increment

$$\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{y}, t)$$

to new increment

$$\mathbf{w}(\mathbf{y}, t) = \mathbf{u}(\mathbf{x}(\mathbf{y}, t), t) - \mathbf{u}(\mathbf{y}, t)$$

to Lagrangian velocity increment

$$\mathbf{v}(\mathbf{y}, t) = \mathbf{u}(\mathbf{x}(\mathbf{y}, t), t) - \mathbf{u}(\mathbf{y}, 0)$$

Eulerian fine grained pdf

$$f_E(v_1, v_2; y, x, t) := \langle \delta(u(y,t) - v_1)\, \delta(u(y+x,t) - v_2) \rangle$$

EL fine grained pdf

$$f_{EL}(v_1, v_2; y, t, \tau) := \langle \delta(u(y, t-\tau) - v_1) \int dx \delta(\tilde{x}(\tau, y, t) - x)\, \delta(u(y,t), t) - v_2)\, \delta(u(y+x, t) - v_1) \rangle$$

Lagrangian fine grained pdf

$$\begin{aligned} f_L(v_1, v_2; y, t, \tau) &:= \langle \delta(u(y, t-\tau) - v_1)\, \delta(u(y + \tilde{x}(y, \tau, t), t) - v_2) \rangle \\ &= \left\langle \int dv \delta(u(y, t-\tau) - v_1) \int dx \delta(\tilde{x}(\tau, y, t) - x)\, \delta(u(y,t), t) - v)\, \delta(u(y+x, t) - v_1) \right\rangle \end{aligned}$$

now use (two times)

$$P(a, b) = P(a|b)P(b)$$

$$\begin{aligned} f_L(v_1, v_2; y, t, \tau) &= \int dv\; p_b(v_1|v, v_2; y, t, \tau) f_{EL}(v, v_2; y, t, \tau) \\ &= \int dv\; p_b(v_1|v, v_2; y, t, \tau) \int dx\; p_a(x|v, v_2; y, t, \tau)\; f_E(v_1, v_2; x, y, t) \end{aligned}$$

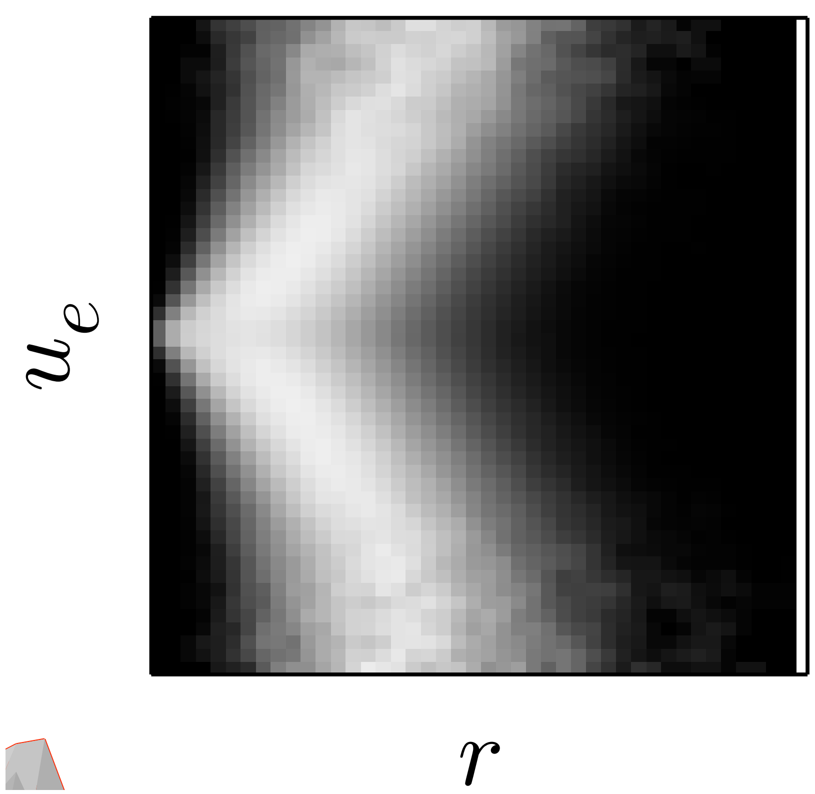
introduce velocity increments

$$u = v_2 - v \quad , \quad w = v - v_1 \implies u_L = u + w$$

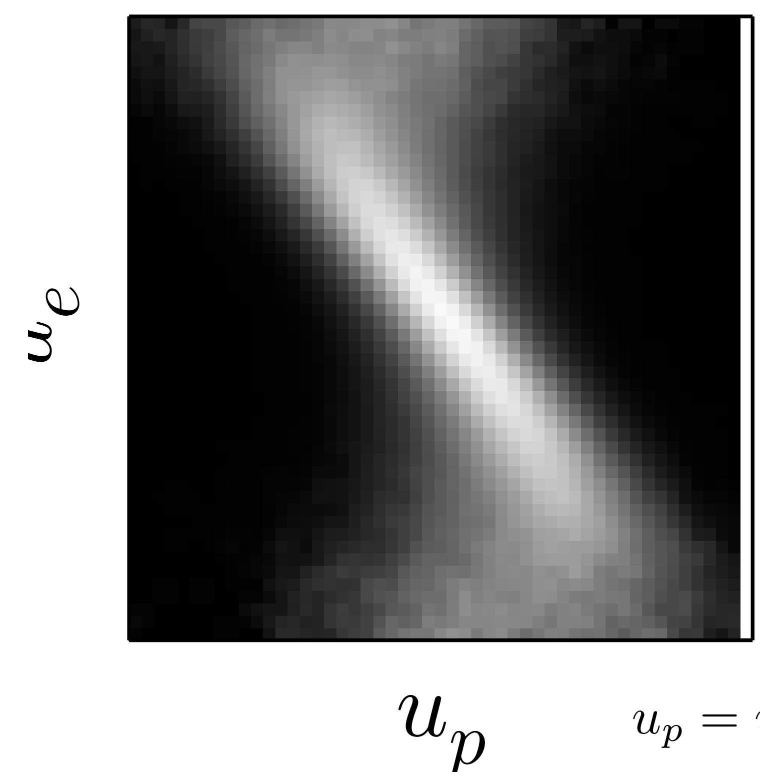
assume isotropic, homogeneous, stationary flows

$$f_L(u_L;\tau) = \int p_b(w|u;\tau) \int_0^\infty dr\; p_a(r|u;\tau)\; f_E(u;r)$$

