Meeting "EULERIAN-LAGRANGIAN", WPI, Vienna, May 2012

Discussion Topic: Mixed Eulerian-Lagrangian approaches for the simulation of fundamental turbulence problems

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This document contains two independent parts:
(i) The first part consists of four pages with a survey on mixed Eulerian-Lagrangian approaches and a short discussion.
(ii) The second part consists of seven pages, and is the logic continuation of the blackboard notes (available in the links provided) about Eulerian-Lagrangian approaches that follow from a unifying variational principle. This seven-page document contains a detailed description of the so-called hybrid Eulerian-Lagrangian approach, originally developed by Soward and Andrews&McIntyre; it discusses its applications in turbulence and mentions briefly the related Euler-α approach introduced by Holm.

Definitions

1. Eulerian methods. These are all numerical methods whose underlying evolution equations are written for a set of fields (scalar, vectorial, tensorial, etc.) that depend on the time variable t and space variables $\mathbf{x} = (x_1, \cdots, x_N)$, where the space variables correspond physically to the spatial position on a time-independent coordinate system.

Examples: Any pseudo-spectral method, finite-difference method, finite-volume method, finite element method, spectral element method, in Cartesian or in mapped coordinates, but **with static spatial grid.** Also, adaptive-mesh methods where the mesh is static piecewise in time: only at some times the mesh is re-generated in order to resolve more certain regions of intense activity.

Advantages of Eulerian: Main advantage is simplicity of coding and logical structure of the spatial points.

Disadvantages of Eulerian: Main disadvantage is that the fixed grid does not take into account the anisotropy and inhomogeneity of the flow, and thus the method has difficulty to resolve sharp features of the flow such as pre-shocks. Moreover, in the case of complicated boundaries (fixed or moving), the mesh in the Eulerian approach is hard to parameterise in the numerical code and has to be treated in an ad-hoc manner.

2. Lagrangian methods. These are all numerical methods whose underlying evolution equations are written for a set of fields (scalar, vectorial, tensorial, etc.) that depend on the time variable t and so-called "label" variables $\mathbf{a} = (a_1, \cdots, a_N)$, where the latter variables correspond physically to the initial spatial positions of a set of "markers" whose positions $\mathbf{r}(t)$ are functions of time and the labels: $\mathbf{r}(t) = (r_1(\mathbf{a},t), \cdots, r_N(\mathbf{a},t))$. The markers' positions satisfy certain set of non-linearly coupled ordinary differential equations. The fields of the Eulerian picture are not independent: they become functions of time via their dependence on the markers.

Examples: Particle methods for fluid dynamics such as:

- Vortex method (VM) [Rosenhead L., The spread of vorticity in the wake behind a cylinder, Proc. R. Soc. A **127**, 590–12 (1930); Krasny R., A study of singularity formation in a vortex sheet by the point vortex approximation, J.Fluid Mech. **167**, 65–93 (1986)].
- Smoothed particle hydrodynamics (SPH) [L.B. Lucy. A numerical approach to the testing of the fission hypothesis. Astron. J., **82**, 1013–1024 (1977); R.A. Gingold and J.J. Monaghan. Smoothed Particle Hydrodynamics: Theory and application to non-spherical stars. Mon. Not. R. Astr. Soc., **181**, 375–389 (1977)], independently suggested by Salmon [R. Salmon. Practical use of Hamilton's principle. J. Fluid Mech., **132** 431–444 (1983)] in the context of geophysical fluid dynamics.
- See the review by P. Koumoutsakos [Multiscale flow simulations using particles, Annu. Rev. Fluid Mech. **37**, 457–87 (2005)].

Advantages of Lagrangian: It usually describes much more accurately natural interfaces such as the dynamically advected boundaries between two different fluids, and vortex sheets. Also, typically resolution is automatically increased in regions of high density. There is a bit more freedom regarding spatial boundary conditions, but there are complications due to the coarsening of the particles near the boundaries, which generate spurious small scales.

Disadvantages of Lagrangian: Problems near solid boundaries and severe particle distortion due to inhomogeneity, both leading to generation of spurious small scales. The problem of computing the Eulerian fields from the Lagrangian markers requires an interpolation algorithm that takes a significant proportion of computer time (unless it is a linear interpolation, which produces a significant amount of errors).

3. Mixed/hybrid Eulerian-Lagrangian methods. This category is more fuzzy. One can say that these are all numerical methods where some of the underlying fields belong to Eulerian category and some others belong to Lagrangian. In some of these methods, only the Lagrangian fields need to be integrated in time, in such a way that some conservation laws are “explicitly conserved” except for the error in the integration of the Lagrangian variables. In others, both Eulerian and Lagrangian fields are evolved. Yet in others, only Eulerian fields are evolved.
In this category we also find so-called hybrid methods, where in some parts of the fluid an Eulerian solver is used (typically near bounding walls) and this is coupled with a Lagrangian solver in other parts of the fluid.

