On Lagrangian Single-Particle Statistics

MAX-PLANCK-GESELLSCHAFT

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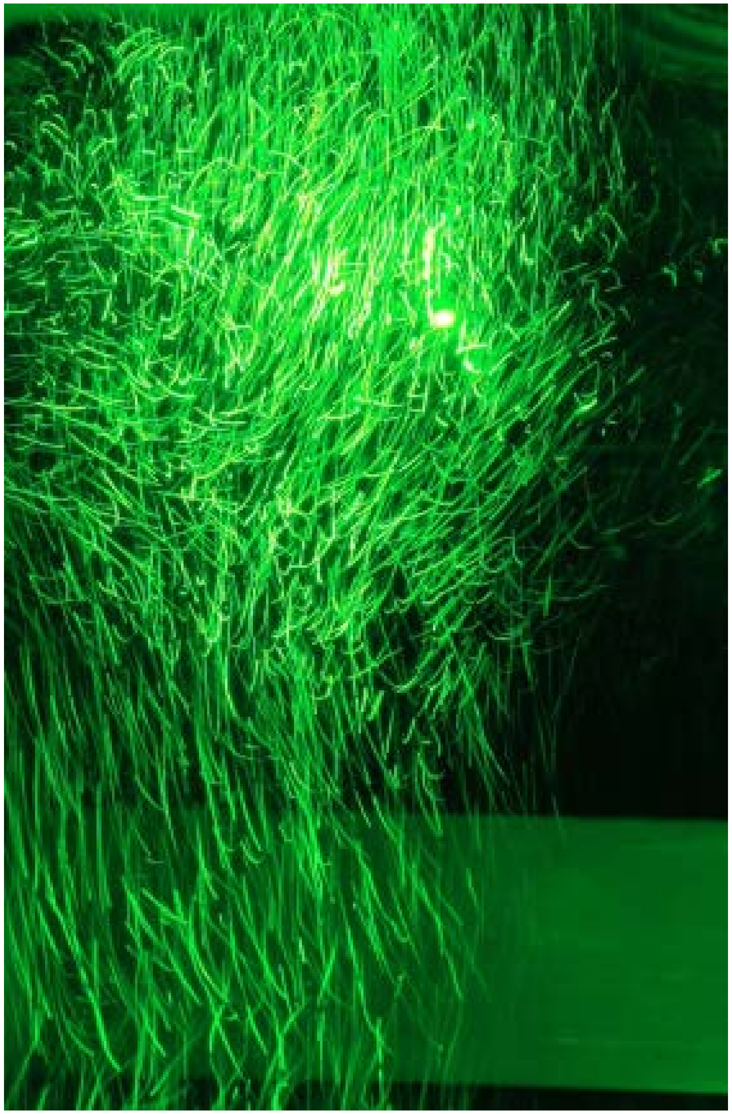
Particle Tracking

- Seed flow with tracer particles

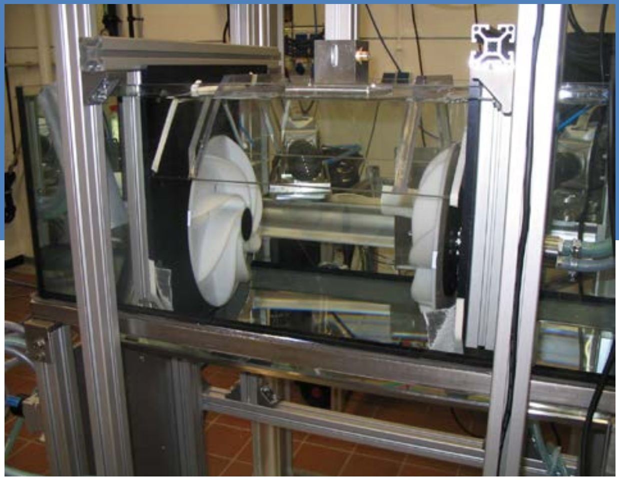
- Follow tracers in 3d in space and time