$p_a(r|u;\tau)$

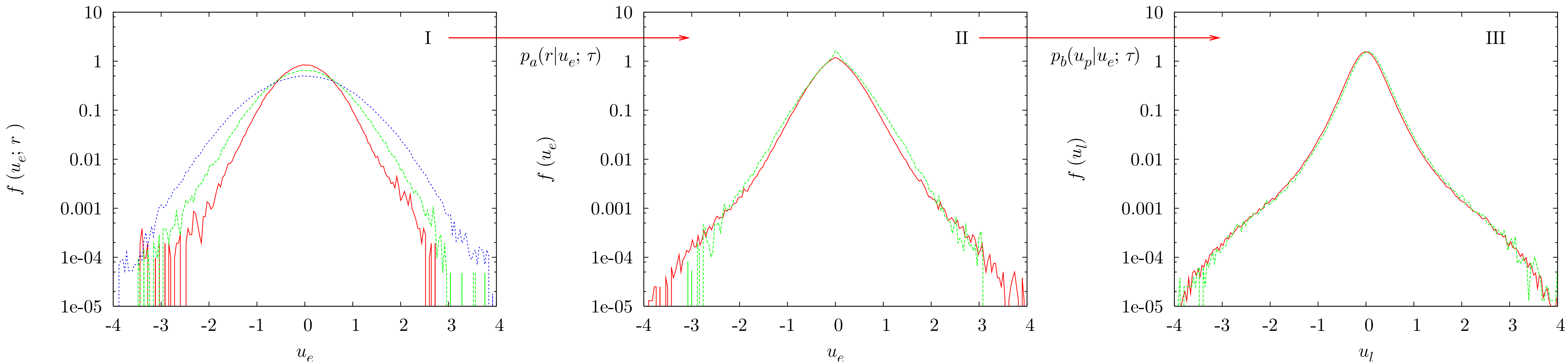


$p_b(w|u;\tau)$

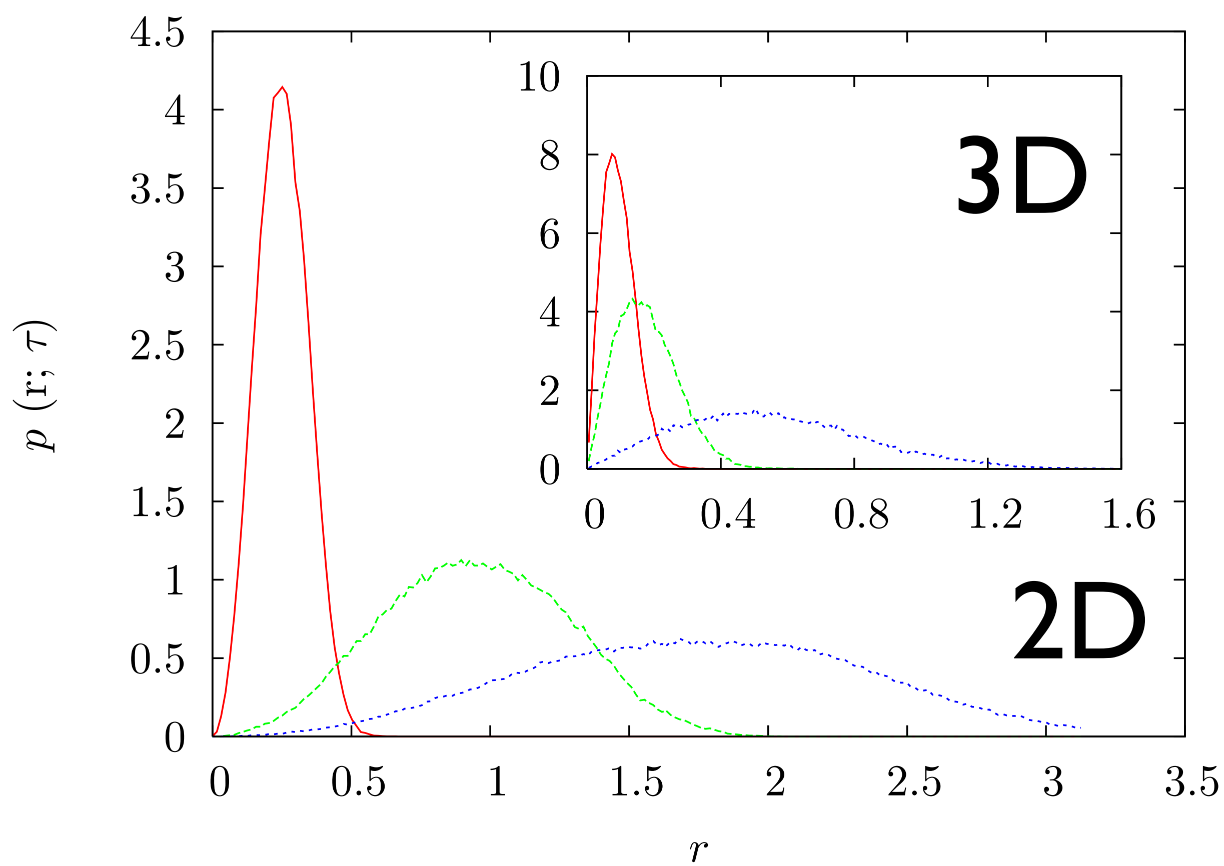


$$u_p = v(\boldsymbol{y},t) - v(\boldsymbol{y},t-\tau)$$

Eulerian EL Lagrangian



r = 0.06,0.12,0.3



$\tau = 3.5\tau_\eta,\ 14\tau_\eta,\ 28\tau_\eta$

Mellin transform:

$$P(\delta_r u_E, r) = \frac{1}{\delta_r u_E} \int_{-i\infty}^{i\infty} dn\, S_E(n) (\delta_r u_E)^{-n}$$

with $S_E(n) = A_E(n) r^{\zeta_E(n)}$.

inverse Mellin transform:

$$\int_0^\infty d(\delta_r u_E)\, (\delta_r u_E)^n P(\delta_r u_E, r) = S(n)$$

Euler-Lagrange translation:

$$f_L(\delta u_L;\tau) = \int d(\delta u_{EL})\, P_b(\delta u_L - \delta u_{EL} | \delta u_{EL};\tau) \int_0^\infty dr\, P_a(r|\delta u_E;\tau) f_E(\delta u_E; r)$$

Borgas ansatz:

$$P_a(r|\delta u_E;\tau) = \delta(r - \delta u_E\, \tau) \quad , \quad P_b(\delta u_L - \delta u_{EL} | \delta u_{EL};\tau) = \delta(\delta u_L - \delta u_E)$$

Thus we have

$$f_L(\delta u_L;\tau) = f_E(\delta u_E; \delta u_E \tau)$$

Now use the Mellin transformation

$$f_L(\delta u_L;\tau) = \frac{1}{\delta_r u_L}\int_{-i\infty}^{i\infty} dn\, A_E(n)\tau^{\zeta_E(n)}(\delta_r u_L)^{\zeta_E(n)-n}$$

to get the Lagrangian structure functions (inverse Mellin)

$$\int_0^\infty d(\delta_r u_L)\,\frac{1}{\delta_r u_L}(\delta_r u_L)^n \int_{-i\infty}^{i\infty} dj\, A_E(j)\tau^{\zeta_E(j)}(\delta_r u_L)^{\zeta_E(j)-j} = *$$