Examples:

- Particle-in-cell algorithm [Harlow, F. H., The particle-in-cell computing method for fluid dynamics, Methods Comput. Phys. **3**, 319–343 (1964)].
- Particle-mesh method, described in the textbooks by Hockney & Eastwood [R.W. Hockney and J.W. Eastwood. Computer Simulations Using Particles. IOP Publisher (1988)] and Birdsall & Langdon [C.K. Birdsall and A.B. Langdon. Plasma Physics via Computer Simulations. McGraw-Hill, New York, (1981)].
- See also a Hamiltonian particle-mesh method in the spirit of the Euler alpha-model in [J. Frank, G. Gottwald, and S. Reich. The Hamiltonian particle-mesh method. In M. Griebel and M.A. Schweitzer, editors, Meshfree Methods for Partial Differential Equations, volume 26 of Lect. Notes Comput. Sci. Eng., pages 131–142, Berlin Heidelberg, 2002. Springer-Verlag.].
- Also, Kuznetsov's method of vortex lines [V. A. Zheligovsky, E. A. Kuznetsov and O. M. Podvigina, Numerical Modeling of Collapse in Ideal Incompressible Hydrodynamics, JETP Letters **74,** 367–370 (2001)] for 3D Euler equations, where the Eulerian vorticity

is obtained in terms of the markers.

Also in this category one finds methods which are purely Eulerian in form but which contain information on Lagrangian variables, such as the evolution of Clebsch-Weber variables [K. Ohkitani and P. Constantin. Numerical study of the Eulerian–Lagrangian formulation of the Navier–Stokes equations. Physics of Fluids **15**, 3251–3254 (2003); C. Cartes, M. D. Bustamante, and M. Brachet. Generalized Eulerian-Lagrangian description of Navier-Stokes dynamics. Physics of Fluids **19**, 077101 (2007)].

Also, adaptive-mesh methods where the mesh not only changes its resolution at discrete times, but also moves with the fluid or in another prescribed manner. For example, Grauer et al. [R. Grauer, C. Marliani, and K. Germaschewski, Adaptive mesh refinement for singular solutions of the incompressible Euler equations, Phys. Rev. Lett. **80**, 4177 (1998)] and Ceniceros&Hou [H. D. Ceniceros and T. Y. Hou, An Efficient Dynamically Adaptive Mesh for Potentially Singular Solutions, J. Comput. Phys. **172**, 609-639 (2000)]

Finally, hybrid particle level set methods where one is interested in following a sharp dynamically moving boundary such as the boundary between two fluids or a vortex patch, treating the boundary as an Eulerian level set but using Lagrangian particles to monitor the level set accuracy and even to determine the level set. See [D. Enright , R. Fedkiw , J. Ferziger , I. Mitchell, A Hybrid Particle Level Set Method for Improved Interface Capturing, J. Comput. Phys **183,** 83–116 (2002)].

Advantages of mixed/hybrid: Typically it is possible to get the best of both Eulerian and Lagrangian worlds using mixed or hybrid approaches.

Hybrid and mixed approaches in fundamental turbulence

Thinking of fundamental turbulence, what is the main argument in favour of hybrid or mixed approaches? The answer to this question depends on what is considered as fundamental turbulence. As is well known, the definition of turbulence is not universal [A. Tsinober, An Informal Conceptual Introduction to Turbulence, Springer (2009)].

If we are looking at only Eulerian statistics in simple geometries then obviously the best choice is a purely Eulerian approach. However if one is interested in Lagrangian statistics such as Lagrangian velocity structure functions or alignment of vorticity with Lagrangian velocity gradients, then it is more natural to try from the start a Lagrangian or a mixed approach, since then there is more control on the accuracy of the conservation of Lagrangian quantities such as mass, vortex lines, magnetic field lines, etc. Conservation of these quantities would be a critical requirement in the case of Lagrangian statistics. Interestingly, the majority of investigations up to date of Lagrangian statistics make use of Eulerian methods and the Lagrangian properties are computed in post-processing [F. Toschi and E. Bodenschatz, Lagrangian Properties of Particles in Turbulence, Annu. Rev. Fluid Mech. **41**, 375 - 404 (2009)]. The reason for this choice is that the current state-of-the-art memory capacities and computer speed of Lagrangian numerical methods, are below the level required for a statistical convergence of the various Lagrangian quantities of interest.

