KINETIC EQUATIONS: STATISTICS AND DYNAMICS IN THE INVERSE CASCADE OF 2 D TURBULENCE

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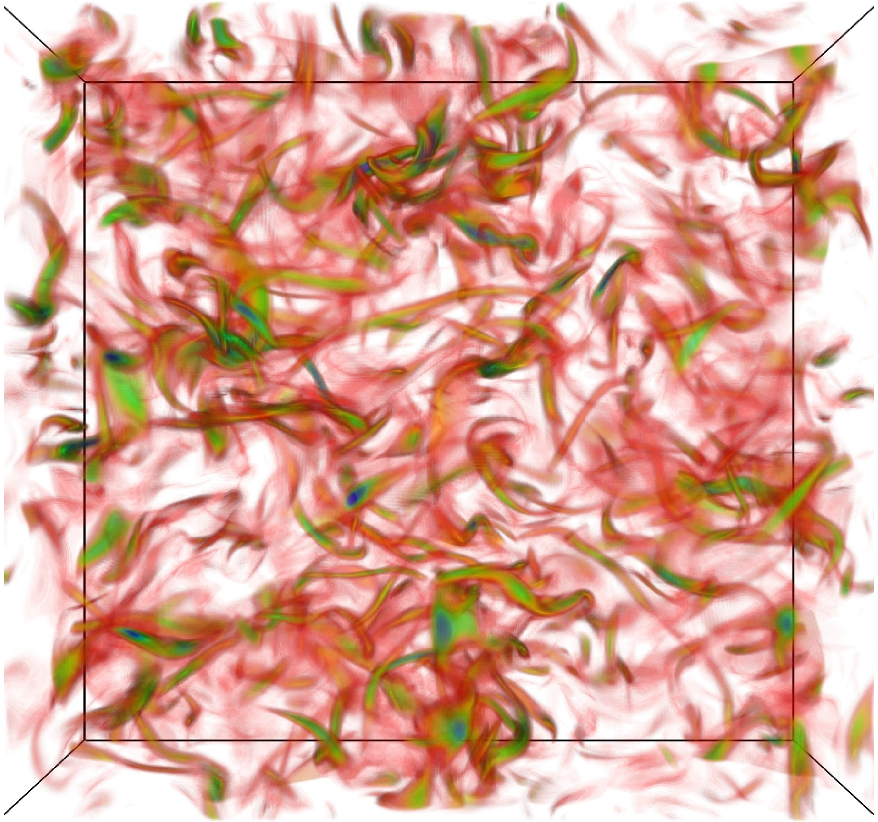
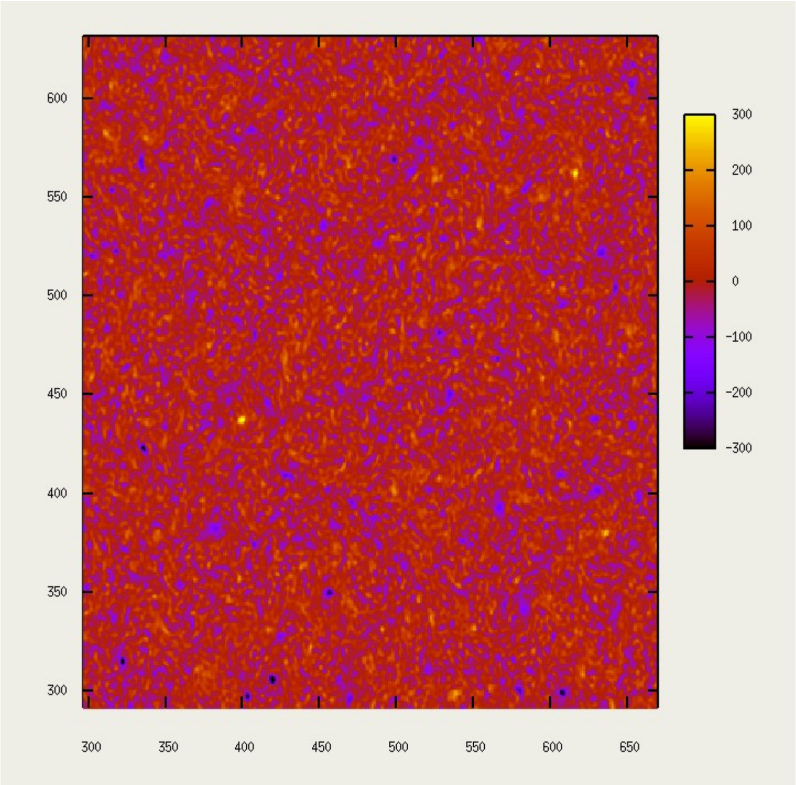
TURBULENT CASCADES





- DIRECT CASCADE
- INVERSE CASCADE

REMINDER ON POINT VORTEX DYNAMCS

IDEAL 2D HYDRODYNAMICS
VORTEX DYNAMICS

$$\frac{\partial}{\partial t}\omega(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\omega(\mathbf{x},t) = 0$$

$$\mathbf{u}(\mathbf{x},t) = \int d\mathbf{x}'\omega(\mathbf{x}',t)\mathbf{e}_z \times \frac{\mathbf{x}-\mathbf{x}'}{2\pi|\mathbf{x}-\mathbf{x}'|^2}$$

- CONSERVATION OF LAGRANGIAN VORTICITY
- BIOT-SAVART'S LAW
- POINT VORTEX

$$\omega(\boldsymbol{x},t) = \Gamma\,\delta(\boldsymbol{x}-\boldsymbol{X}(t))$$

$$\boldsymbol{u} = \frac{\Gamma}{2\pi}\boldsymbol{e}_z \times \frac{\boldsymbol{x}-\boldsymbol{X}(t)}{|\boldsymbol{x}-\boldsymbol{X}(t)|^2}$$

IDEAL 2D HYDRODYNAMICS VORTEX DYNAMICS

- POINT VORTEX
- LAGRANGIAN PICTURE
- HAMILTONIAN SYSTEM

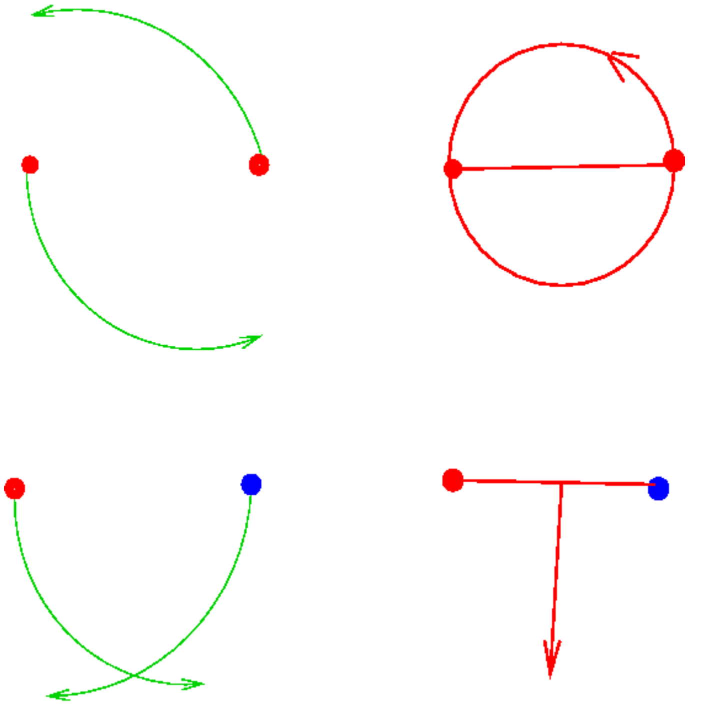
$$\omega(\mathbf{x},t) = \sum_j \Gamma_j \delta\left(\mathbf{x} - \mathbf{X}_j(t)\right)$$

$$\mathbf{u}(\mathbf{x},t) = \sum_j \Gamma_j \mathbf{e}_z \times \frac{\mathbf{x} - \mathbf{X}_j(t)}{2\pi|\mathbf{x} - \mathbf{X}_j(t)|^2}$$

$$H = -\frac{1}{4\pi}\sum_{i\neq j} \Gamma_i \Gamma_j ln|\mathbf{X}_i - \mathbf{X}_j|$$

$$\dot{\mathbf{X}}_j(t) = \sum_{k\neq j} \Gamma_k \mathbf{e}_z \times \frac{\mathbf{X}_j(t) - \mathbf{X}_k(t)}{2\pi|\mathbf{X}_j(t) - \mathbf{X}_k(t)|^2}$$

TWO-VORTEX MOTION



$$\dot{\mathbf{X}}_1 = \Gamma_2 \mathbf{e}_z \times \frac{\mathbf{X}_1 - \mathbf{X}_2}{2\pi|\mathbf{X}_1 - \mathbf{X}_2|^2}$$

$$\dot{\mathbf{X}}_2 = \Gamma_1 \mathbf{e}_z \times \frac{\mathbf{X}_2 - \mathbf{X}_1}{2\pi|\mathbf{X}_2 - \mathbf{X}_1|^2}$$

$$\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$$

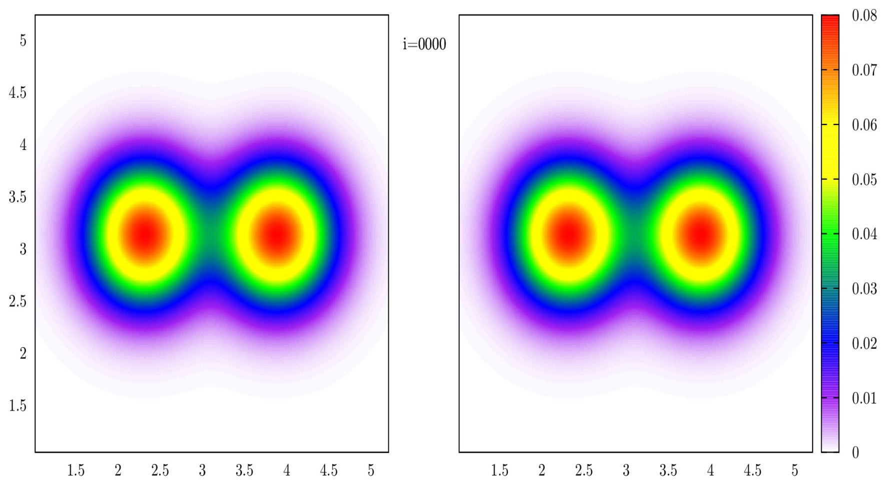
$$\dot{\boldsymbol{r}} = (\Gamma_1 + \Gamma_2)\,\boldsymbol{e}_z \times \frac{\boldsymbol{r}}{r^2}$$

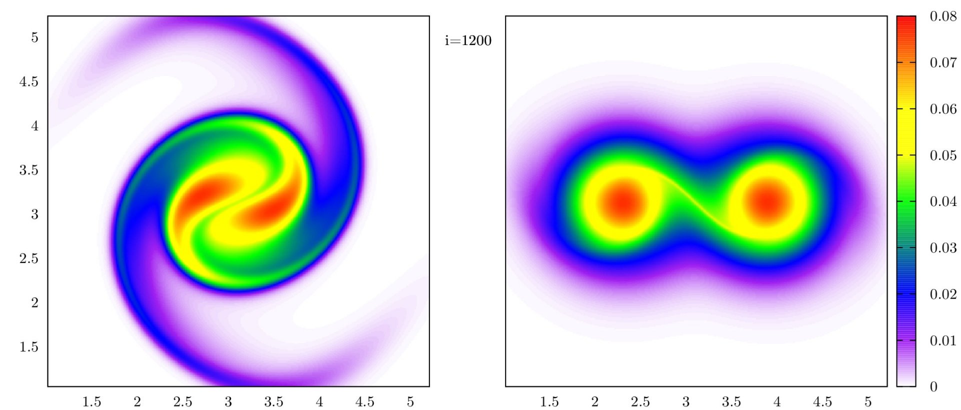
$$\frac{d}{dt}|\boldsymbol{r}| = const$$