Substitute $j'(j) = j - \zeta_E(j)$, $dj' = (1-\partial_j\zeta_E(j))dj$ and denote the inverse function by $j(j')$. Thus we have

$$* = \int_0^\infty d(\delta_r u_L)\,\frac{1}{\delta_r u_L}(\delta_r u_L)^n \int_{-i\infty}^{i\infty} dj'\,\frac{A_E(j(j'))}{1-\partial_j\zeta_E(j(j'))}\tau^{\zeta_E(j(j'))}(\delta_r u_L)^{-j'}$$

$$* = \int_0^\infty d(\delta_r u_L)\,\frac{1}{\delta_r u_L}(\delta_r u_L)^n \int_{-i\infty}^{i\infty} dj'\, S_L(j')(\delta_r u_L)^{-j'} = S_L(n)$$

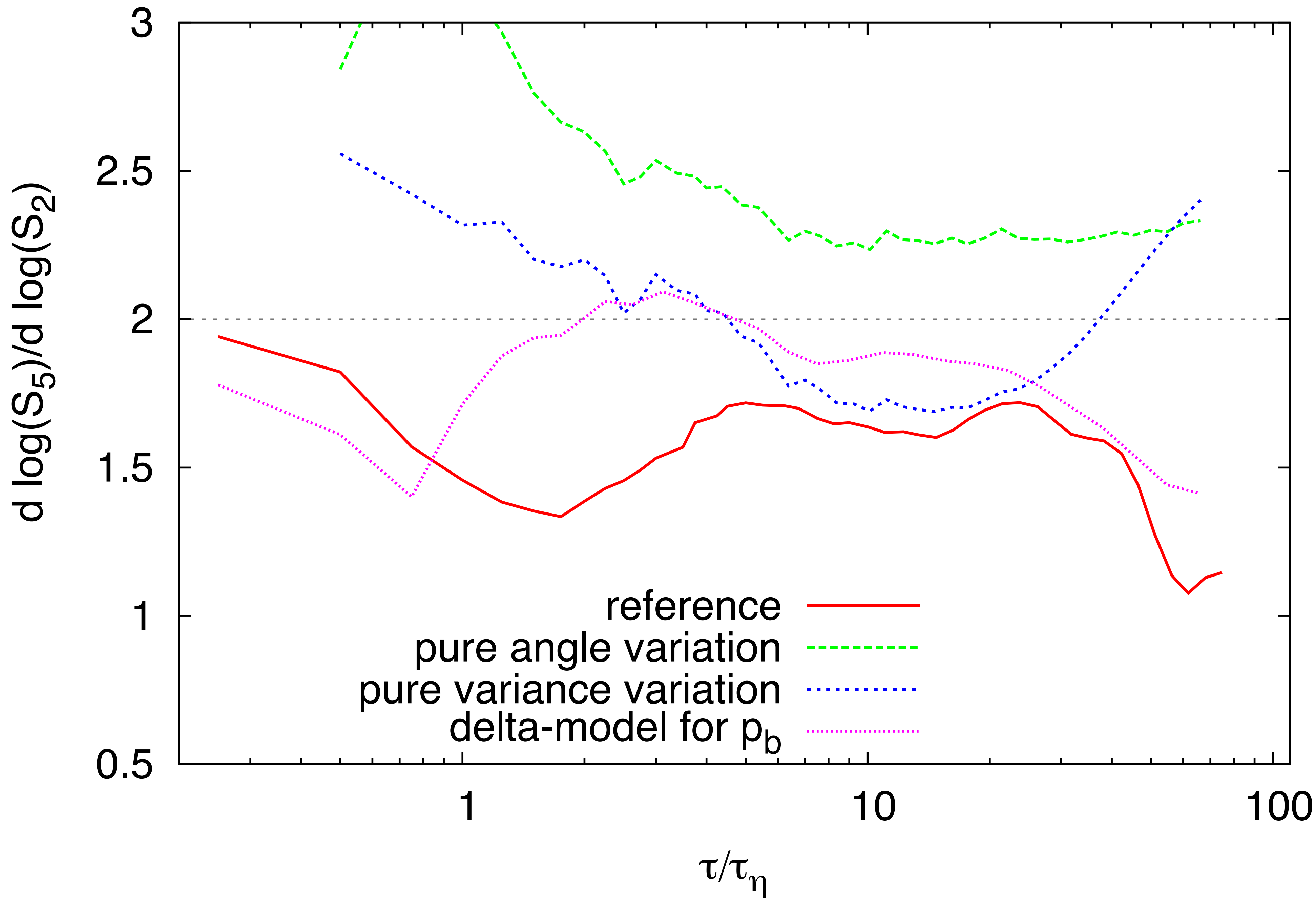
with $S_L(j') = \frac{A_E(j)}{1-\partial_j\zeta_E(j)}\tau^{\zeta_E(j)}$ and $j' = j - \zeta_E(j)$ and we obtain for the exponents:

$$\zeta_L(n-\zeta_E(n)) = \zeta_E(n)$$

- same result as Biferale et al, but much easier formula
- can prove monotonicity property

Using the Mellin transform one can show

- τ-dependent tilt decreases intermittency
- τ-dependent variance increases intermittency



Conditional Eulerian and Lagrangian PDFs

Gagne, Marchand, Castaing 1994

Energy dissipation: $$\epsilon(x) = \nu \sum_{i,j} [\partial_j u_i(x) + \partial_i u_j(x)]^2$$

longitudinal increment: $$\delta_l^{||} u = (u(x+l) - u(x)) \cdot \hat{l}$$

average over scale l: $$\epsilon_l = \frac{1}{l} \int_0^l \epsilon(x + s\,\hat{l}) ds$$