Even for Eulerian statistics alone, sometimes one is looking at conditional statistical quantities whose main contributions come from localised spatial regions where intense velocity gradients appear. There, adaptive-mesh schemes or hybrid Eulerian-Lagrangian schemes could be essential in order to maintain the required high level of accuracy.

Correspondingly, if one is interested in explaining experimental observations, then depending on the nature of the experiment one will prefer Eulerian to Lagrangian methods. For example, if the measurements are at fixed positions, then the natural method is Eulerian and if the measurements use particle tracking or similar methods, then the natural method is a Lagrangian or a mixed approach. Here, the fact that real particles are not point particles leads to increased complexity in the modelling of the particle advection. This feeds back on the numerical modelling and numerical methods, both in Eulerian and Lagrangian approaches.

Open Discussion: Paradigms

Let us consider the general research area of statistical properties of turbulence, without going into any details regarding the flow properties or even the type of physical fluids.

Rather than concentrating on “fundamental” turbulence questions within this area, it is perhaps useful to decide or discuss a set of paradigms regarding what is necessary to achieve in order to improve significantly the state-of-the-art in numerical turbulence research. We propose the following paradigms:

P1. One should aim at finding the numerical method that provides the most accurate numerical solution to the underlying fluid equations, given an “arbitrary” initial condition. Thus, if accuracy is optimised for a single arbitrary realisation of the dynamical fields, then this accuracy will transfer to the statistical description.

P2. One should consider a set of validation tests of numerical methods in the light of paradigm P1. Ideally, these validation tests should be defined a priori, or in any case, independently of the knowledge of Eulerian or Lagrangian methods. This is in order to avoid any potential biases, conscious or unconscious, in favour of certain types of numerical approaches.

Notice that the first paradigm is broken by any ad-hoc stochastic model of turbulence. But the current evidence is that turbulence statistics does not follow any ad-hoc rule.

Open Discussion: Some Questions

In the light of the above paradigms P1 and P2, we plan to consider a couple of questions whose answer might require a mixed Eulerian/Lagrangian numerical method, or better a comparison of this mixed method with purely Eulerian methods and purely Lagrangian methods.

Q1. How to capture consistently reconnection events in fluids such as 3D Navier-Stokes or 2D / 3D resistive MHD?

Q2. How to accurately describe the evolution of intense vorticity events in 3D Euler or 3D Navier-Stokes?

Notice that these two questions do not involve directly a statistical computation, so their answer can be obtained by comparing single realisations of solutions of the underlying partial differential equations.

Some Eulerian-Lagragian models or approaches in fluid mechanics

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Lecture Notes for the Eulerian-Lagrangian Meeting at WPI Vienna, 07-10 May 2012

This is a continuation of the lecture on Tuesday 08 May 2012 (blackboard pictures should be available).

Remark. The basic formulation of action principles for the incompressible fluid equations (density is taken equal to one for simplicity), presented on Tuesday 08 May 2012, justifies the use of geometrical structures for describing the evolution of the fields:

$$\frac{\partial}{\partial t}\mathbf{U} + \mathcal{L}_{\mathbf{u}}\mathbf{U} = -\nabla\Pi\,, \tag{1}$$

where $\mathbf{U}$ is to be understood as a momentum field (technically, a 1-form field) and $\mathbf{u}$ as a velocity field, and the "Lie derivative of $\mathbf{U}$ along $\mathbf{u}$" is defined by:

$$(\mathcal{L}_{\mathbf{u}}\mathbf{U})_i \equiv \mathbf{u}\cdot\nabla U_i + U_j\nabla_i u^j$$

(Einstein convention of sum over repeated indices), where $\nabla_i \equiv \frac{\partial}{\partial x^i}$, and $\mathbf{a}\cdot\mathbf{B} \equiv a^j B_j$.

CONTENTS

I. 3D EULER FLUID EQUATIONS

We consider the familiar case of the 3D Euler fluid equations for an incompressible fluid in an Euclidean base space. There, the relation between $\mathbf{U}$ and $\mathbf{u}$ is, in Cartesian coordinates, simply $U_i = \delta_{ij}u^j$ so there is a perfect component-by-component identification between momentum and velocity, only in Cartesian coordinates. The incompressibility condition is, in Cartesian coordinates, $\nabla\cdot\mathbf{u} = 0$.