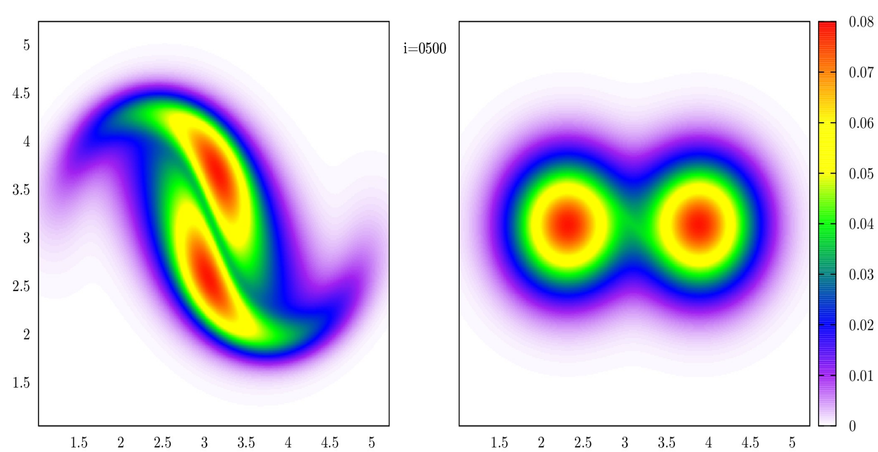
- THREE POINT VORTEX MOTION INTEGRABLE, FOUR POINT MOTION CHAOTIC
- DISSIPATION

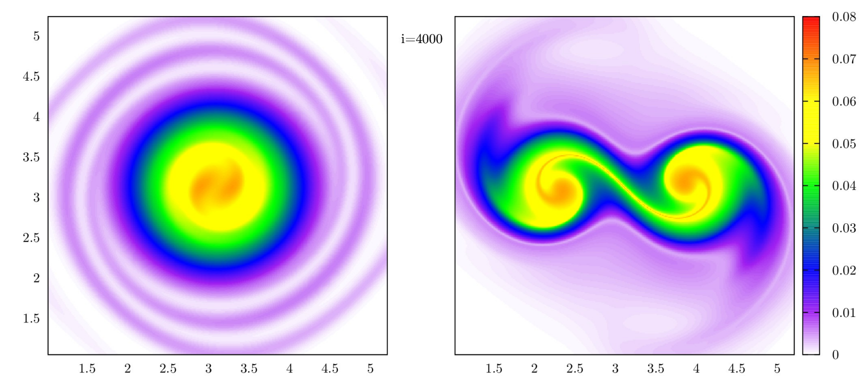
TWO-VORTEX MOTION: EFFECT OF DISSIPATION EULERIAN VERSUS LAGRANGIAN PICTURE

with A. Daitche

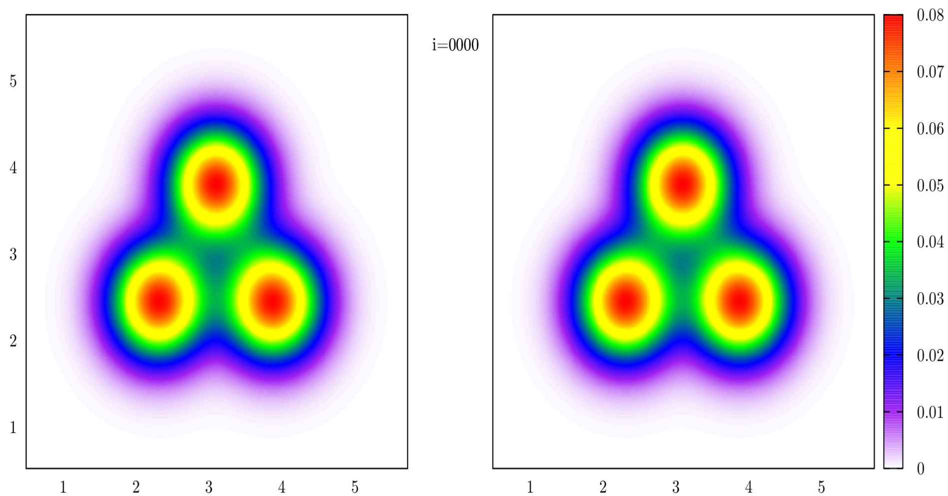


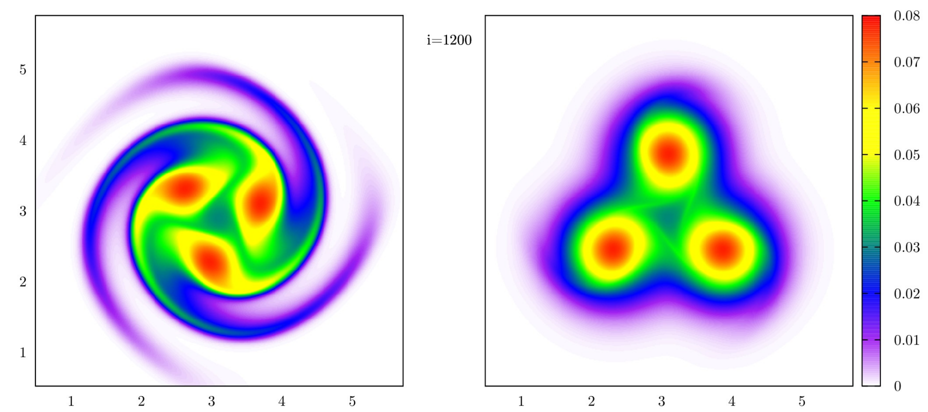


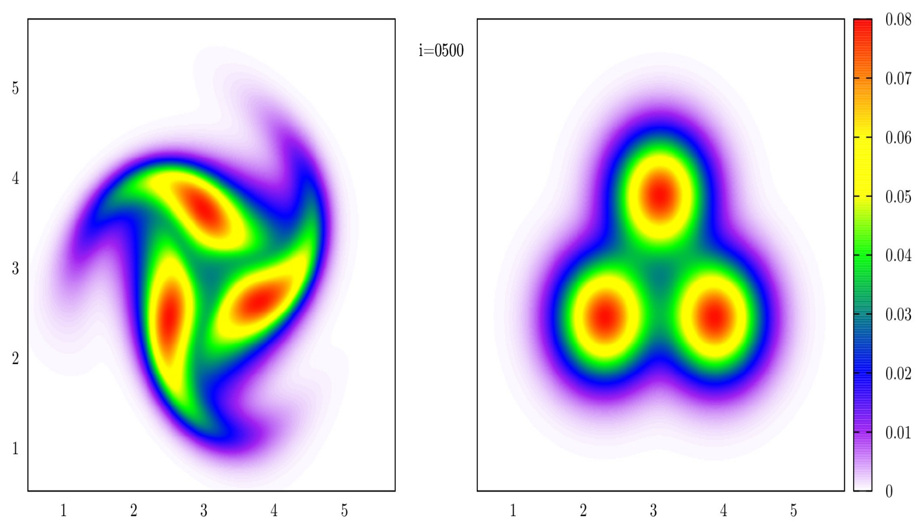


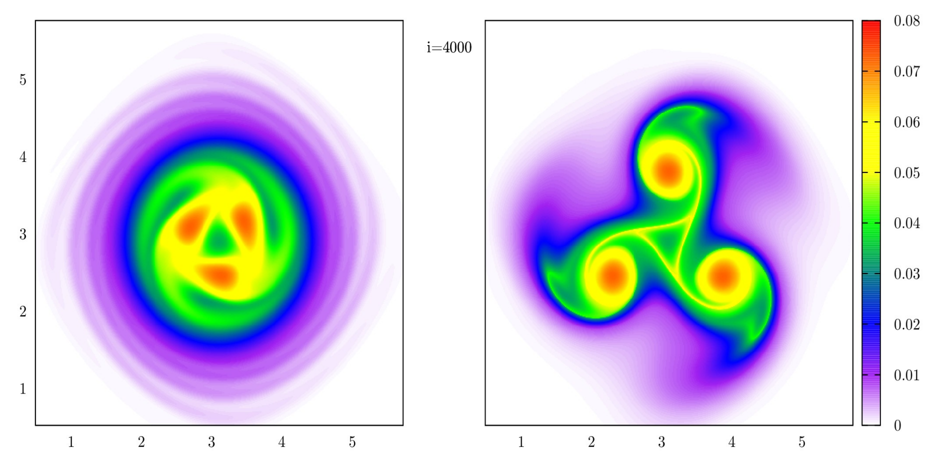


THREE-VORTEX MOTION









L. ONSAGER: STATISTICS OF POINT VORTEX SYSTEMS

- **POINT VORTEX SYSTEM: HAMILTONIAN SYSTEM**
- **STATISTICAL TREATMENT**
- **EQUILIBRIUM ENSEMBLE**

$$F(\mathbf{x}_1, .., \mathbf{x}_N) = Z^{-1}(\beta) e^{-\beta H}$$

- **KINETIC EQUATIONS: LIOUVILLE EQUATION, BBGKY-HIERARCHY**

$$[\frac{\partial}{\partial t} + \sum_{i \neq j} \left(\Gamma_j \mathbf{e}_z \times \frac{\mathbf{x}_i - \mathbf{x}_j}{2\pi |\mathbf{x}_i - \mathbf{x}_j|^2} \cdot \nabla_i \right)] F(\mathbf{x}_1, .., \mathbf{x}_N, t) = 0$$

VORTICITY STATISTICS IN THE INVERSE CASCADE OF 2D TURBULENCE

with O. KAMPS, M. VOSSKUHLE

GENERATING A STATIONARY CASCADE: FLUX EQUILIBRIUM

$$\frac{\partial}{\partial t}\omega(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\omega(\mathbf{x},t) = L(-\Delta)\omega(\mathbf{x},t) + F(\mathbf{x},t)$$