probability distribution

$$P(\delta_l^{||} u | \epsilon_l)$$

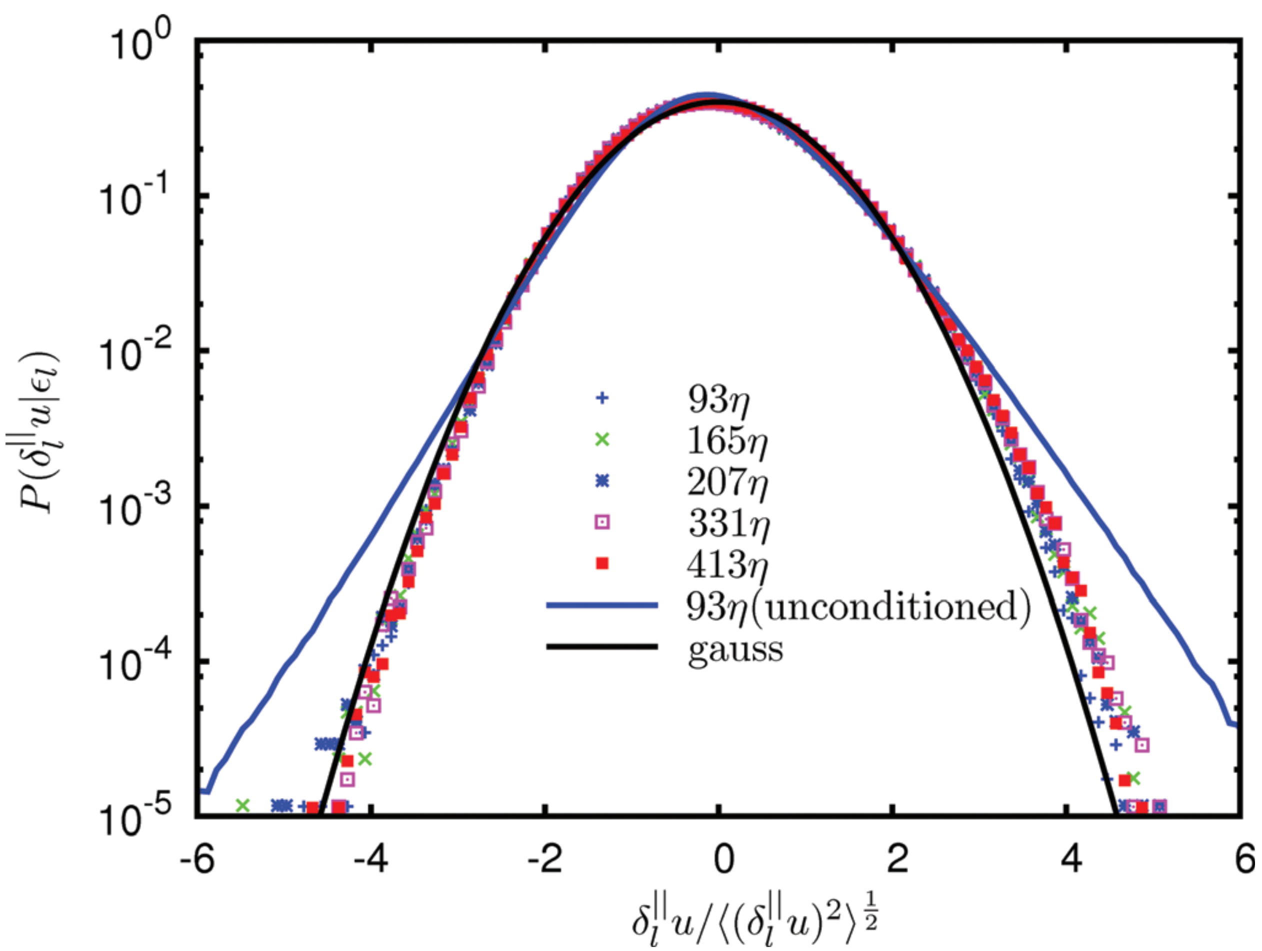
nearly Gaussian from dissipation- up to integral-scale.

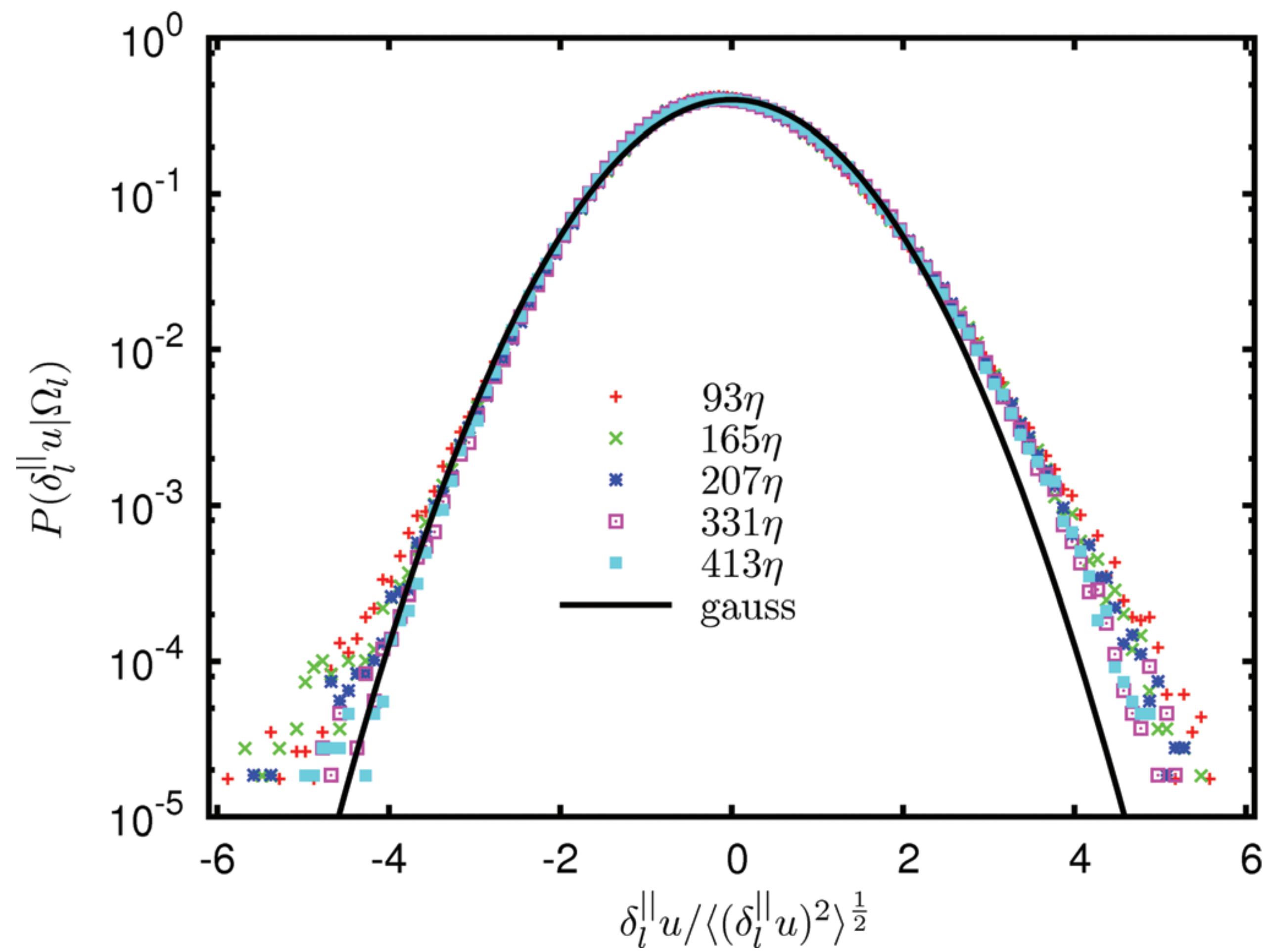
other possible conditionings:

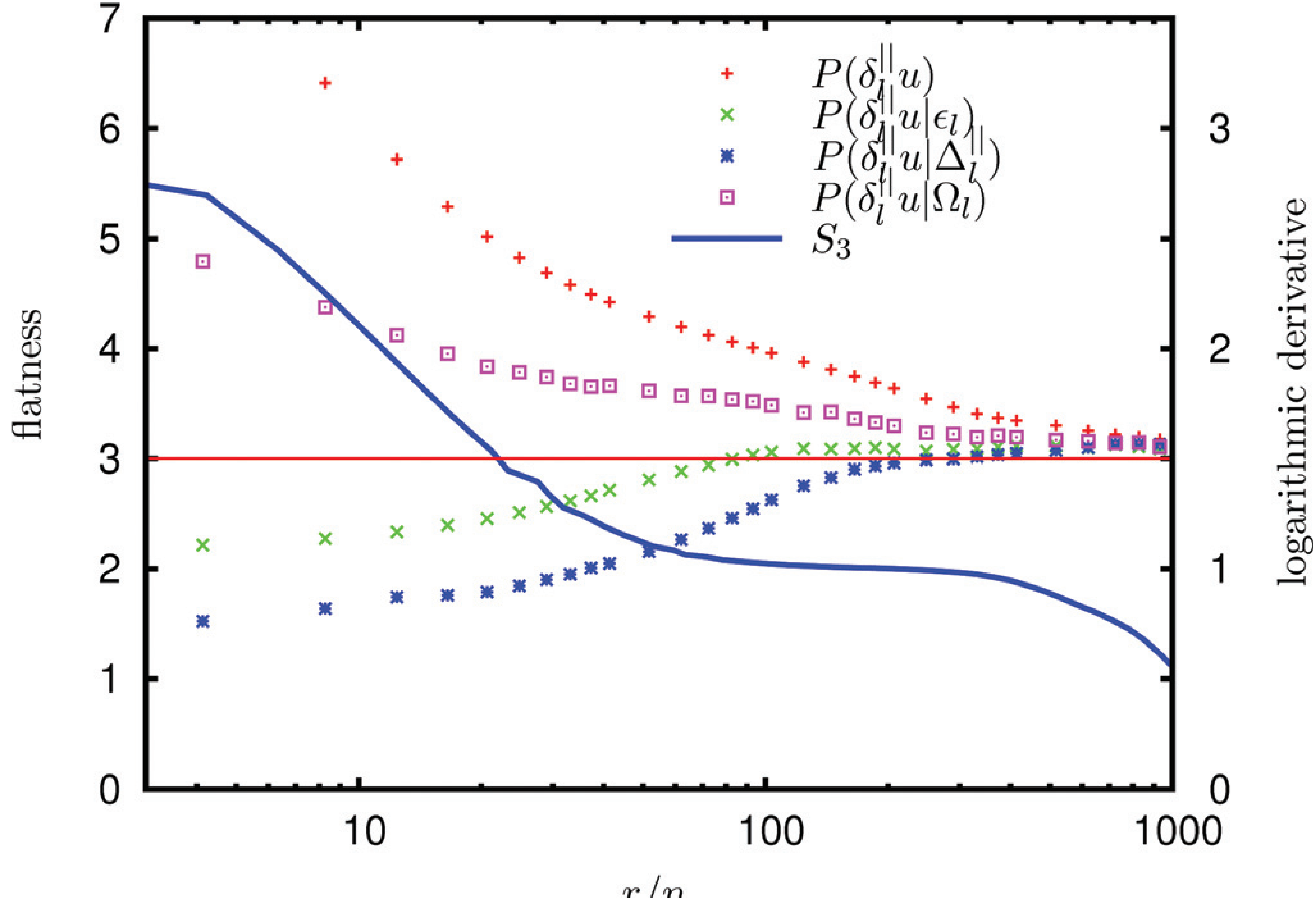
$$\Omega_l = \frac{1}{l}\int_0^l ds\, \nu |\omega(x + s\,\hat{l})|^2$$

$$\Delta_l^{||} = \frac{1}{l}\int_0^l ds\, \nu |\hat{\boldsymbol{l}} \cdot \nabla \boldsymbol{u}(\boldsymbol{x} + s\,\hat{\boldsymbol{l}})|^2.$$