The formulation (1) has several advantages. First, notice that under an arbitrary invertible coordinate transformation $x^i \to \widetilde{x}^i(x^1, x^2, x^3)$ we have the result

$$\frac{\partial}{\partial t}\widetilde{\mathbf{U}} + \mathcal{L}_{\widetilde{\mathbf{u}}}\widetilde{\mathbf{U}} = -\widetilde{\nabla}\widetilde{\Pi}\,, \tag{2}$$

where now tilded variables are defined as follows: $\widetilde{u}^j = \frac{\partial \widetilde{x}^j}{\partial x^k}u^k$, $\widetilde{U}_j = \frac{\partial x^k}{\partial \widetilde{x}^j}U_k$, and of course

$$\left(\mathcal{L}_{\widetilde{\mathbf{u}}}\widetilde{\mathbf{U}}\right)_i \equiv \widetilde{\mathbf{u}}\cdot\widetilde{\nabla}\widetilde{U}_i + \widetilde{U}_j\widetilde{\nabla}_i\widetilde{u}^j,$$

where $\widetilde{\nabla}_j = \frac{\partial x^k}{\partial \widetilde{x}^j}\nabla_k$. So the evolution equation has the same form in any coordinate system.

On the other hand, the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ transforms, in the tilded coordinate system, to the equation

$$\widetilde{\nabla} \cdot \left(\det \left[\frac{\partial x^k}{\partial \widetilde{x}^j} \right] \widetilde{\mathbf{u}} \right) = 0,$$

which is more complicated than the Cartesian version.

The second advantage is that the Kelvin circulation theorem (conservation of circulation) is best written in terms of the momentum $\mathbf{U}$, again in a way that does not depend on the coordinate system. For 3D Euler we have the conservation of the line integral:

$$\frac{d}{dt} \oint_{\mathcal{C}(t)} \mathbf{U} \cdot \mathbf{dx} = 0\,,$$

where $\mathcal{C}(t)$ is any closed loop that is advected by the flow, as a "material" loop. Correspondingly, in the new coordinate system $\widetilde{x}^j$, the theorem is

$$\frac{d}{dt} \oint_{\widetilde{\mathcal{C}}(t)} \widetilde{\mathbf{U}} \cdot \mathbf{d}\widetilde{\mathbf{x}} = 0\,.$$

The third advantage is that this formulation can be extended to the MHD equations in a similar fashion.

II. THE HYBRID EULERIAN-LAGRANGIAN (HEL) APPROACH

Remark: change of notation. From here on, the notation for the Cartesian coordinates will be $\widetilde{x}^i, \quad i = 1, 2, 3,$ and the notation for curved coordinates will be $x^i\,, \quad i = 1, 2, 3$. The reason for this change of notation is that, in this formalism, all equations end up being written in the curved coordinate system x^i rather than the original system.

A. Paradigms

This approach deals with flows that have an important "mean part" plus fluctuations. The first paradigm of the model is to understand the fluctuations of the fluid's velocity solely in terms of a "fluctuating" coordinate transformation. Recall that there is a direct relation between Lagrangian coordinates and Eulerian velocities, and this model exploits this relation. We define the variables $\widetilde{x}^i$ as the actual position coordinates of the particles, and assume that they are Cartesian coordinates and also 'fluctuating' coordinates. We denote by x^i their averages, so we write:

$$\widetilde{x}^i = x^i + \xi^i(x^1, x^2, x^3, t)\,, \quad i = 1, 2, 3, \tag{3}$$

where on average,

$$\left\langle \xi^i(x^1, x^2, x^3, t) \right\rangle = 0\,, \quad i = 1, 2, 3, \tag{4}$$

$$\left\langle x^i \right\rangle = x^i\,, \quad i = 1, 2, 3, \tag{5}$$

so x^i is non-fluctuating and $\langle \widetilde{x}^i \rangle = x^i$. This average can be understood in terms of ensembles, but formally the theory should work if we replace the ensemble average by any linear operator with the above properties. Notice that $\xi^i(x^1, x^2, x^3, t)$ is not necessarily small.

Coordinates $\widetilde{x}^i$ are termed "HEL" coordinates (hybrid Eulerian-Lagrangian), and any function of these coordinates is called an "HEL" variable [1] [2]. In this document we just call them

[1] D. G. Andrews and M. E. McIntyre, *JFM* **89**, 609–646 (1978)
[2] A. M. Soward and P. H. Roberts, *JFM* **661**, 45–72 (2010)

'fluctuating' coordinates. Our notation $\widetilde{x}^i$ differs from the conventional notation $(x^i)^L$, used in the cited references.

In this way, equation (3) defines a transformation from average coordinates x^i to fluctuating coordinates $\widetilde{x}^i$. The rest of this Section deals with the implications of this coordinate transformation on: (i) the transformation law from the velocity field defined on the Cartesian coordinate system $\widetilde{x}^i$ and the curvilinear coordinate system x^i, (ii) the transformation law of the dynamical equations from the Cartesian coordinate system to the curvilinear one, and (iii) the amazing way in which the ensemble-averaged equations get simplified in the curvilinear coordinate system.