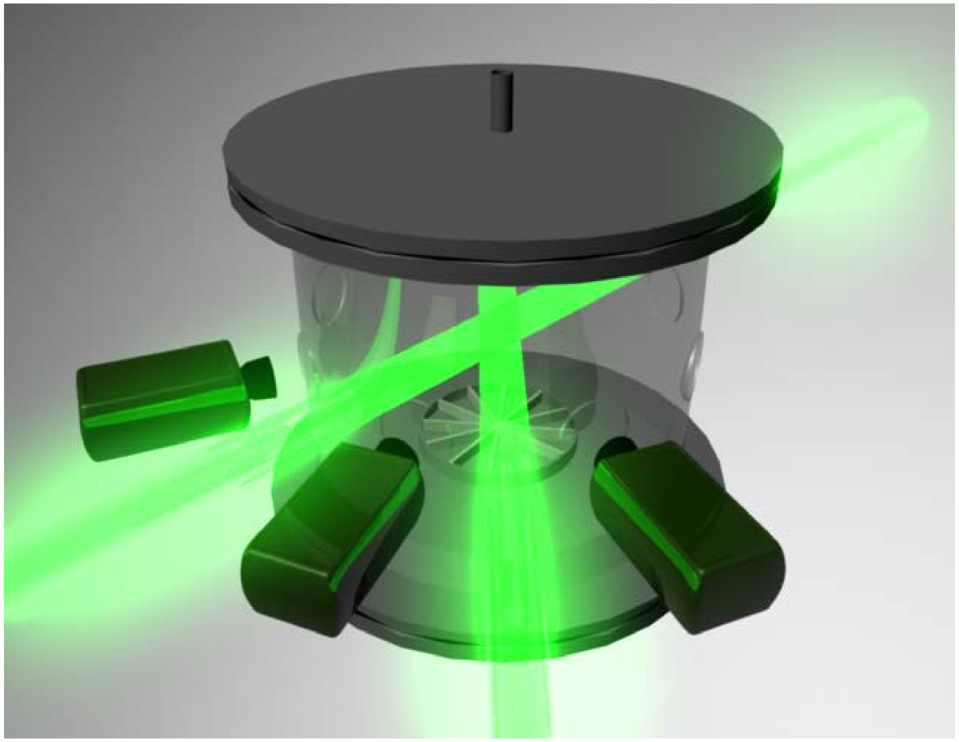
$\Rightarrow$ particle tracks

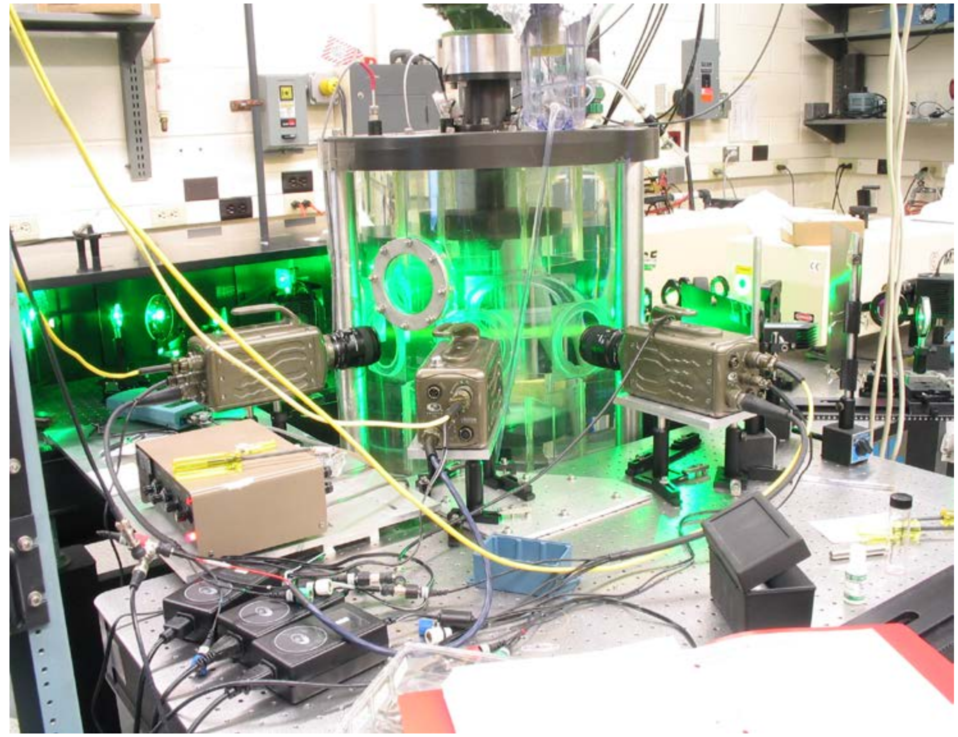


Why is it so difficult?

Apparatus	P (bar)	ν (m^2/s)	u' (m/s)	ϵ (m^2/s^3)	ℓ (m)	λ (μm)	η (μm)	τ_η (ms)	R_λ
SF_6 tunnel	15	1.5×10^{-7}	1.0	1.2	0.45	1400	7.3	0.36	9600