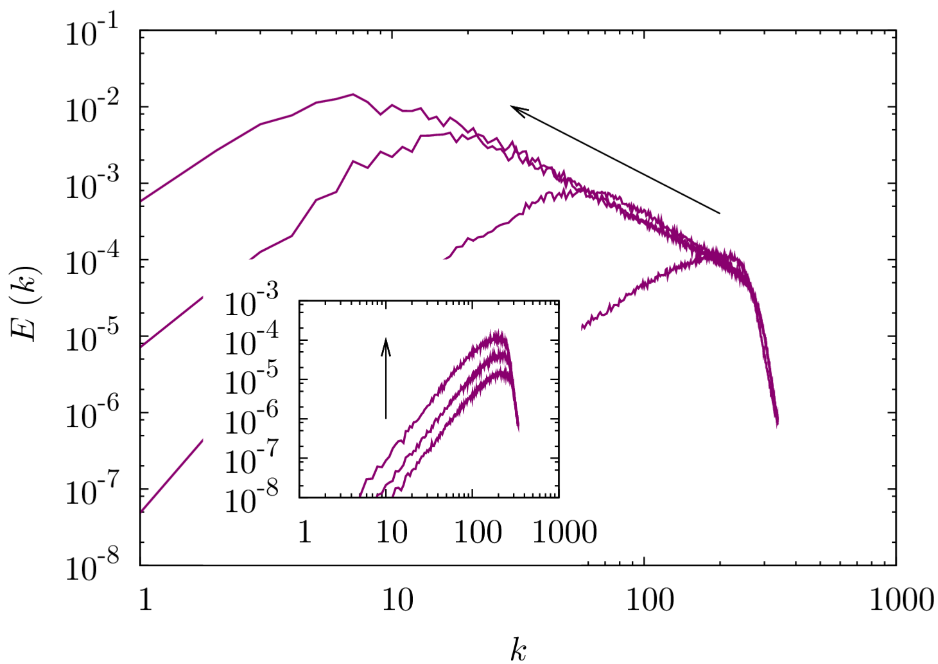
$$\langle F(\mathbf{x},t)F(\mathbf{x}',t')\rangle = Q(|\mathbf{x}-\mathbf{x}'|)\delta(t-t')$$

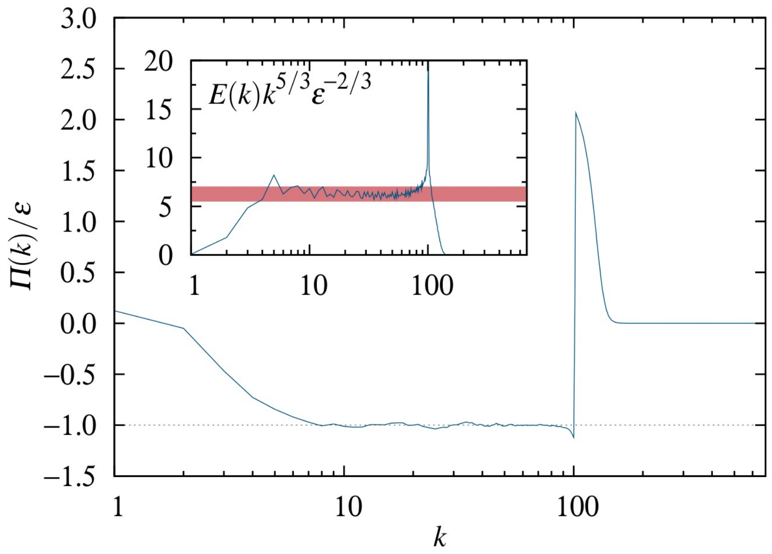
$$L(-\Delta) = -\gamma - \nu(-\Delta)^{\alpha}$$

- F: Small-Scale Stirring = Excitation of Point Vortices of Circulation (with Gaussian Statistics)
- L: Large-Scale Damping

MOVIES: CLUSTERING OF VORTICITY

ENERGY SPECTRUM, SPECTRAL ENERGY FLUX INERTIAL RANGE

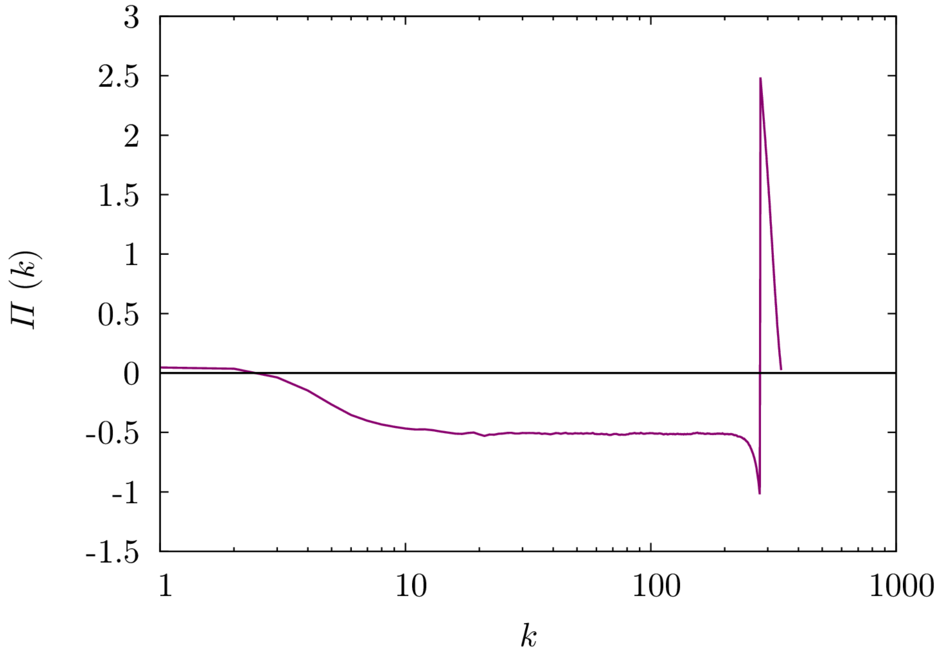




$$\mathbf{u}(\mathbf{x},t) = \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{u}_{\mathbf{k}}(t)$$

$$E(k) = 2\pi k \langle \mathbf{u}_{\mathbf{k}} \mathbf{u}_{-\mathbf{k}} \rangle$$

ENERGY SPECTRUM, SPECTRAL ENERGY FLUX

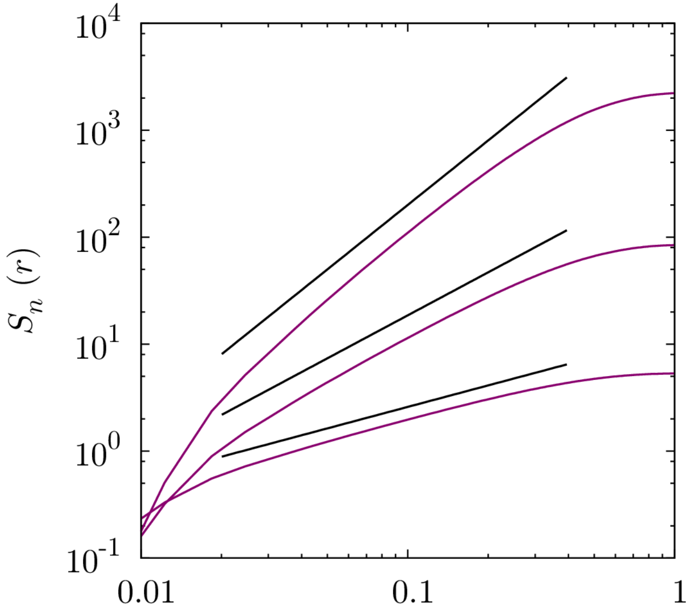


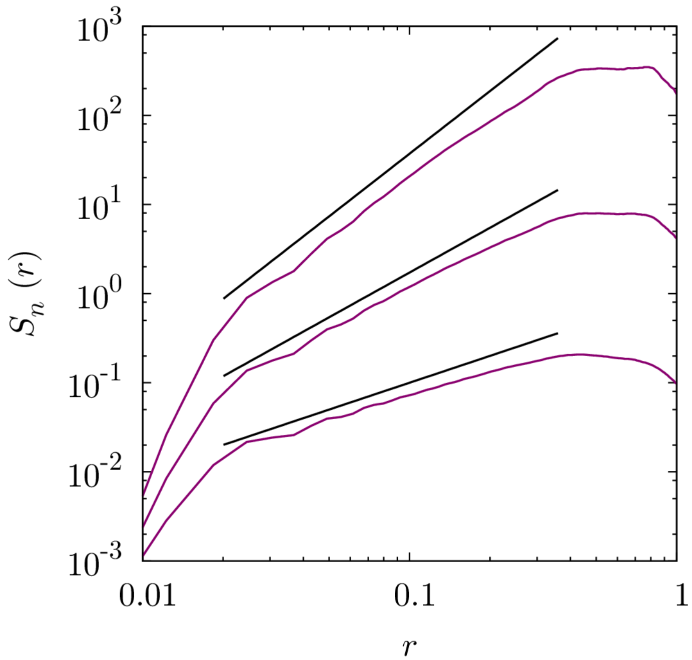
- CONSTANT SPECTRAL ENERGY FLUX
- O. KAMPS

$$E(k) = C_K k^{-5/3}$$

$$\frac{\partial}{\partial t}E(k,t) + \frac{\partial}{\partial k}\Pi(k,t) = L(k^2)E(k,t) + Q(k)$$

STRUCTURE FUNCTIONS





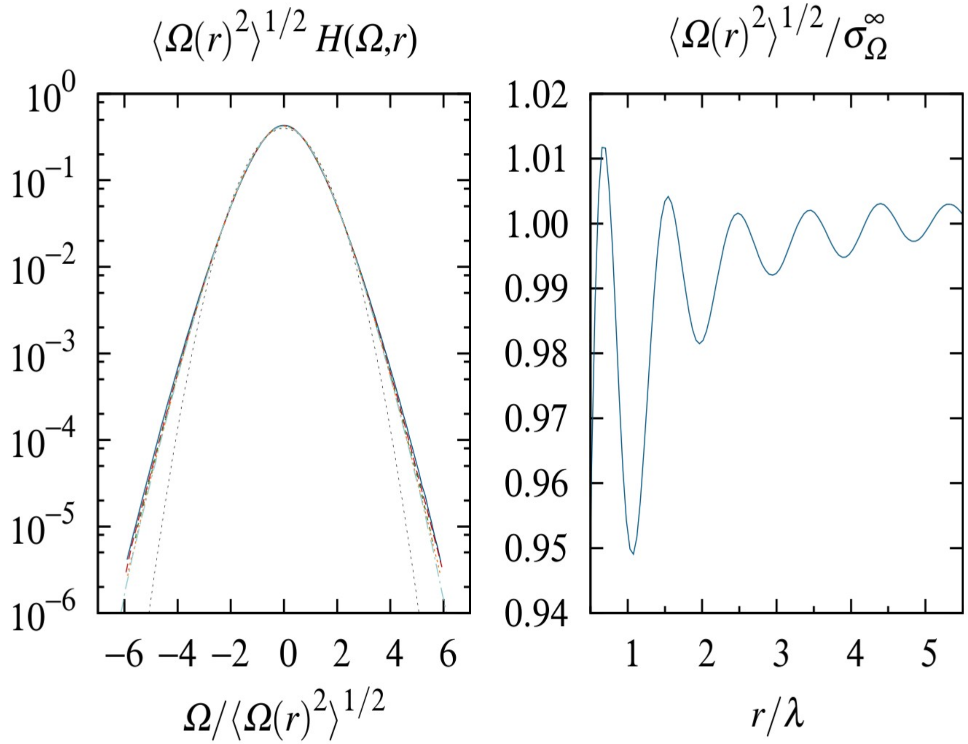
$$v_r(r,t)=\boldsymbol{e}_r\cdot[\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)]$$