There will be a slight extra complexity in the transformation of the velocity fields due to the explicit time-dependence of the coordinate transformation, but this is easy to deal with. It follows from the basic chain rule that due to the time-dependent transformation (3), the velocity field components in the coordinate system x^i are equal to

$$u^i(x^1, x^2, x^3, t) = \frac{\partial x^i}{\partial \widetilde{x}^j}\left(\widetilde{u}^j(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3, t) - \frac{\partial \xi^j}{\partial t}(x^1, x^2, x^3, t)\right), \quad i = 1, 2, 3, \tag{6}$$

where $\widetilde{u}^j(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3, t)$ is the actual velocity field of the particle in the Cartesian coordinates $\widetilde{x}^i$. The extra term $\frac{\partial \xi^j}{\partial t}$ comes from the explicit time dependence of the coordinate transformation.

The definition of the average coordinates x^i as non-fluctuating, implies by a consistency argument that the velocity field $\mathbf{u}(x^1, x^2, x^3, t)$ is non-fluctuating:

$$\big\langle \mathbf{u}(x^1, x^2, x^3, t)\big\rangle = \mathbf{u}(x^1, x^2, x^3, t). \tag{7}$$

A second paradigm of the model is to use Euler equations rather than Navier-Stokes, as an approximation for the dynamics of the fluid at the 'inertial' scales, i.e., where dissipation is not important. This is a simplicity paradigm, not an essential paradigm in the model.

A third paradigm of the model is to assume that the "fluctuating" coordinate system $\widetilde{x}^i$ is Cartesian, i.e., our familiar coordinate system. Therefore, in the fluctuating coordinates the form of the Euler equations is just the usual familiar formulation because $\widetilde{\mathbf{U}}$ can be identified with $\widetilde{\mathbf{u}}$, component by component. So the evolution equations for the velocity field in Cartesian, fluctuating coordinates, are assumed to be the usual 3D Euler equations:

$$\frac{\partial}{\partial t}\widetilde{\mathbf{U}} + \mathcal{L}_{\widetilde{\mathbf{u}}}\widetilde{\mathbf{U}} = -\widetilde{\nabla}\widetilde{\Pi}\,, \tag{8}$$

where the incompressibility condition is given by the usual equation $\widetilde{\nabla} \cdot \widetilde{\mathbf{u}} = 0$.

B. Result 1: Euler equations in the curvilinear coordinate system, and their averages

It is possible to show, after some straightforward calculations, starting from equations (8), that the evolution equations in the x^i coordinate system (which is curved but non-fluctuating) are:

$$\frac{\partial}{\partial t}\mathbf{U} + \mathcal{L}_{\mathbf{u}}\mathbf{U} = -\nabla\Pi\,, \tag{9}$$

$$U_j = \frac{\partial \widetilde{x}^k}{\partial x^j}\widetilde{U}_k, \tag{10}$$

and the relation between $\mathbf{u}$ and $\widetilde{\mathbf{u}}$ is given in equation (6).

The *raison d'être* of the HEL approach is that the ensemble average of equation (9) does not involve quadratic quantities: from the fact that the velocity field $\mathbf{u}(x^1, x^2, x^3, t)$ is non-fluctuating, it follows that the average of $\mathcal{L}_{\mathbf{u}}\mathbf{U}$ is simply

$$\langle \mathcal{L}_{\mathbf{u}}\mathbf{U}\rangle = \mathcal{L}_{\mathbf{u}}\langle \mathbf{U}\rangle.$$

We remark that this equation is not an approximation. The average of equation (9) gives the exact result

$$\frac{\partial}{\partial t}\langle \mathbf{U}\rangle + \mathcal{L}_{\mathbf{u}}\langle \mathbf{U}\rangle = -\nabla\langle \Pi\rangle\,, \tag{11}$$

which is formally the same type of equation as Euler equations but now the relationship between $\langle \mathbf{U}\rangle$ and $\mathbf{u}$ is more involved due to the combined effects of: (i) the fact that x^i is a curvilinear coordinate system, and (ii) the use of ensemble averages to obtain $\langle \mathbf{U}\rangle$. In Subsection II C we will look at this relationship in detail. Historically, equations (11) are called "Generalized Lagrangian mean equations" or GLM equations since Andrews & McIntyre (1978, *op.cit.*).

We remark that Kelvin's circulation theorem applies to the averaged equations, so we have

$$\frac{d}{dt}\oint_{\mathcal{C}(t)}\langle \mathbf{U}\rangle\cdot\mathbf{dx} = 0\,,$$

where $\mathcal{C}(t)$ is a closed curve that moves with the non-fluctuating fluid velocity $\mathbf{u}$.