air tunnel	1	1.5×10^{-5}	1.2	3.9	0.4	9100	172	2.0	730
SF_6 tank	15	1.5×10^{-7}	1.0	5.5	0.094	648	5.0	0.17	4360
water tank	1	8×10^{-7}	2.2	59	0.094	1000	9.7	0.12	2800







typical in 3d:

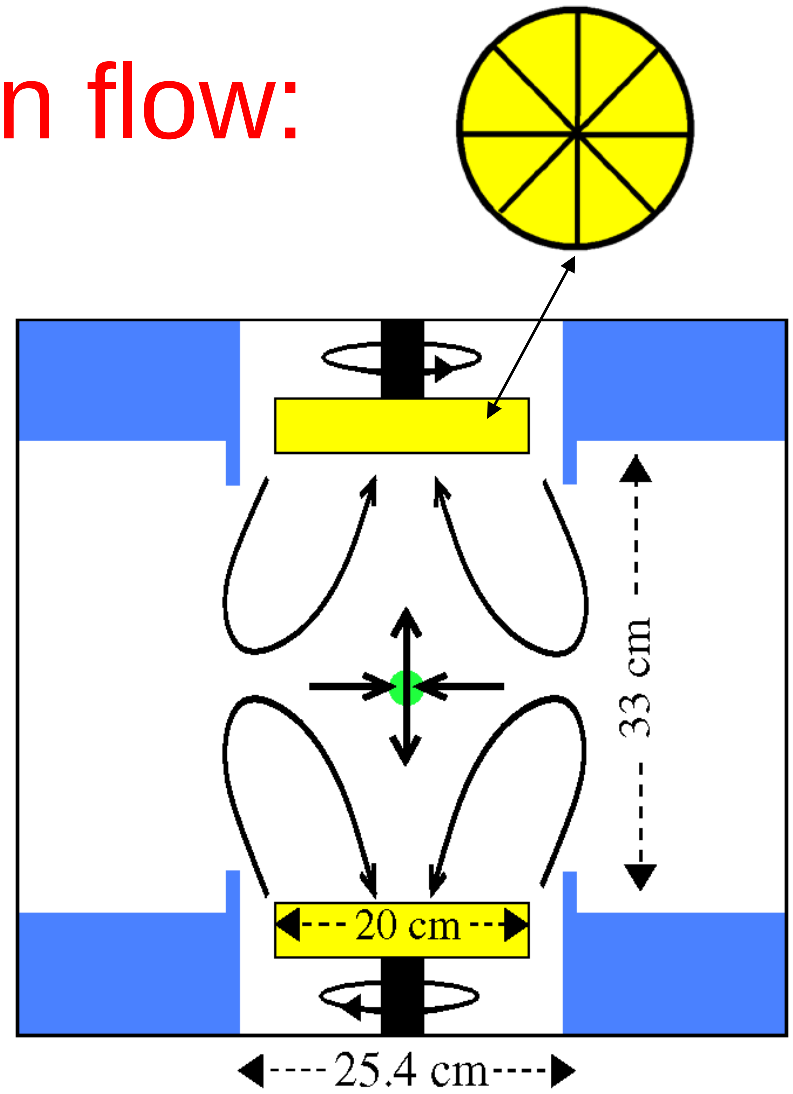
6000fps	20000 particles
70000 fps	300 particles