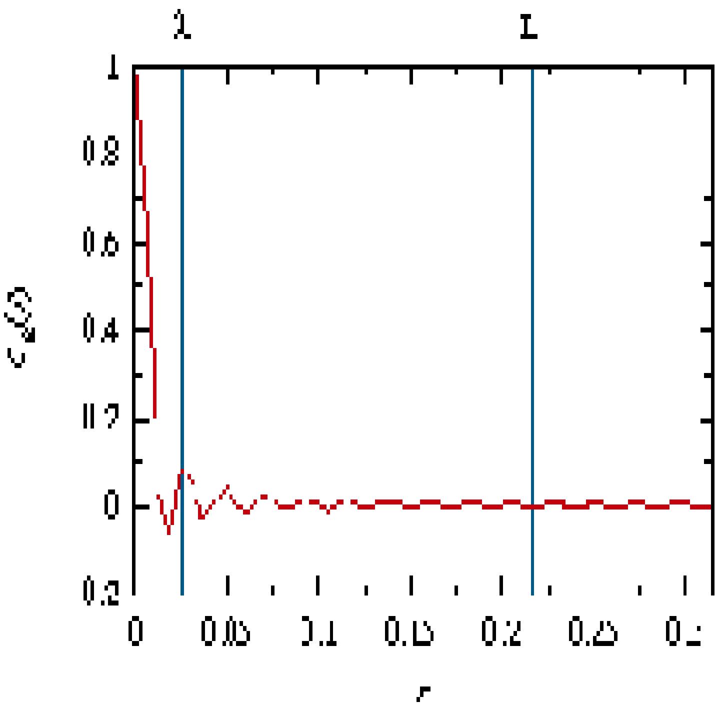
$$S_r^n(r)=\langle v_r(r,t)^n\rangle=s_n r^{n/3}$$

$$S_r^3(r)=\langle v_r(r,t)^3\rangle=+s_3 r$$

- SCALING BEHAVIOUR IN THE INERTIAL RANGE
- NO INTERMITTENCY
- KRAICHNAN-PREDICTIONS

VORTICITY INCREMENT STATISTICS





$$\Omega(\mathbf{r}, t) = \omega(\mathbf{x} + \mathbf{r}, t) - \omega(\mathbf{x}, t)$$

- **DESPITE NEAR-GAUSSIANITY: NO DERIVATION OF SCALING BEHAVIOUR**

ANALYTICAL DERIVATION OF ENERGY SPECTRUM

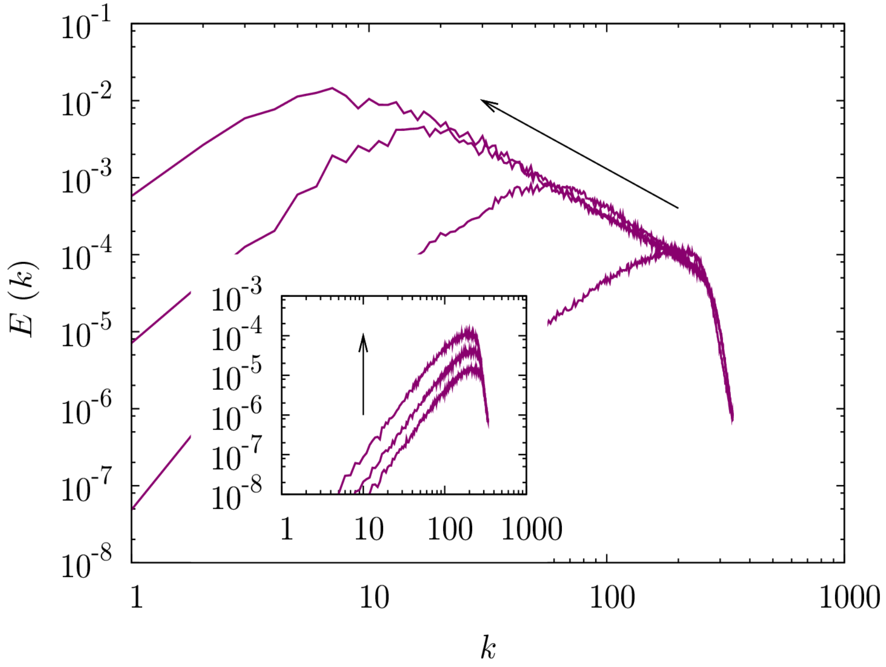
- DESPITE NEAR-GAUSSIANITY: NO THEORETICAL DERIVATION OF CORRELATION FUNCTIONS
- RENORMALIZED PERTURBATION THEORIES
- RENORMALIZATION GROUP THEORIES

A.A. MIGDAL (1993): After trying for few years to do something with the Wyld approach I conclude that this is a dead end. The best bet here would be the renormalization group, which magically works in statistical physics. Those critical phenomena were close to Gaussian... There is no such luck in turbulence. The nonlinear effects are much stronger...No! These old tricks are not going to work, we have to invent the new ones

NONPERTURBATIVE TREATMENT!

WHAT IS THE NATURE OF THE TRANSPORT PROCESS IN THE INVERSE CASCADE

INVERSE CASCADE: NON-FICKIAN TRANSPORT IN SCALE



- **TRANSPORT OF ENERGY UPHILL!**
- **STATISTICAL TREATMENT + NONLINEAR DYNAMICS: KINETIC EQUATIONS**

KINETIC EQUATIONS FOR 2D-TURBULENCE: DYNAMICS AND STATISTICS

- KINETIC EQUATIONS FOR TURBULENT FIELDS
- MONIN, LUNDGREN, NOVIKOV
- SIMILAR TO BBGKY HIERARCHY OF CLASSICAL MECHANICS
- RECENT APPLICATIONS:

  M.WILCZEK, R.F. (VORTICITY FIELD 3D DIRECT CASCADE, PHYS. REV. E (2010))

  M. WILCZEK, A. DAITCHE, R.F. (VELOCITY FIELD 3D DIRECT CASCADE, J.FLUID MECH. (2011))

  J. LÜLF, M. WILCZEK, R.F. (RAYLEIGH-B'ENARD CONVECTION, NEW JOUR.. PHYS. (2011))

  M. VOSSKUHLE, O. KAMPS, M. WILCZEK, R.F., 2D-INVERSE CASCADE arXiv (2011)

KINETIC EQUATIONS FOR TURBULENT VORTICITY

$$f(\omega, \mathbf{x}, t) = \langle \delta(\omega - \omega(\mathbf{x}, t)) \rangle$$

$$\frac{\partial}{\partial t} f(\omega, \mathbf{x}, t) = \langle \frac{\partial}{\partial t} \delta(\omega - \omega(\mathbf{x}, t)) \rangle$$
$$= -\frac{\partial}{\partial \omega} \langle \dot{\omega}(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle$$

$$\frac{\partial}{\partial t} f(\omega(\mathbf{x}, t) + \nabla_x \cdot \langle \mathbf{u}(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle$$
$$= -\frac{\partial}{\partial \omega} \nu \langle L(-\Delta) \omega(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t) \rangle - \frac{\partial}{\partial \omega} \langle F(\mathbf{x}, t) \delta(\omega - \omega(\mathbf{x}, t)) \rangle$$

- CLOSURE PROBLEM OF NONLINEAR FIELD THEORIES
- COUPLING TO TWO-POINT FUNCTIONS (HIERARCHY)
- INTRODUCTION OF CONDITIONAL EXPECTATIONS

KINETIC EQUATIONS FOR TURBULENT VORTICITY

$$\frac{\partial}{\partial t} f(\omega(\mathbf{x},t) + \nabla_x \cdot \langle \mathbf{u}(\mathbf{x},t)\delta(\omega - \omega(\mathbf{x},t))\rangle$$
$$= -\frac{\partial}{\partial \omega}\nu\langle L(-\Delta)\omega(\mathbf{x},t)\delta(\omega - \omega(\mathbf{x},t)\rangle - \frac{\partial}{\partial \omega}\langle F(\mathbf{x},t)\delta(\omega - \omega(\mathbf{x},t))\rangle$$

- CONDITIONAL EXPECTATIONS

$$\langle \mathbf{u}(\mathbf{x},t)\delta(\omega - \omega(\mathbf{x},t))\rangle = \langle \mathbf{u}(\mathbf{x},t)|\omega,\mathbf{x}\rangle f(\omega,\mathbf{x},t)$$

$$\langle (Diss + Forc)\delta(\omega - \omega(\mathbf{x},t))\rangle = \langle \mu(\mathbf{x}|\omega,\mathbf{x}\rangle f(\omega,\mathbf{x},t)$$

TWO-POINT VORTICITY STATISTICS $f(\omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2; t)$

- Kinetic Equation

$$[\frac{\partial}{\partial t} + \nabla_{x_1} \cdot \langle \mathbf{u}(\mathbf{x}_1) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle + \nabla_{x_2} \cdot \langle \mathbf{u}(\mathbf{x}_1) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle] f(\omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2; t)$$

$$= -[\frac{\partial}{\partial \omega_1} \langle \mu(\mathbf{x}_1 | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle + \frac{\partial}{\partial \omega_2} \langle \mu(\mathbf{x}_2 | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle] f(\omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2; t)$$

- First Order Partial Differential Equation
- Solvable by Methods of Characteristics
- CONDITIONAL EXPECTATIONS ACCESSIBLE BY DIRECT NUMERICAL SIMULATIONS

TWO-POINT VORTICITY STATISTICS $f(\omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2; t)$

- Kinetic Equation

$$[\frac{\partial}{\partial t} + \nabla_{x_1} \cdot \langle \mathbf{u}(\mathbf{x}_1) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle + \nabla_{x_2} \cdot \langle \mathbf{u}(\mathbf{x}_1) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle] f(\omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2; t)$$

$$= -[\frac{\partial}{\partial \omega_1} \langle \mu(\mathbf{x}_1 | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle + \frac{\partial}{\partial \omega_2} \langle \mu(\mathbf{x}_2 | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle] f(\omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2; t)$$

- Characteristic Equations

$$\dot{\mathbf{x}}_1 = \langle \mathbf{u}(\mathbf{x}_1) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle$$

$$\dot{\mathbf{x}}_1 = \langle \mathbf{u}(\mathbf{x}_2 | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle$$

$$\dot{\omega}_1 = \langle \mu(\mathbf{x}_1) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle$$

$$\dot{\omega}_2 = \langle \mu(\mathbf{x}_2 | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle$$

EXTENSION TO N-POINT STATISTICS

$$\dot{\mathbf{x}}_i = \langle \mathbf{U}(\mathbf{x}_i) | \{\omega_k, \mathbf{x}_k\} \rangle \qquad \dot{\omega}_i = \langle \mu(\mathbf{x}_i) | \{\omega_k, \mathbf{x}_k\} \rangle$$

- **N TO INFINITY: APPROACHING LAGRANGIAN DESCRIPTION OF FLUID MOTION**

$$\dot{\mathbf{X}}(\mathbf{y}, t) = \int d\mathbf{y}' \Omega(\mathbf{y}', t) \mathbf{e}_z \times \frac{\mathbf{X}(\mathbf{y}, t) - \mathbf{X}(\mathbf{y}', t)}{2\pi |\mathbf{X}(\mathbf{y}, t) - \mathbf{X}(\mathbf{y}', t)|^2}$$

$$\dot{\Omega}(\mathbf{y}, t) = [\nu \Delta \omega(\mathbf{x}, t) + F(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{X}(\mathbf{y}, t)}$$