Finally, the density of the fluid in the curved, non-fluctuating coordinates, is not trivial. We have by definition:

$$\rho(x^1,x^2,x^3,t) \equiv \det\left(\frac{\partial\widetilde{x}^i}{\partial x^j}\right)\,,$$

and this density satisfies the continuity equation, in curved coordinates:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

Averaging this equation gives the result:

$$\frac{\partial\langle\rho\rangle}{\partial t} + \nabla\cdot(\langle\rho\rangle\mathbf{u}) = 0. \tag{12}$$

C. Result 2: Explicit relation between the average momentum $\langle \mathbf{U}\rangle$ and the non-fluctuating fluid velocity $\mathbf{u}$

It would appear to be fundamentally unjustified to simply identify the non-fluctuating fluid velocity components $u^i(x^1,x^2,x^3,t)$ (defined in curvilinear, non-fluctuating coordinates x^i) with the average fluid velocity components $\langle\widetilde{u}^i(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3,t)\rangle$ (defined in Cartesian, fluctuating coordinates $\widetilde{x}^i$). The reason being that these velocity fields 'live' in different coordinate systems. Think for example of a coordinate transformation that took us from Cartesian to spherical coordinates. Would it make sense to relate, say, the radial component to any particular, fixed, Cartesian component? Obviously not.

However, it is quite illuminating to see that such identification can be justified —and, more importantly, mathematically validated— in the HEL approach, by the fact that the coordinate transformation from x^i to $\widetilde{x}^i$ is not too singular.[3] After all, by definition, the average of this transformation is just the identity transformation (but things are not trivial, as we will see below). So let us use the defining transformation (6) to compute explicitly the averages $\langle\widetilde{u}^i(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3,t)\rangle$. First, we solve equation (6) for $\widetilde{u}^i$ to get:

$$\widetilde{u}^i(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3,t) = \frac{\partial\xi^i}{\partial t}(x^1,x^2,x^3,t) + \frac{\partial\widetilde{x}^i}{\partial x^j}(x^1,x^2,x^3,t)\,u^j(x^1,x^2,x^3,t)\,,\quad i=1,2,3. \tag{13}$$

Second, we use equation (3) to find: $\frac{\partial\widetilde{x}^i}{\partial x^j} = \delta^i_j + \frac{\partial\xi^i}{\partial x^j}$. Third, we take average of equation (13) and use the facts that

$$\left\langle\frac{\partial\xi^i}{\partial t}(x^1,x^2,x^3,t)\right\rangle = 0$$

[3] *singular* in the sense that spherical coordinates are singular at the z-axis.

and

$$\left\langle \frac{\partial \xi^i}{\partial x^j}(x^1,x^2,x^3,t)\, u^j(x^1,x^2,x^3,t) \right\rangle = \left\langle \frac{\partial \xi^i}{\partial x^j}(x^1,x^2,x^3,t) \right\rangle u^j(x^1,x^2,x^3,t) = 0.$$

We deduce readily:

$$\langle \widetilde{u}^i(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3,t)\rangle = u^i(x^1,x^2,x^3,t)\,, \quad i=1,2,3,$$

component by component.

This result agrees with original heuristic motivation that the field $u^i(x^1,x^2,x^3,t)$ is the "mean" velocity fluid. However, this result is not useful for computing the dynamics: what we need is $\langle U_i \rangle$, which is what appears in the dynamical equation (11).

We have, from equation (10):

$$U_i(x^1,x^2,x^3,t) = \frac{\partial \widetilde{x}^j}{\partial x^i}\widetilde{U}_j = \left(\delta_i^j + \frac{\partial \xi^j}{\partial x^i}\right)\delta_{jk}\widetilde{u}^k\,,$$

and using equation (13) we get

$$U_i(x^1,x^2,x^3,t) = \frac{\partial \widetilde{x}^j}{\partial x^i}\widetilde{U}_j = \left(\delta_i^j + \frac{\partial \xi^j}{\partial x^i}\right)\delta_{jk}\left(\frac{\partial \xi^k}{\partial t} + \frac{\partial \widetilde{x}^k}{\partial x^m}\,u^m\right)\,,$$

so, expanding the terms and summing up the Kronecker deltas,

$$\begin{aligned} U_i(x^1,x^2,x^3,t) &= \delta_{ik}\frac{\partial \xi^k}{\partial t}(x^1,x^2,x^3,t) + \delta_{im}u^m(x^1,x^2,x^3,t) + \delta_{ik}\frac{\partial \xi^k}{\partial x^m}\,u^m(x^1,x^2,x^3,t) \\ &+ \frac{\partial \xi^j}{\partial x^i}\delta_{jk}\frac{\partial \xi^k}{\partial t}(x^1,x^2,x^3,t) + \frac{\partial \xi^j}{\partial x^i}\delta_{jm}u^m(x^1,x^2,x^3,t) + \frac{\partial \xi^j}{\partial x^i}\delta_{jk}\frac{\partial \xi^k}{\partial x^m}u^m(x^1,x^2,x^3,t). \end{aligned}$$