longitudinal increments

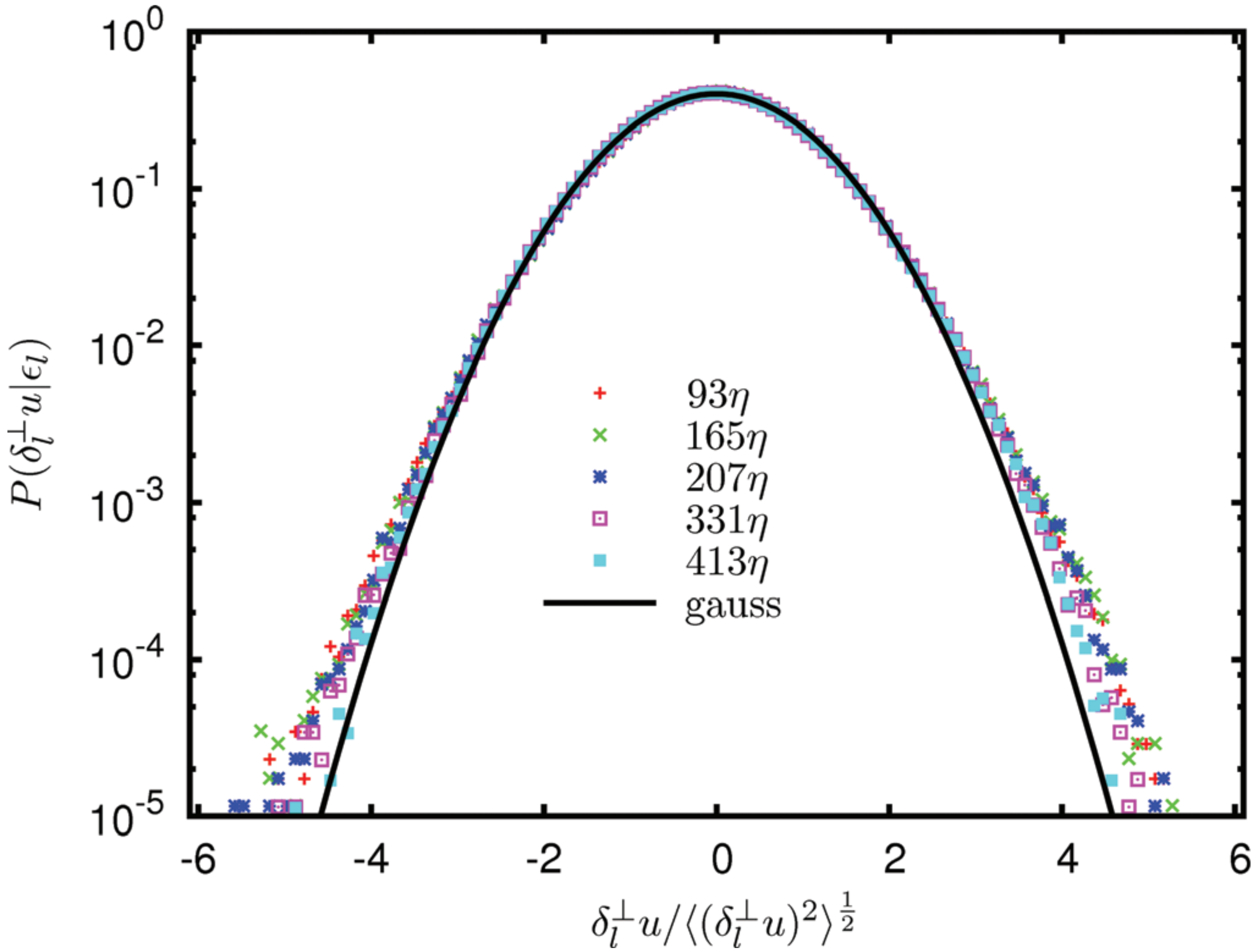


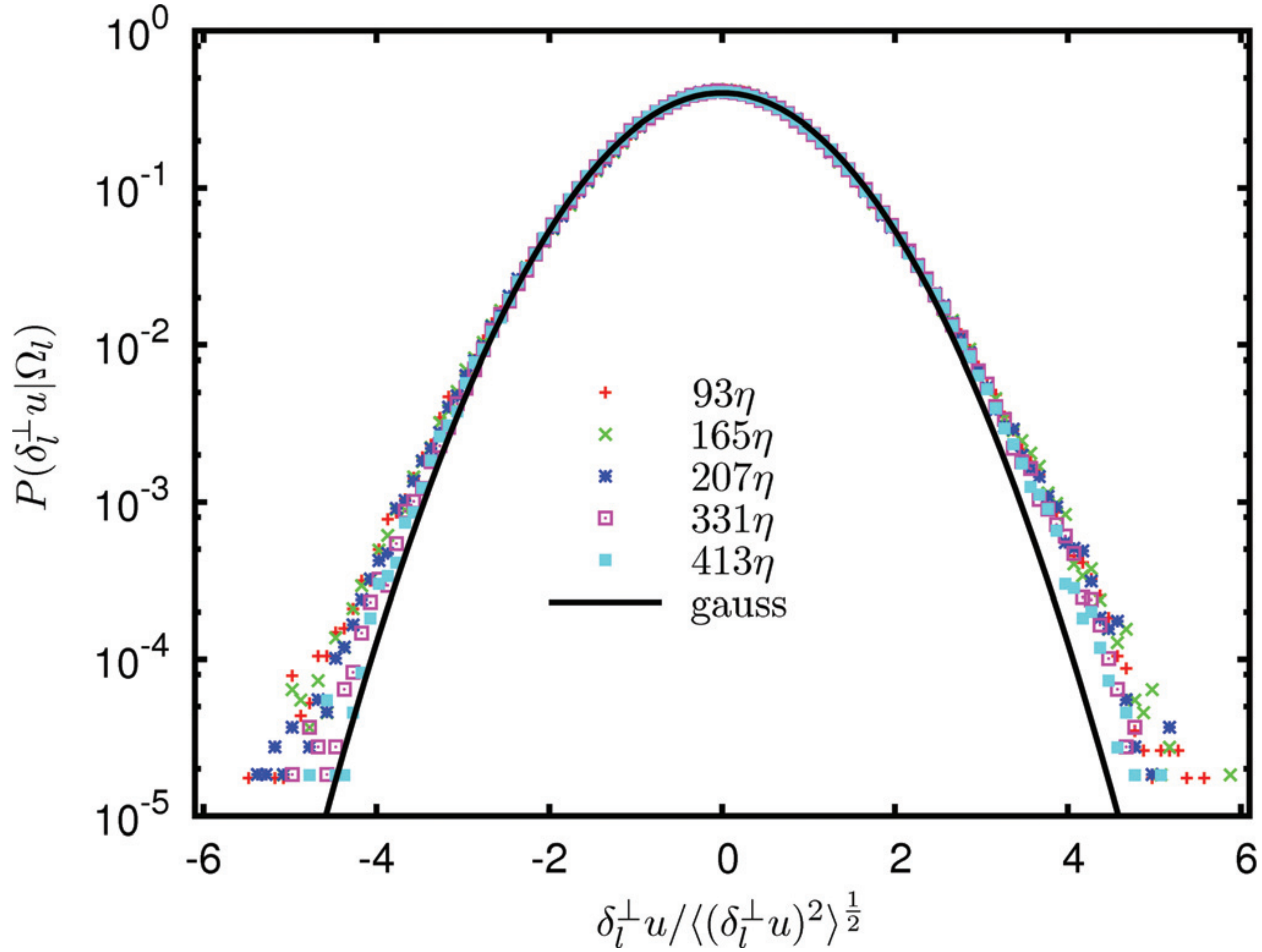




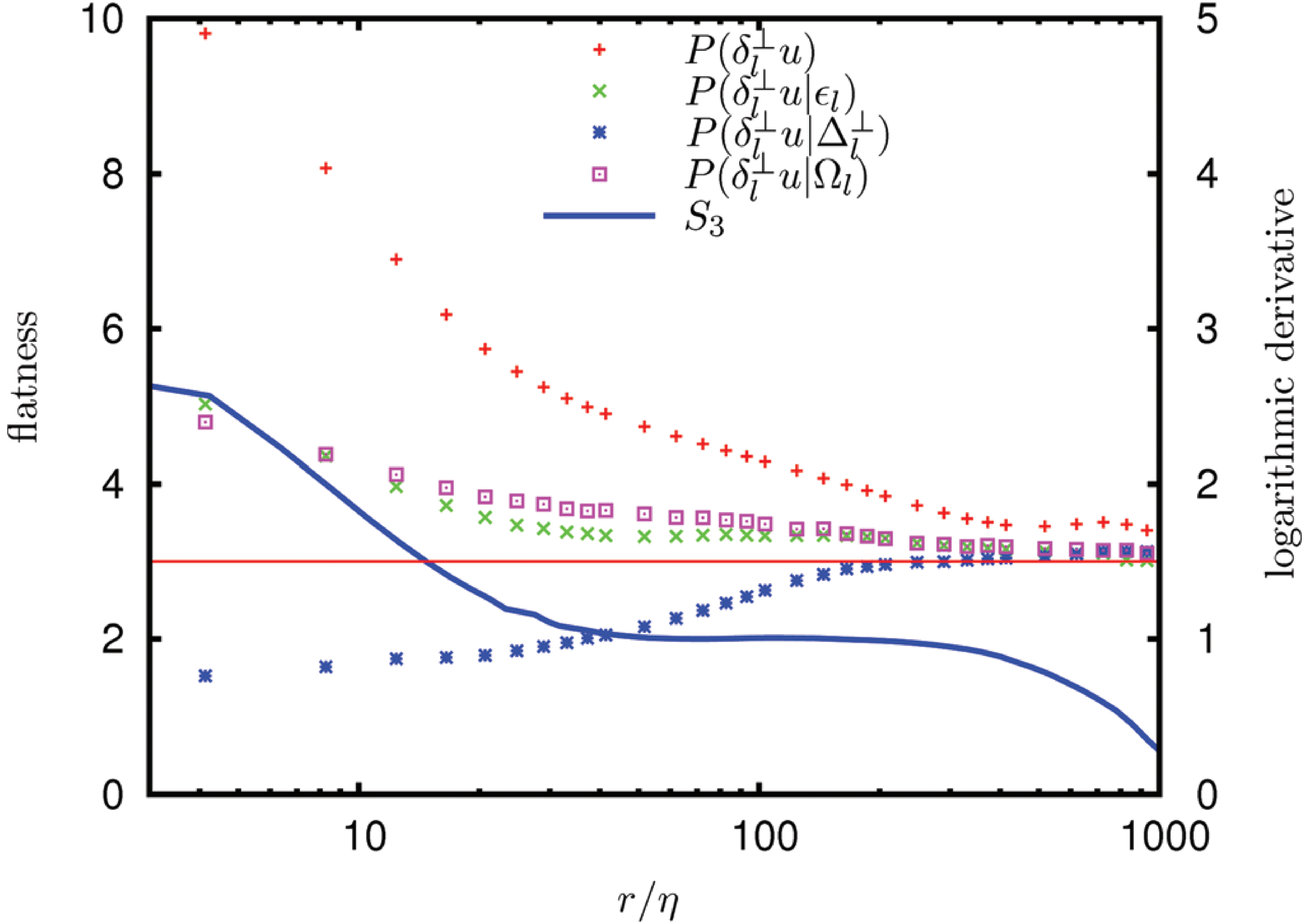
7
6
5
4
3
2
1
0
flatness
3
2
1
0
logarithmic derivative
10
100
1000
r/η
$P(\delta_l^{||}u)$
$P(\delta_l^{||}u|\epsilon_l)$
$P(\delta_l^{||}u|\Delta_l^{||})$
$P(\delta_l^{||}u|\Omega_l)$
S_3