CAN WE DETERMINE CONDITIONAL EXPECTATIONS BY AB INITIO CALCULATIONS

A FIRST GUESS ON CONDITIONAL VELOCITY FIELDS:
ON THE WAY TO SUBGRID-MODELING

- CONDITIONAL STATISTICS: ASSUME GAUSSIAN STATISTICS, CORRELATION FUNCTION C(x-x')

$$\langle \mathbf{U}(\mathbf{x},t)|\omega_1,\mathbf{x}_1;\omega_2,\mathbf{x}_2)\rangle = \int d\mathbf{x}' \langle \omega(\mathbf{x}')|\omega_1,\mathbf{x}_1;\omega_2,\mathbf{x}_2\rangle \mathbf{e}_z \times \frac{\mathbf{x}-\mathbf{x}'}{2\pi|\mathbf{x}-\mathbf{x}'|^2}$$

$$\langle \omega(\mathbf{x})|\omega_1,\mathbf{x}_1;\omega_2,\mathbf{x}_2\rangle = \int d\omega' \omega' p(\omega',\mathbf{x}'|\omega_1,\mathbf{x}_1;\omega_2,\mathbf{x}_2\rangle$$

- CONDITIONAL VORTICITY: SUPERPOSITION OF CIRCULAR VORTICES

$$\langle \omega(\mathbf{x})|\omega_1,\mathbf{x}_1;\omega_2,\mathbf{x}_2\rangle = [C(\mathbf{x}-\mathbf{x}_1)C^{-1}(0)\omega_1 + C(\mathbf{x}-\mathbf{x}_2)C^{-1}(0)\omega_2]$$

DRESSED VORTICES

$$\langle \mathbf{U}(\mathbf{x}) | \omega_1, \mathbf{x}_1; \omega_2, \mathbf{x}_2 \rangle = \omega_1 \mathbf{e}_z \times \nabla \chi(\mathbf{x} - \mathbf{x}_1) + \omega_2 \mathbf{e}_z \times \nabla \chi(\mathbf{x} - \mathbf{x}_2)$$

$$\Delta \chi(\mathbf{x} - \mathbf{x}_1) = C^{-1}(0) C(\mathbf{x} - \mathbf{x}_1)$$

- **BARE POINT VORTEX:**

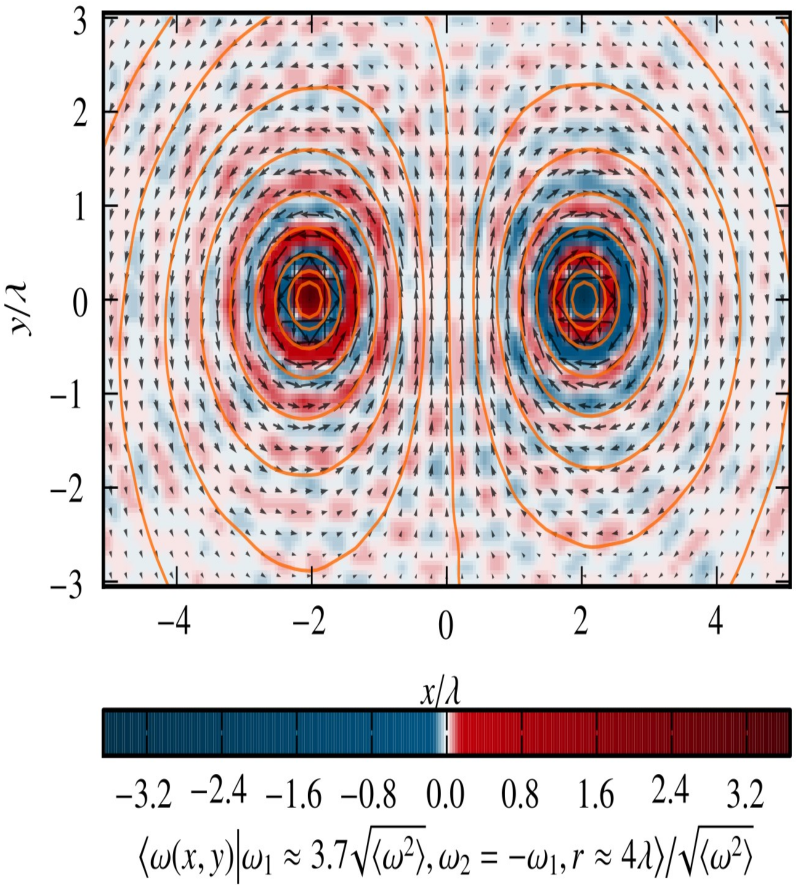
$$C^{-1}(0) C(\mathbf{r}) \to \delta(\mathbf{r}) \qquad \mathbf{U} = \mathbf{e}_z \times \frac{\mathbf{r}}{r^2}$$

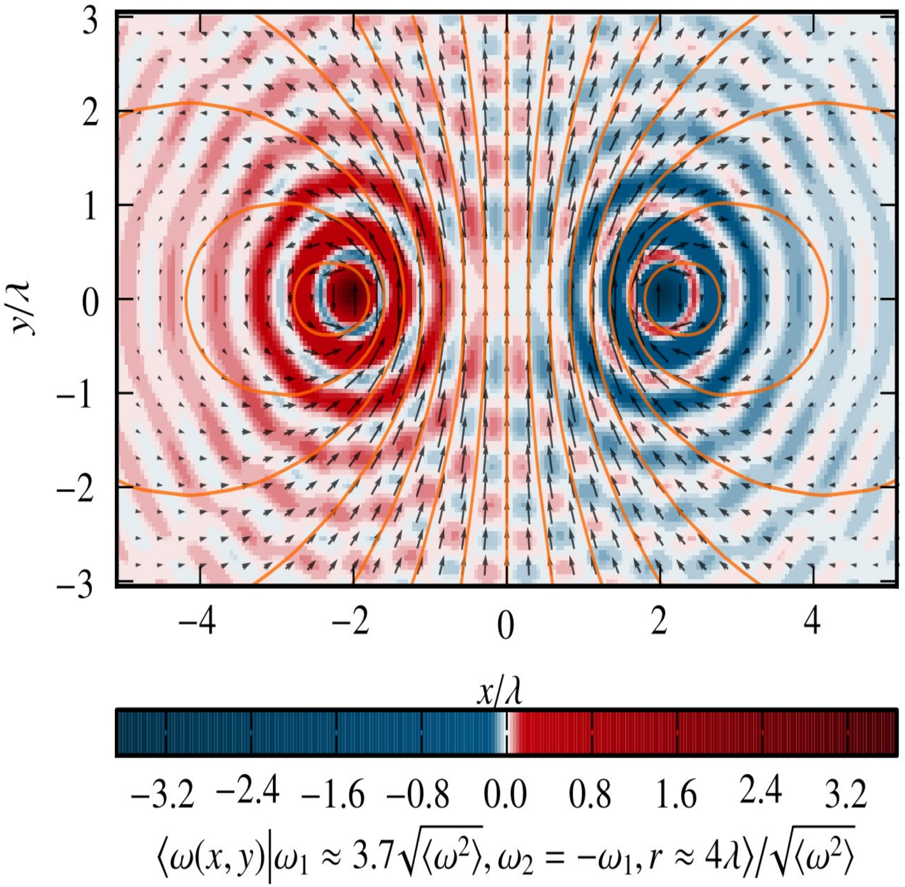
- **DRESSED VORTEX:**

$$\mathbf{U} = \mathbf{e}_z \times \frac{\mathbf{r}}{h(r)}$$

- **DIFFERENT BIOT-SAVART'S LAW**
- **LANDAU QUASI-PARTICLES**

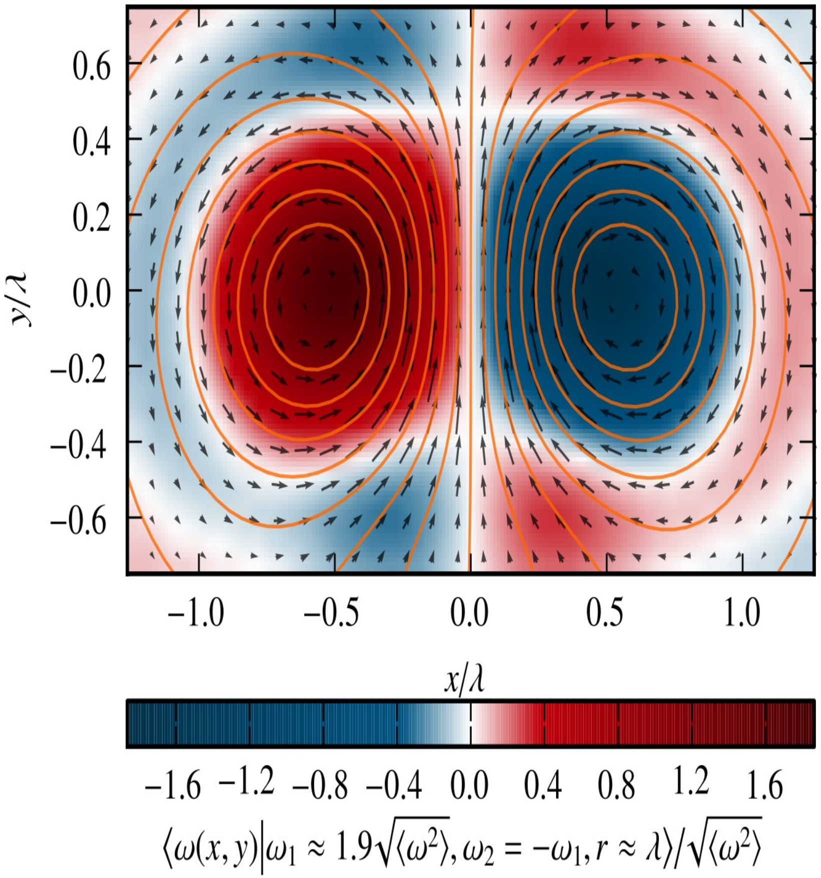
CONDITIONAL GAUSSIAN APPROXIMATION

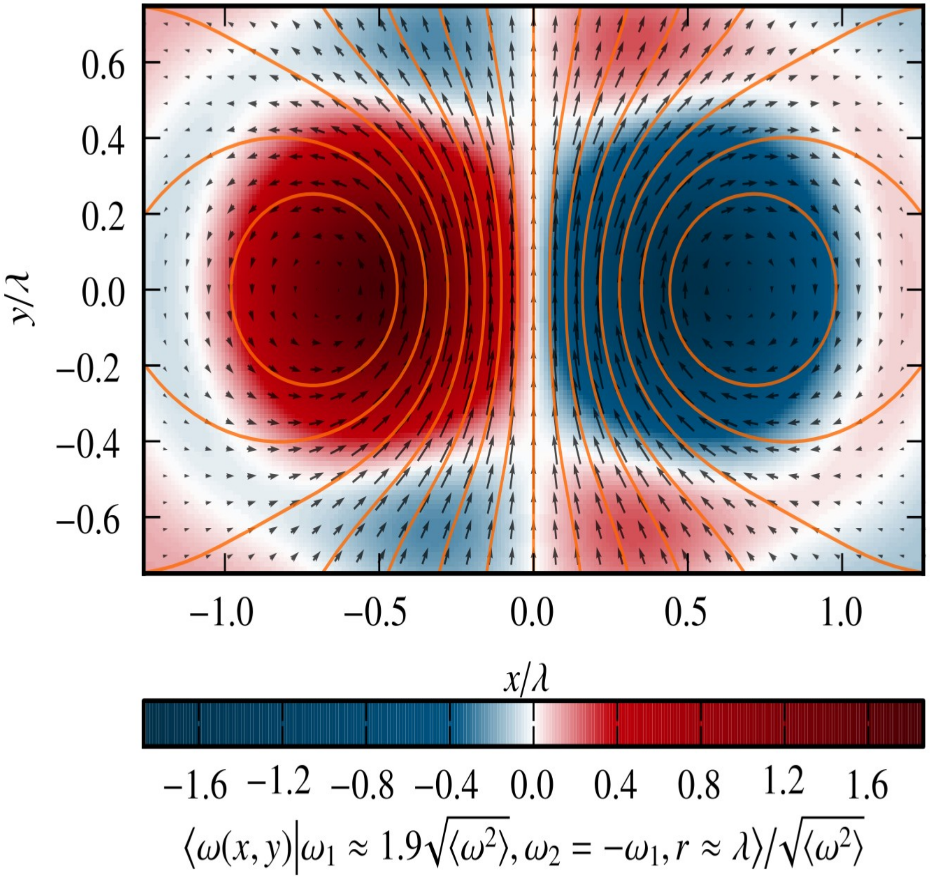




- MEASUREMENT APPROXIMATION

CONDITIONAL GAUSSIAN APPROXIMATION





- MEASUREMENT APPROXIMATION

FAILURE OF THE GAUSSIAN APPROXIMATION

$$\dot{\mathbf{x}}_1 = \mathbf{e}_z \times \frac{\mathbf{x}_1 - \mathbf{x}_2}{h(|\mathbf{x}_1 - \mathbf{x}_2|)}\omega_2$$