Karman flow:

- closed container
- no meanflow in middle
- driven by 1kW DC motors
- temperature controlled to 50mK
- water filtered to 0.3 microns

$Re_\lambda = 1000$

$(Re = 70.000)$



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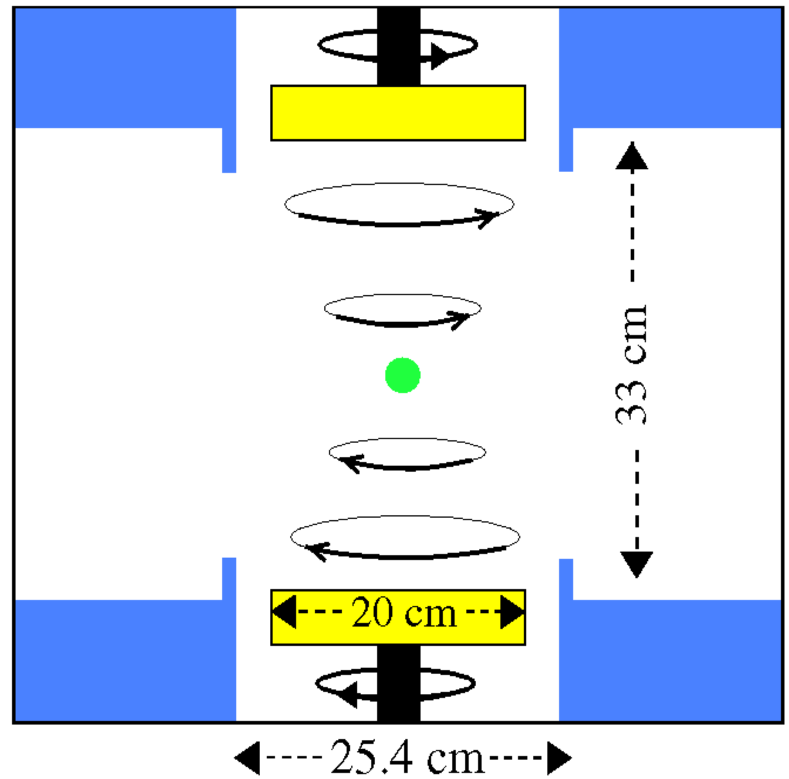
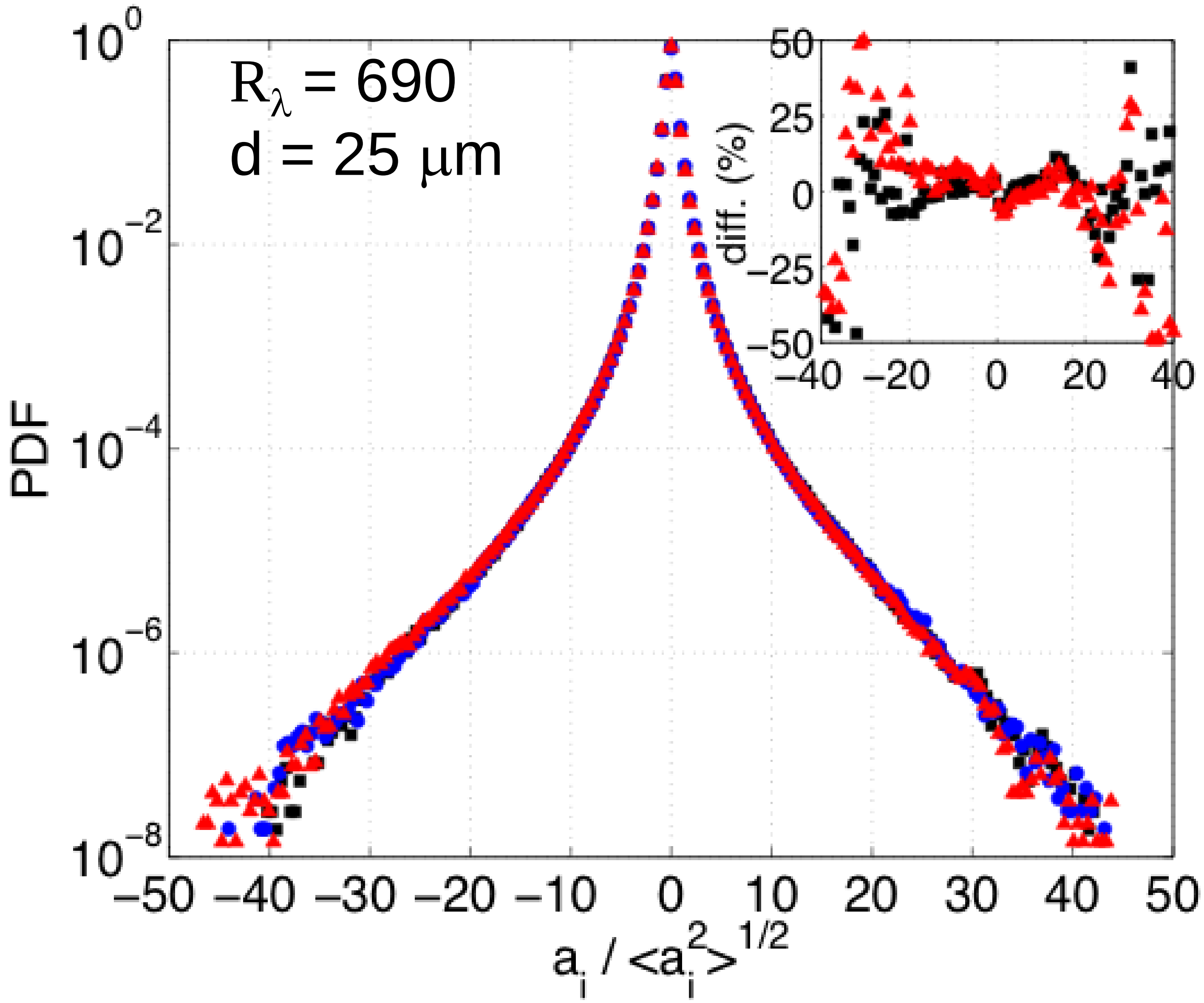
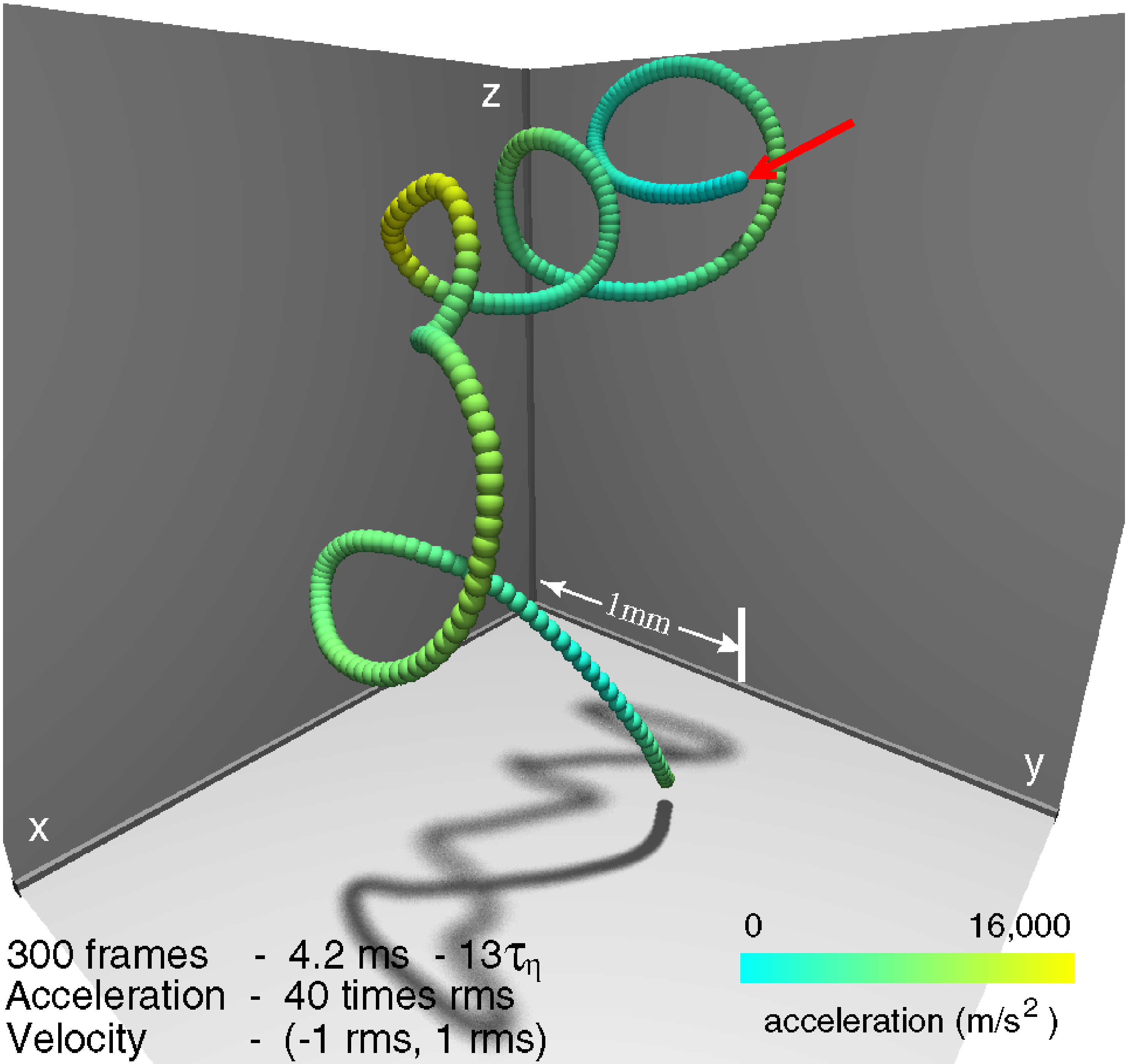