Finally we average this equation and observe that in the RHS, the first, third and fifth terms vanish after averaging. Therefore we get

$$\langle U_i(x^1,x^2,x^3,t)\rangle = \delta_{jk}\left\langle \frac{\partial \xi^j}{\partial x^i}\frac{\partial \xi^k}{\partial t}\right\rangle + \delta_{im}u^m(x^1,x^2,x^3,t) + \delta_{jk}\left\langle \frac{\partial \xi^j}{\partial x^i}\frac{\partial \xi^k}{\partial x^m}\right\rangle u^m(x^1,x^2,x^3,t).$$

Physically, the fluctuating coordinates have generated an extra drift in the momentum, due to the time dependence of the fluctuations, and also a deformation (with respect to the Euclidean metric) of the relation between velocity components and momentum components, due to the spatial dependence of the fluctuations.

We remark that the density of the fluid in curved, non-fluctuating coordinates is given explicitly in terms of the fluctuations $\xi^i(x^1,x^2,x^3,t)$:

$$\rho(x^1,x^2,x^3,t) = \det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = \det\left(\delta_j^i + \frac{\partial \xi^i}{\partial x^j}\right)\,,$$

so the average of the density can be computed explicitly in terms of the averages of powers of fluctuations. In general, the averages of up to third powers of fluctuations would appear. We remark that in the literature, different authors make different simplifying extra assumptions about this density. We will not discuss this.

III. SUMMARY OF THE HYBRID EULERIAN-LAGRANGIAN (HEL) APPROACH

In summary, starting from the usual 3D Euler fluid equations for the velocity field in a fluctuating Cartesian coordinate system, we arrive without approximations at the following average equations of motion in the curved, non-fluctuating coordinate system:

$$\frac{\partial}{\partial t}\langle \mathbf{U}\rangle + \mathcal{L}_{\mathbf{u}}\langle \mathbf{U}\rangle = -\nabla\langle \Pi\rangle \,, \tag{14}$$

where $\mathbf{u} \equiv \mathbf{u}(x^1, x^2, x^3, t)$ is the non-fluctuating mean fluid velocity, and

$$\big\langle U_i(x^1,x^2,x^3,t)\big\rangle = \delta_{jk}\left\langle \frac{\partial \xi^j}{\partial x^i}\frac{\partial \xi^k}{\partial t}\right\rangle + \delta_{im}u^m(x^1,x^2,x^3,t) + \delta_{jk}\left\langle \frac{\partial \xi^j}{\partial x^i}\frac{\partial \xi^k}{\partial x^m}\right\rangle u^m(x^1,x^2,x^3,t). \tag{15}$$

This system is complemented with the continuity equation for the average mass density:

$$\frac{\partial \langle\rho\rangle}{\partial t} + \nabla\cdot(\langle\rho\rangle\mathbf{u}) = 0, \tag{16}$$

where

$$\big\langle \rho(x^1,x^2,x^3,t)\big\rangle = \left\langle \det\left(\delta^i_j + \frac{\partial \xi^i}{\partial x^j}\right)\right\rangle .$$

IV. LOOKING FORWARD: CAN WE MODEL TURBULENCE WITH THE HEL APPROACH?

Is the HEL model with its fluctuating coordinates a good approach to the turbulent fluctuations, so-called E-turbulence? I do not know if this question has been discussed in detail. But I can only guess that McIntyre, Soward and Holm have thought about it.

The inclusion of dissipation in the model is possible but requires the introduction of extra technical concepts. We avoid this extra complexity and see what we can conclude about the dynamics at 'inertial-range' scales, where dissipation is not too important so we can keep using Euler rather than Navier-Stokes.

First we notice that the average quantities appearing in equations (14)–(16) depend on moments of the fluctuations $\xi^i(x^1, x^2, x^3, t)$, and these moments are unknown. The question is: can we measure these moments? I do not know if this has been discussed in the literature, but I think it could be a good idea to try and relate these moments to measurements, say, of structure functions, average particle lengths, average particle distances, etc.