$$\dot{\mathbf{x}}_2 = \mathbf{e}_z \times \frac{\mathbf{x}_2 - \mathbf{x}_1}{h(|\mathbf{x}_2 - \mathbf{x}_1|)}\omega_1$$

$$\dot{\omega}_1 = -\gamma\,\omega_1 + \eta_1(t)$$

$$\dot{\omega}_2 = -\gamma\,\omega_1 + \eta_2(t)$$

$$\boxed{\frac{\partial}{\partial t}|\mathbf{x}_1 - \mathbf{x}_2| = 0}$$

- **NO CASCADE, NO ENERGY FLUX**
- **DEVIATIONS FROM GAUSSIANITY!**
- **INTERACTION OF QUASI-PARTICLES**

$$S_r^3(r) = 0$$

EXTENSION OF THE GAUSSIAN APPROXIMATION

$$\dot{\mathbf{X}}_1 = \mathbf{e}_z \times \frac{\mathbf{X}_1 - \mathbf{X}_2}{h(|\mathbf{X}_1 - \mathbf{X}_2|)}\omega_2 + \mathbf{e}_r U(|\mathbf{X}_1 - \mathbf{X}_2|, \omega_1, \omega_2)$$

$$\dot{\mathbf{X}}_2 = \mathbf{e}_z \times \frac{\mathbf{X}_2 - \mathbf{X}_1}{h(|\mathbf{X}_2 - \mathbf{X}_1|)}\omega_1 + \mathbf{e}_r U(|\mathbf{X}_2 - \mathbf{X}_1|, \omega_2, \omega_1)$$

- ATTRACTIVE, REPULSIVE INTERACTION
- BREAKING OF PARITY-TIME-INVARIANCE

$$U(r, \omega_1, \omega_2) = U(r, -\omega_1, -\omega_2)$$

LANDAU'S QUASI-PARTICLES QUASI-VORTICES

- TWO-POINT PROBABILITY DISTRIBUTION
- KINETIC EQUATION (REDUCED EULERIAN STATISTICS)
- CHARACTERISTIC EQUATION (AVERAGED LAGRANGIAN STATISTICS)
- QUASI-VORTICES:

  SCREENED BIOT-SAVARTŚ LAW

  RELATIVE MOTION (LINKED TO CASCADE, KARMAN-HOWARTH RELATION)
- WHERE DOES THE GLUING FORCE COME FROM
-

STATISTICS OF RELATIVE MOTION FROM DNS

RELATIVE MOTION: DYNAMICS OF VORTICITY INCREMENTS

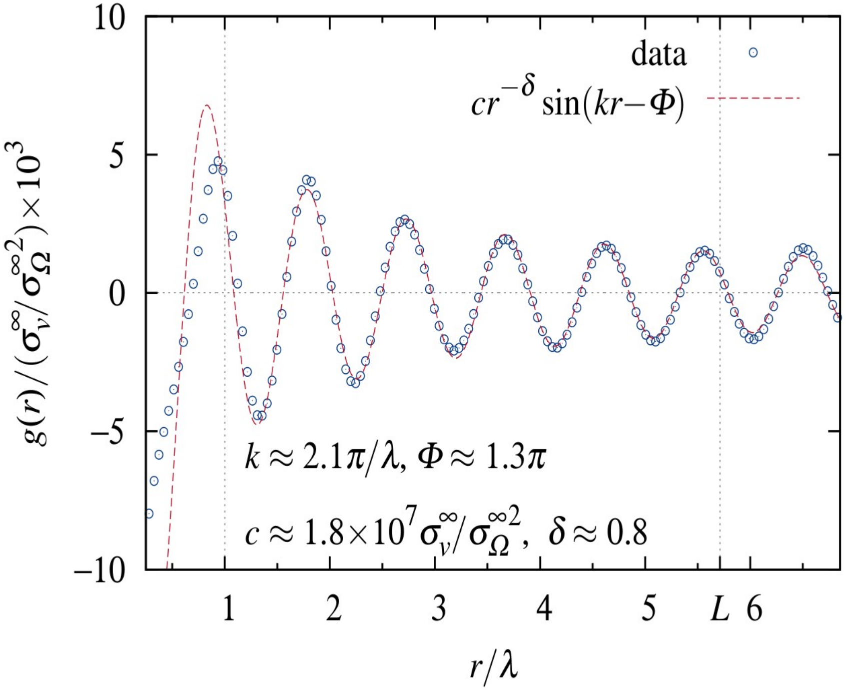
- **VORTICITY INCREMENT**

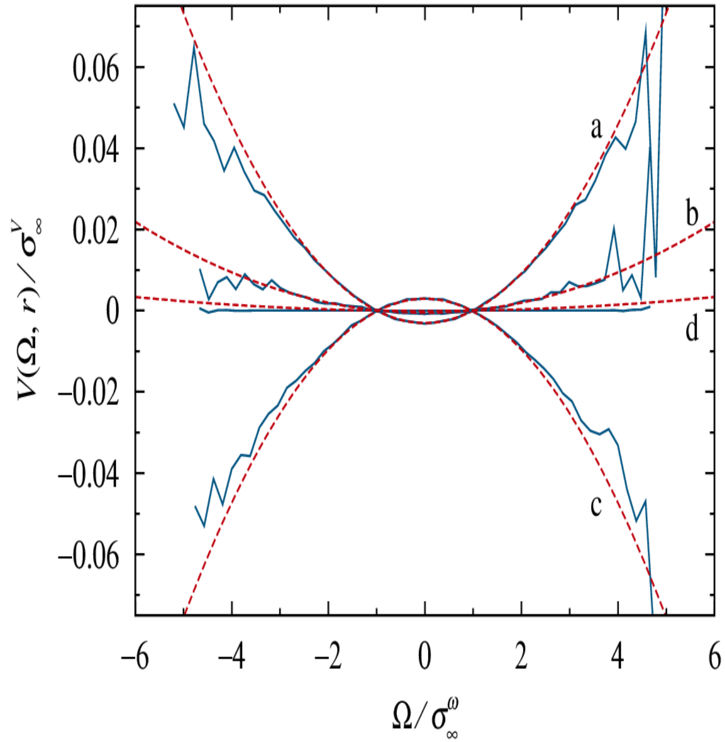
$$\Omega = \omega(\mathbf{x}+\mathbf{r},t) - \omega(\mathbf{x},t)$$

$$[\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\langle U(r)|\Omega\rangle] f(\Omega,r,t) = \frac{\partial}{\partial \Omega}\mu(r,\Omega) f(\Omega,r,t)$$

- **CONDITIONAL LONGITUDINAL VELOCITY INCREMENT** $\langle U(r)|\Omega\rangle$
- **FLUX IN THE CASCADE:**

WITH A LITTLE HELP FROM DIRECT NUMERICAL SIMULATION

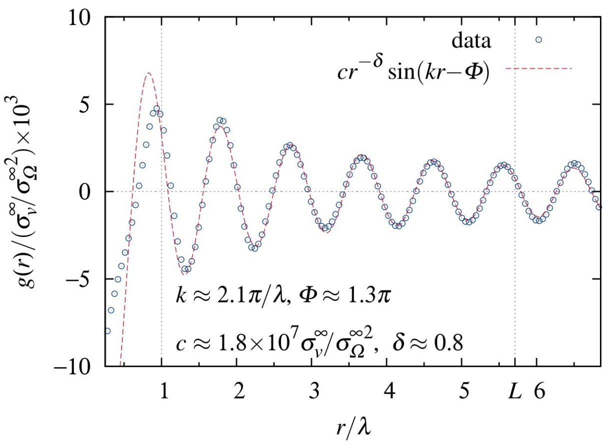


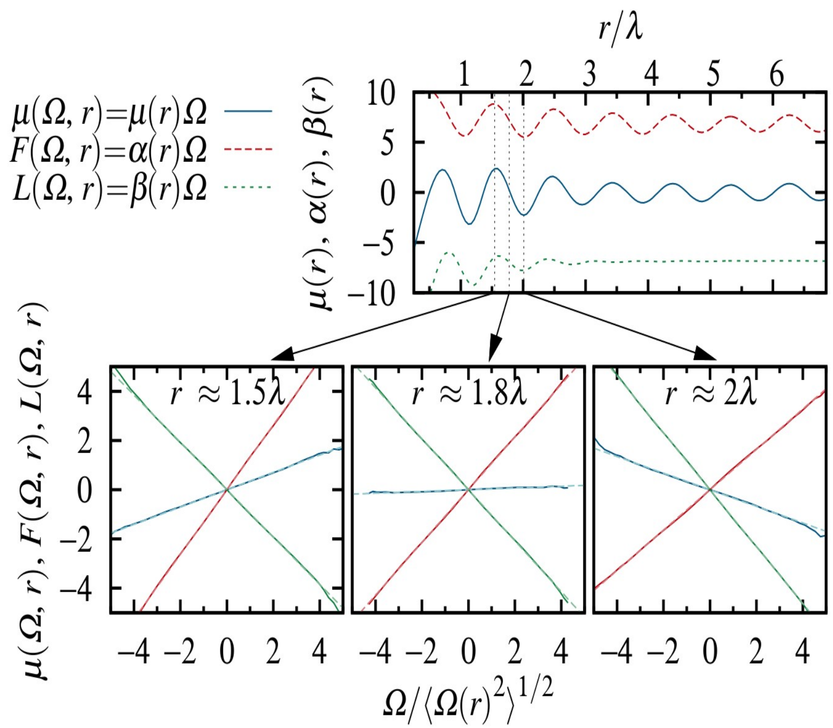


$$U(r,\Omega)=g(r)[\Omega^2-\langle\Omega(r)^2\rangle]$$

$$\Omega=\omega_1-\omega_2$$

DNS





$$\langle U(r)|\Omega\rangle = g(r)[\Omega^2 - \langle\Omega(r)^2\rangle]$$

$$\mu(r,\Omega) = \mu(r)\Omega$$

COMPUTERS ARE BORING: THEY ONLY GIVE YOU ANSWERS

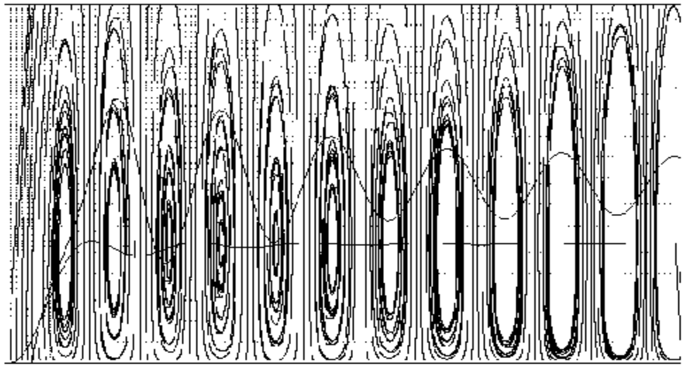
DYNAMICS OF VORTICITY INCREMENTS

- KINETIC EQUATION

$$\frac{\partial}{\partial t} f(\Omega, r, t) + \frac{\partial}{\partial r} g(r)[\Omega^2 - \langle \Omega(r,t)^2 \rangle] f(\Omega, r, t) = -\frac{\partial}{\partial \Omega} \mu(r) \Omega f(\Omega, r, t)$$