transversal increments





ϵ_l not worse than Ω_l



10
8
6
4
2
0
5
4
3
2
1
0
10
100
1000
flatness
logarithmic derivative
r/η
$P(\delta_l^\perp u)$
$P(\delta_l^\perp u|\epsilon_l)$
$P(\delta_l^\perp u|\Delta_l^\perp)$
$P(\delta_l^\perp u|\Omega_l)$
S_3

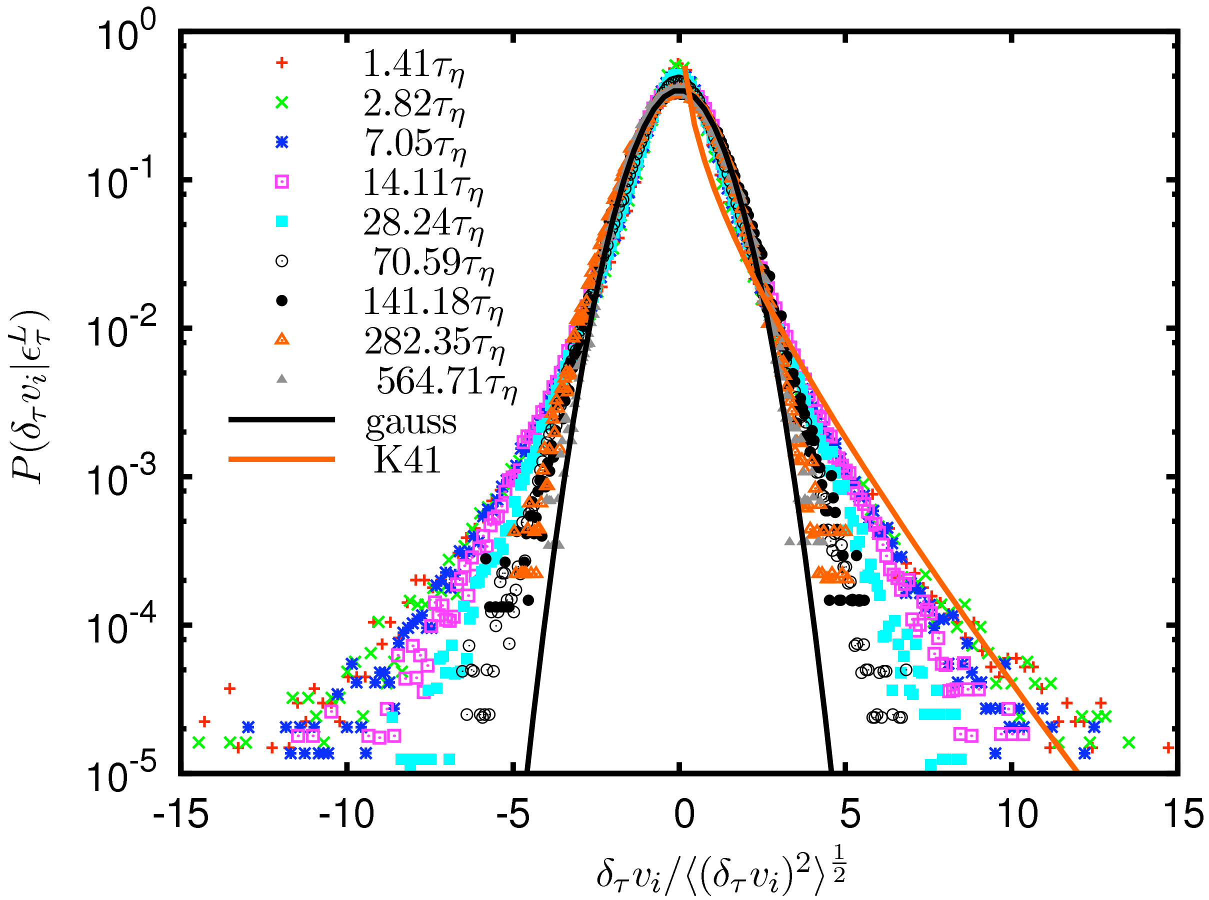
Lagrangian conditional statistics

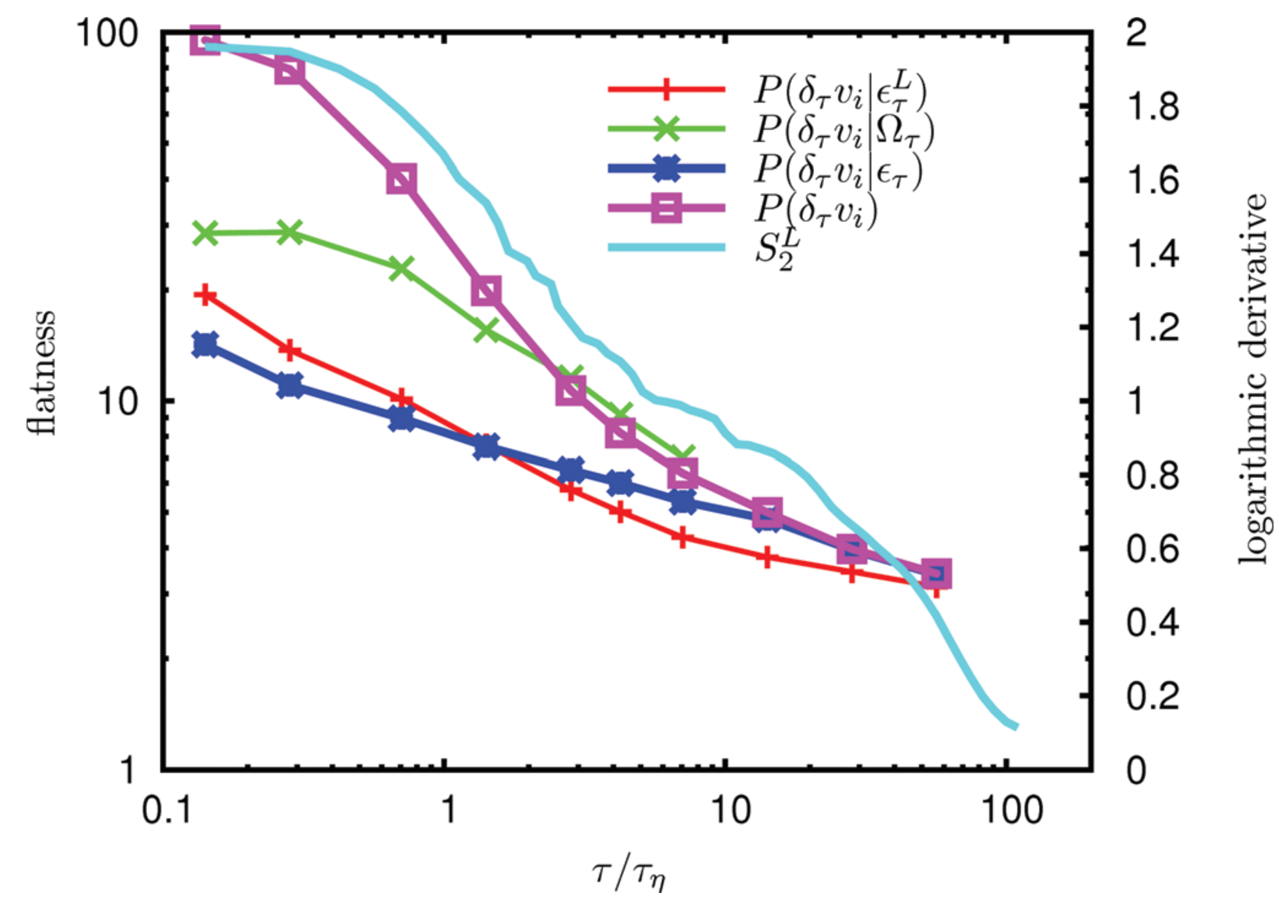
$$\epsilon_\tau = \frac{1}{\tau}\int_0^\tau \epsilon(\boldsymbol{X}(\boldsymbol{x}_0, t))dt \qquad \Omega_\tau = \frac{1}{\tau}\int_0^\tau |\boldsymbol{\omega}(\boldsymbol{X}(\boldsymbol{x}_0, t))|dt.$$