A second remark is that if we think of the averaged coordinates x^i, as the coordinates of the averaged position of a particle, then the equation of motion for these "averaged" trajectories is

$$\frac{d}{dt}X^i(t) = u^i(X^1(t), X^2(t), X^3(t), t)\,, \quad i = 1,2,3,$$

where u^i is the mean velocity field, a non-fluctuating function of its arguments. As discussed by Arkady Tsinober, even for simple velocity fields $u^i(x^1, x^2, x^3, t)$ such as the ABC flow or the Lorenz system, the solutions of these equations of motion give rise to complicated trajectories, typically of chaotic nature. So the observation is that even though we have "eliminated the E-turbulence" by moving to the non-fluctuating coordinate system, the "purely Lagrangian" turbulence (or L-turbulence) has remained. I think this is a good feature of the model, and sheds some light on one of the questions in the discussion session on Thursday.

V. HOLM'S EULER-α AND NAVIER-STOKES-α MODELS

We depart a little bit from the historical construction of this model, and look at it from the point of view of the HEL construction. Consider equation (14) but with $\langle \mathbf{U} \rangle$ replaced by some other momentum field $\mathbf{V}$, to be defined later:

$$\frac{\partial}{\partial t}\mathbf{V} + \mathcal{L}_{\mathbf{u}}\mathbf{V} = -\nabla P\,, \tag{17}$$

where $\mathbf{u} \equiv \mathbf{u}(x^1, x^2, x^3, t)$ is the non-fluctuating mean fluid velocity which is assumed incompressible: $\nabla \cdot \mathbf{u} = 0$. Of course this incompressibility assumption is not in good agreement with the continuity equation (16), unless the mass density ρ is set to constant and uniform.

Notice that irrespective of the choice for the momentum field $\mathbf{V}$, a Kelvin's circulation theorem should hold:

$$\frac{d}{dt}\oint_{\mathcal{C}(t)} \mathbf{V} \cdot \mathbf{dx} = 0\,,$$

where $\mathcal{C}(t)$ is any closed loop that is advected by the velocity field $\mathbf{u}$.

The crucial aspect of the Euler-α model is the relation between $\mathbf{V}$ and $\mathbf{u}$:

$$V_i \equiv \delta_{ij}\left(u^i - \alpha \triangle u^i\right)\,,$$

where $\alpha > 0$ is a constant and $\triangle$ is the usual Laplacian in Cartesian coordinates.

So as a PDE system the Euler-α model is simply

$$\begin{aligned} \frac{\partial}{\partial t}\left(\mathbf{u} - \alpha\triangle\mathbf{u}\right) + \mathbf{u}\cdot\nabla\left(\mathbf{u} - \alpha\triangle\mathbf{u}\right) - \alpha(\triangle u_j)\nabla u^j &= -\nabla p\,,\\ \nabla\cdot\mathbf{u} &= 0\,. \end{aligned}$$

Obviously, in the limit $\alpha \to 0$ we recover the usual 3D Euler equations.

Correspondingly, the Navier-Stokes-α model is

$$\begin{aligned} \frac{\partial}{\partial t}\left(\mathbf{u} - \alpha\triangle\mathbf{u}\right) + \mathbf{u}\cdot\nabla\left(\mathbf{u} - \alpha\triangle\mathbf{u}\right) - \alpha(\triangle u_j)\nabla u^j &= -\nabla p + \nu\triangle\left(\mathbf{u} - \alpha\triangle\mathbf{u}\right)\,,\\ \nabla\cdot\mathbf{u} &= 0\,, \end{aligned}$$

where $\nu > 0$ is the viscosity.

It is perhaps worth to mention that, unlike the HEL approach, there is no derivation of the Euler-α model from first principles starting from 3D Euler. There exists in the literature an attempt of derivation of the model using a technique of averaging of the underlying Hamilton's principle, however Soward showed recently that this derivation had a flaw.

In any case, this does not prevent us from using the Euler-α or the Navier-Stokes-α models as turbulence closure models. This has been done in the literature. The interpretation that has been successful, is that the parameter α is not an intrinsic property of the fluid, but rather a flow-regime parameter, which should depend on the Reynolds number. The main advantage of these α-models is the fact that numerically the number of degrees of freedom needed to resolve the small scales is lower than in the usual Navier-Stokes case, because the energy spectrum typically decays faster (as k^{-3}) due to the α term in the equations. See [4] for more details.

Finally we mention that the Euler-α model has its own Hamilton's principle. In one spatial dimension, the evolution equations are equivalent to the so-called Camassa-Holm system, which is an integrable system with soliton-type solutions.

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[4] C. Foias, D. D. Holm and E. S. Titi, *Physica D* **152**, 505–519 (2001)