- CHARACTERISTIC EQUATION

$$\dot{r} = g(r)[\Omega^2 - \langle \Omega(r,t)^2 \rangle]$$
$$\dot{\Omega} = \mu(r)\Omega$$



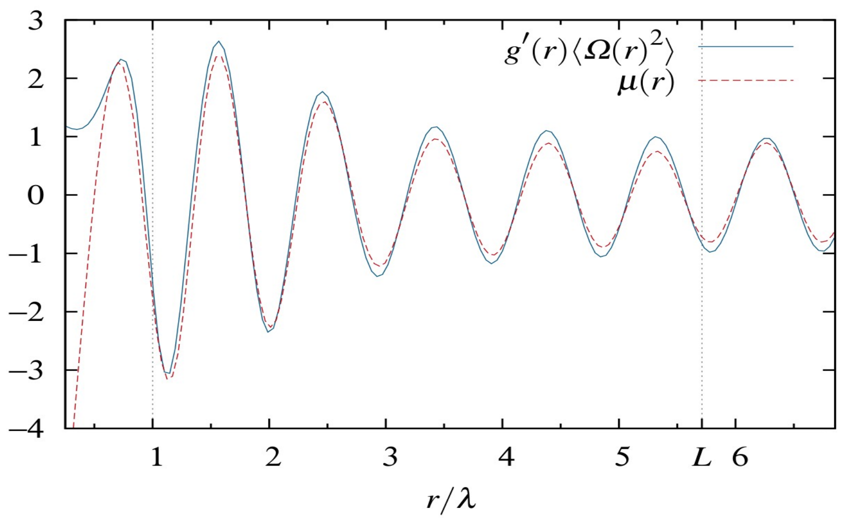
SOLUTION FOR THE VORTICITY PDF

$$g(r)(\Omega^2-\langle\Omega(r)^2\rangle)\frac{\partial}{\partial r}f(\Omega,r)+g'(r)(\Omega^2-\langle\Omega(r)^2\rangle)f(\Omega,r)$$
$$=\frac{\partial}{\partial\Omega}\mu(r)\Omega f(\Omega,r)$$

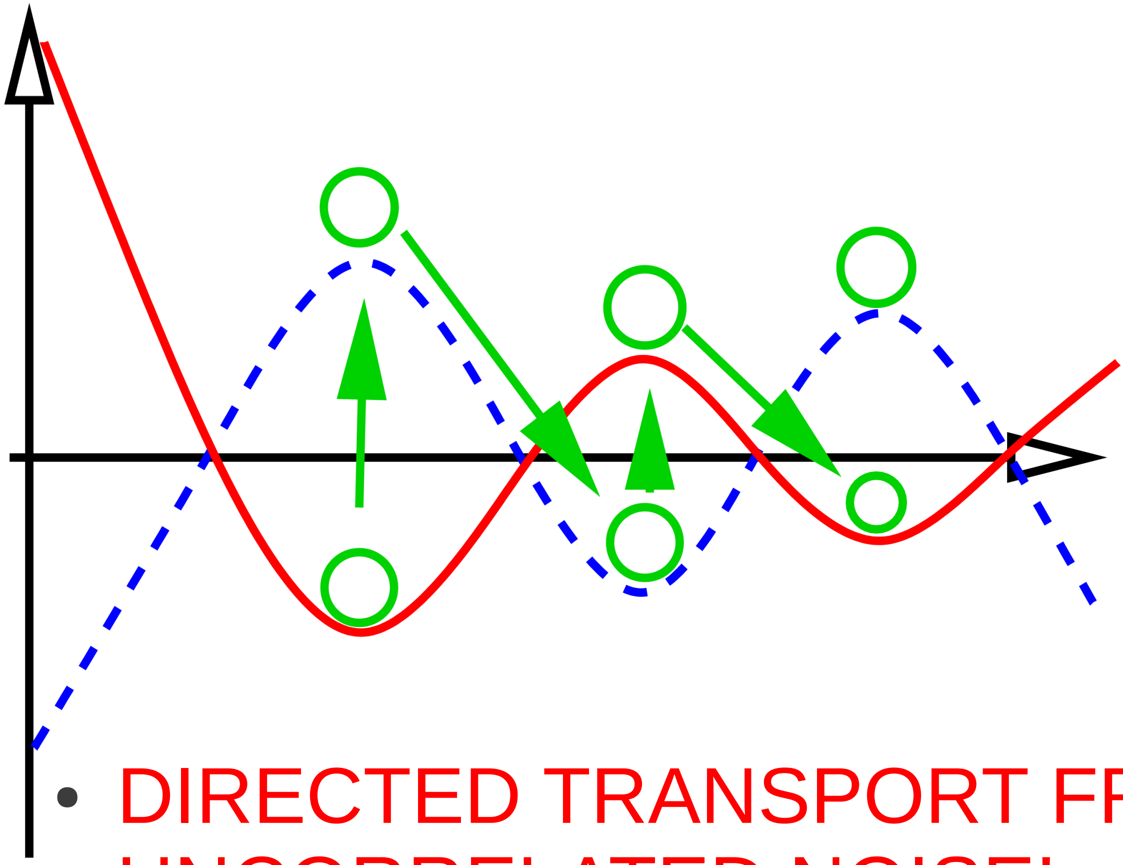
- Gaussian Solution at g'(r)=0 !

$$f(\Omega,r)=Z^{-1}e^{-\frac{g'(r)}{2\mu(r)}\Omega^2}$$

$$\langle\Omega^2(r)\rangle=\frac{\mu(r)}{g'(r)}$$



STOCHASTIC INTERPRETATION OF CHARACTERISTICS: RATCHET EFFECT



$$\dot{r} = g(r)\left[\Omega^2 - \langle \Omega^2(r) \rangle\right] + \eta$$
$$= -\frac{\partial}{\partial r} V(r, \Omega) + \eta$$

$$\dot{\Omega} = -\gamma \Omega + F$$

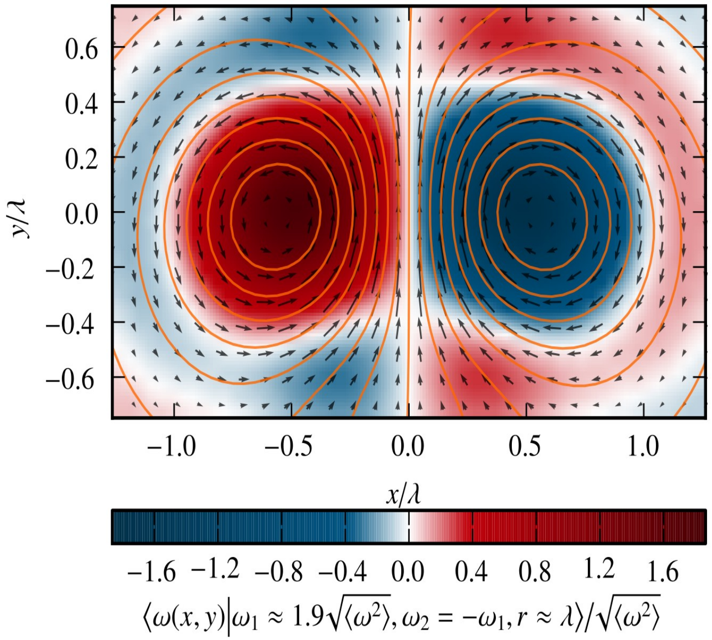
- DIRECTED TRANSPORT FROM UNCORRELATED NOISE!
- RATCHET!

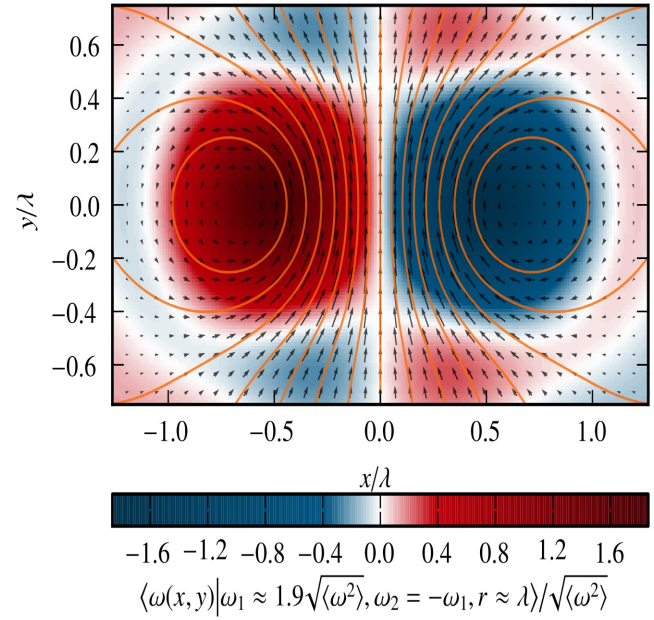
INVERSE CASCADE WITHIN A SIMPLE VORTEX MODEL

(TOGETHER WITH J. FRIEDRICH)

WHERE DOES THE RELATIVE VELOCITY FIELD U COMES FROM?