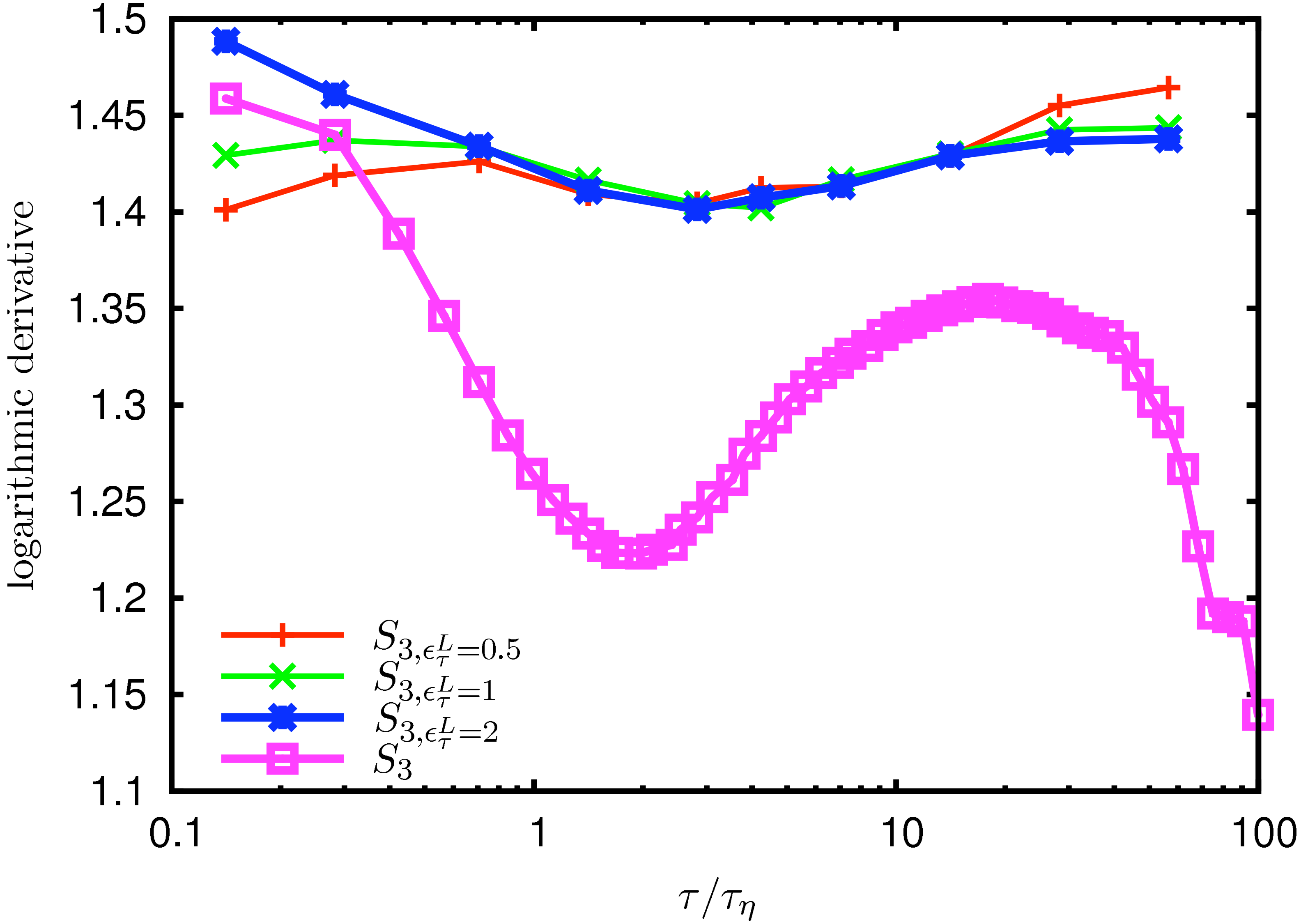
Benzi, Biferale et al (2009):	ϵ_τ works better
Yu and Menveau: (2010):	conditioned the velocity increments on a spatially averaged energy dissipationrate at one foot-point of the increments

Homann, Schulz, Grauer 2010:

$$\epsilon_\tau^L = \frac{1}{2} \int dt \sum_{i,j} [u_j\, \partial_j u_i + u_i\, \partial_i u_j]^2$$







logarithmic derivative
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
0.1
1
10
100
τ/τ_η
$S_{3,\epsilon_\tau^L=0.5}$
$S_{3,\epsilon_\tau^L=1}$
$S_{3,\epsilon_\tau^L=2}$
S_3

Thank You