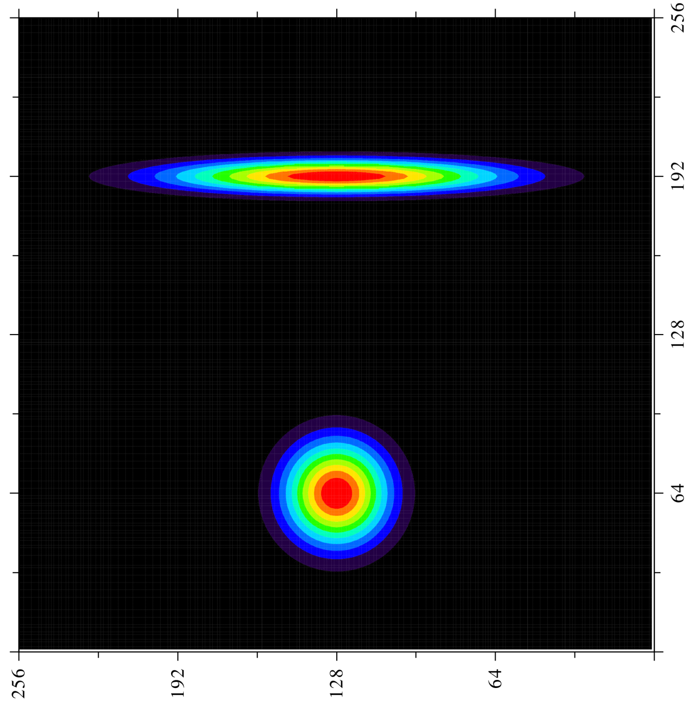
ORIGIN OF INVERSE CASCADE: BEYOND LINEAR GAUSSIAN APPROXIMATION



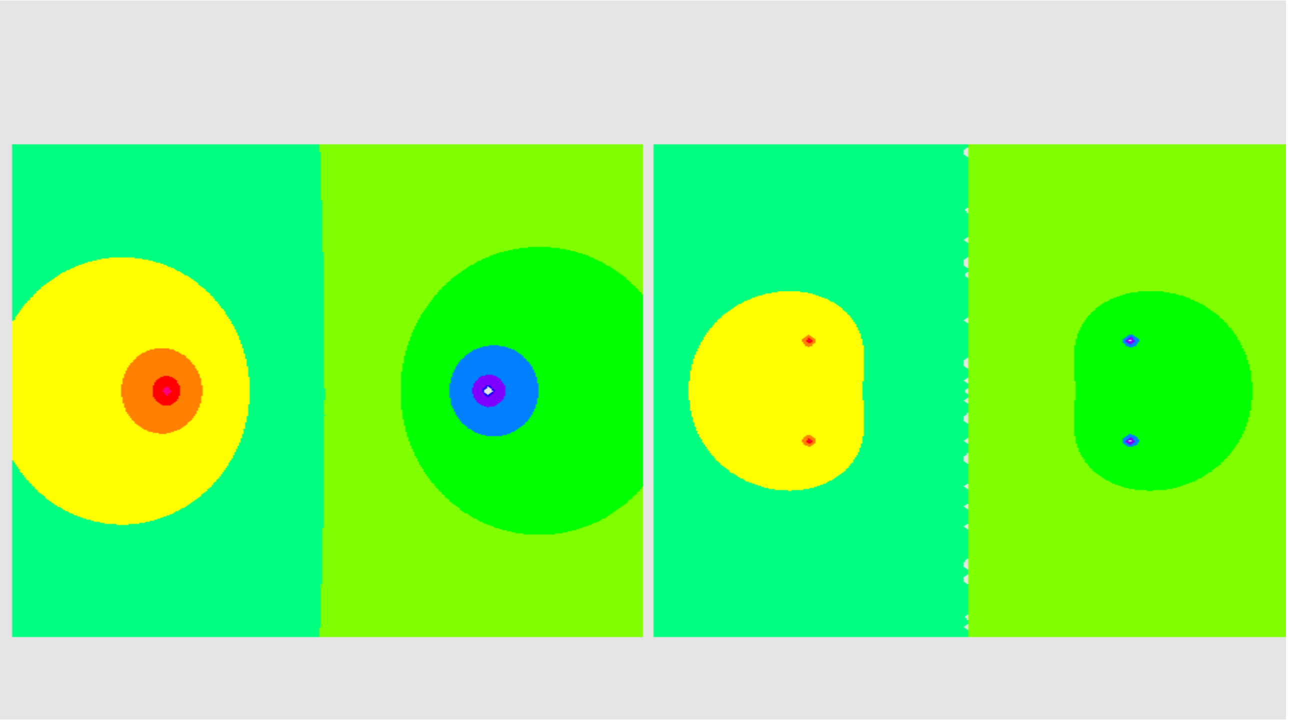


- EXTENSION TO DRESSED ELLIPTICAL VORTICES

VORTEX THINNING:
INNER DEGREE'S OF FREEDOM



GENERATING ELLIPTICAL VORTICES



ELLIPTICAL VORTEX: TWO INELASTICALLY COUPLED POINT VORTICES (EFFECT OF SHEAR)

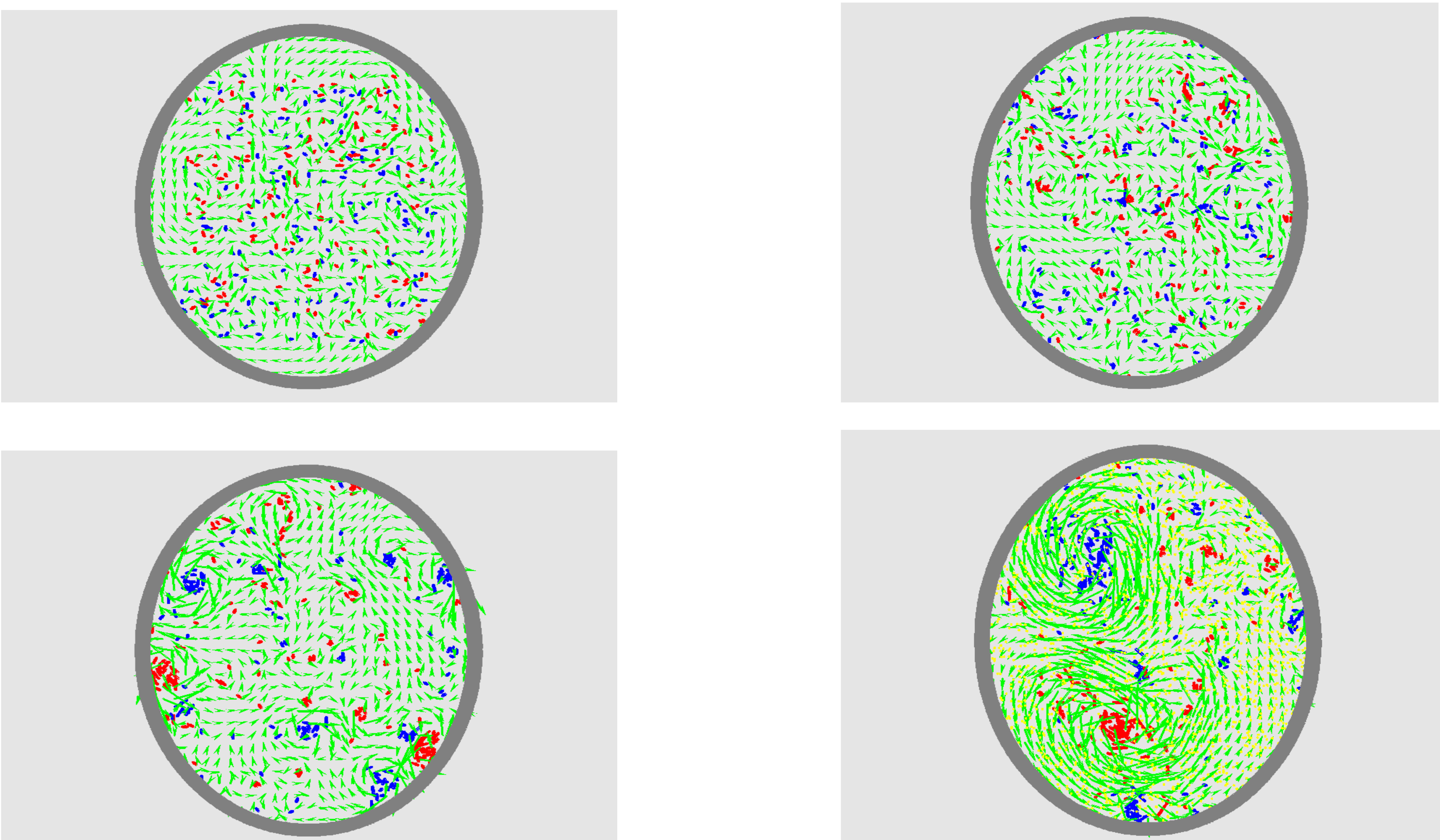
POINT VORTEX PAIR MODEL FOR INVERSE CASCADE

$$\dot{\boldsymbol{x}}_i = \sum_j \Gamma_j \boldsymbol{e}_z \times \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right|^2} + \gamma \left(D_0 - D_{i,i+1}\right) \frac{\boldsymbol{x}_i - \boldsymbol{x}_{i+1}}{\left|\boldsymbol{x}_i - \boldsymbol{x}_{i+1}\right|}$$

$$D_{ik} = \left|\boldsymbol{x}_i - \boldsymbol{x}_k\right|$$

- NONPERTURBATIVE PART: ENERGY INPUT AT SMALL SCALES D_0
- NONLINEAR DYNAMICS TREATMENT
- ACTIVE FLUIDS, ACTIVE PARTICLES

INVERSE CASCADE IN POINT VORTEX SYSTEMS



ONSAGER – HAMILTONIAN POINT VORTEX SYSTEM

INVERSE CASCADE IN POINT VORTEX SYSTEMS

CLUSTER FORMATION OF ELLIPTICAL VORTICES:

$$\frac{d}{dt} R(t) = \frac{c\, d^4}{\gamma} \frac{(\Gamma_1 + \Gamma_2)^2}{R^5(t)}$$

- ATTRACTION OF ROTORS BY A LARGE VORTEX AT THE ORIGIN
- CLUSTER PHYSICS!
- RESULTS OBTAINED BY AVERAGING TECHNIQUES (fast rotations, interesting nonlinear dynamics, J. Friedrich, R. F. ArXiv (2012))
-

TWO PAIRS OFELLIPTICAL VORTICES: GLUING FIELD OF THE CASCADE

$$\frac{d}{dt} R(t) = \frac{c\, d^4}{\gamma} \frac{(\Gamma_1 + \Gamma_2)^2}{R^5(t)}$$

- **ATTRACTION BETWEEN VORTICES WITH THE SAME CIRCULATION**
- **NO INTERACTION BETWEEN VORTICES WITH OPPOSITE CIRCULATION**
- **RESULT OBTAINED BY AVERAGING TECHNIQUES (fast rotations, interesting nonlinear dynamics)**

RECENT WORK ON INVERSE CASCADE BY POINT VORTEX DYNAMICS

$$\omega = \sum_j \Gamma_j N e^{-(\mathbf{x}-\mathbf{x}_j(t))C^{-1}(t)(\mathbf{x}-\mathbf{x}_j(t))}$$

$$\dot{\mathbf{x}}_i = \sum_j \Gamma_j \mathbf{u}(\mathbf{x}_i - \mathbf{x}_j) + \sum_j \Gamma_j \nabla_i (C_i + C_j) \nabla_i \mathbf{u}(\mathbf{x}_i - \mathbf{x}_j)$$

$$\dot{C}_i = C_i \nabla \mathbf{U}(\mathbf{x}_i) + \nabla \mathbf{U}(\mathbf{x}_i) C_i + \gamma (C_0 - C_i(t))$$

- **Approximate derivation of evolution equation for Position and Shape from Navier-Stokes**
- **Relationship to Lagrangian Coherent Structures**
- **Instanton Calculations (Solution of instanton equations by elliptical point vortex ansatz, Variational ansatz with elliptical vortices for MSR action)**

SUMMARY

TWO-POINT STATISTICS OF THE INVERSE CASCADE

- LANDAU QUASI-VORTICES AND THEIR INTERACTION
- ELLIPTICAL DEFORMATIONS OF THE VORTICES: ATTRACTION OF DRESSED VORTICES
- PAIR POINT VORTEX MODEL FOR INVERSE CASCADE
- SUBGRID MODEL FOR 2D TURBULENCE