Acceleration is extremely intermittent



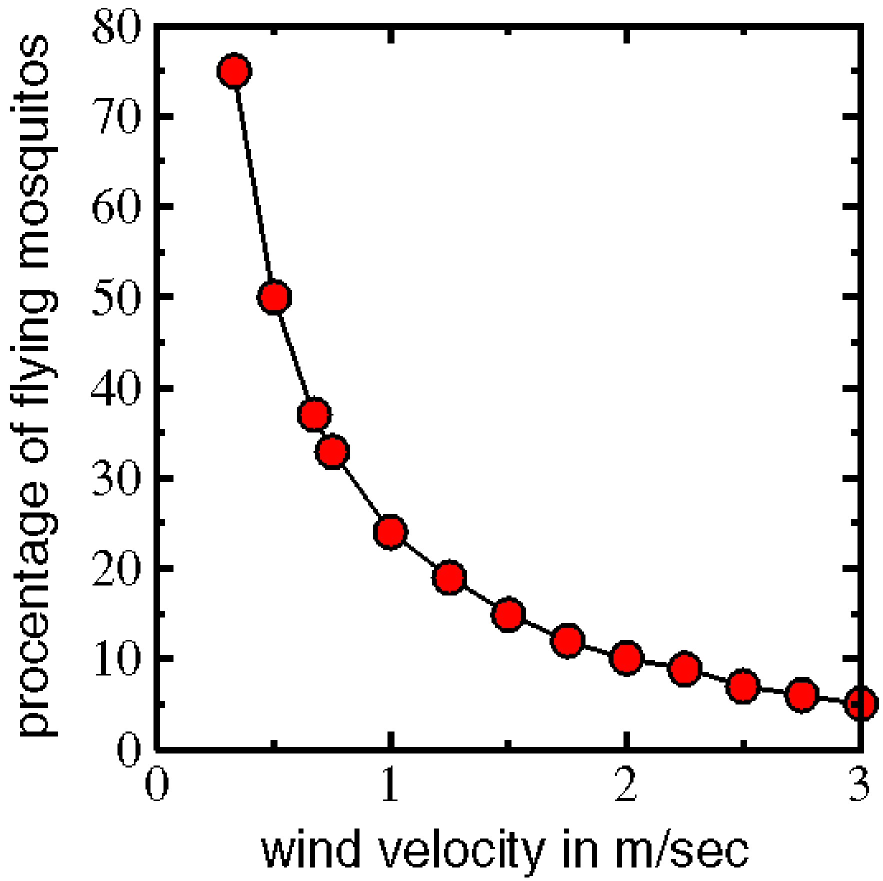
JFM 469 (2002), Nature 409 (2001), DNS Biferale et al.Phys. Fluids 2005



- wind speed 18km/h (5m/sec)
- height above ground 1m
- roughness height 0.05m
  (farmland with few trees in summer time)

$\Downarrow$ τ_η = 5 msec and η = 0.5 mm

every 15 sec > 15g acceleration



Bidlingmayer, W. L., Day, J. F., and Evans, D. G. Effect of wind velocity on suction trap catches of some Florida mosquitos. *J. Am. Mosquito Contr.* **11**, 295–301 (1995).

Lagrangian Velocity Statistics

G. Falkovich, H. Xu, A. Pumir, EB,
L. Biferale, G. Boffetta, A. Lanotte, F. Toschi

(to appear in Phys. Fluids)

Eulerian

Exact flux law
(Kolmogorov Eq.)

$$\langle (\delta_r^L u)^3 \rangle = -\frac{12}{d(d+2)} \epsilon r$$

Eulerian velocity increments
(K41 scaling)

$$\delta_r u \sim (\epsilon r)^{1/3}$$

Lagrangian

Lagrangian velocity increments
(dimensional argument)

$$\delta_\tau u \sim (\epsilon \tau)^{1/2}$$

Assumption:

$$\langle (\delta_\tau u)^2 \rangle \overset{?}{=} C_0 \epsilon \tau$$

Sign of the flux

$$\epsilon > 0 \quad \text{for d=3}$$

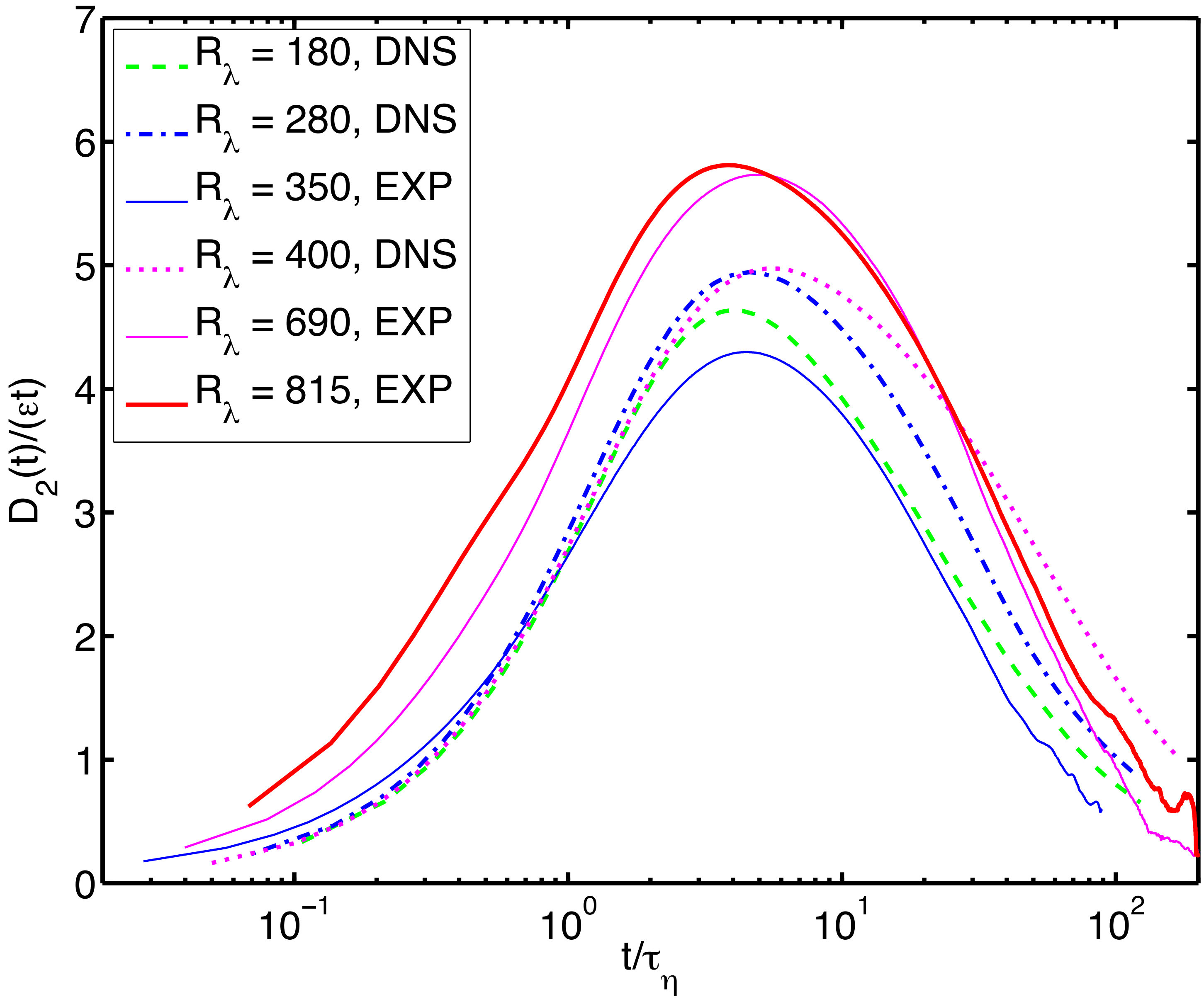
$$\epsilon < 0 \quad \text{for d=2}$$

But $$D_2(\tau) \equiv \langle (\delta_\tau u)^2 \rangle \geq 0$$

Moreover, if reverse time $t \to -t$

The sign of $D_2(\tau)$ remain unchanged!

$$D_2(\tau) \equiv \langle (\delta_\tau u)^2 \rangle \overset{?}{\sim} \epsilon\tau$$

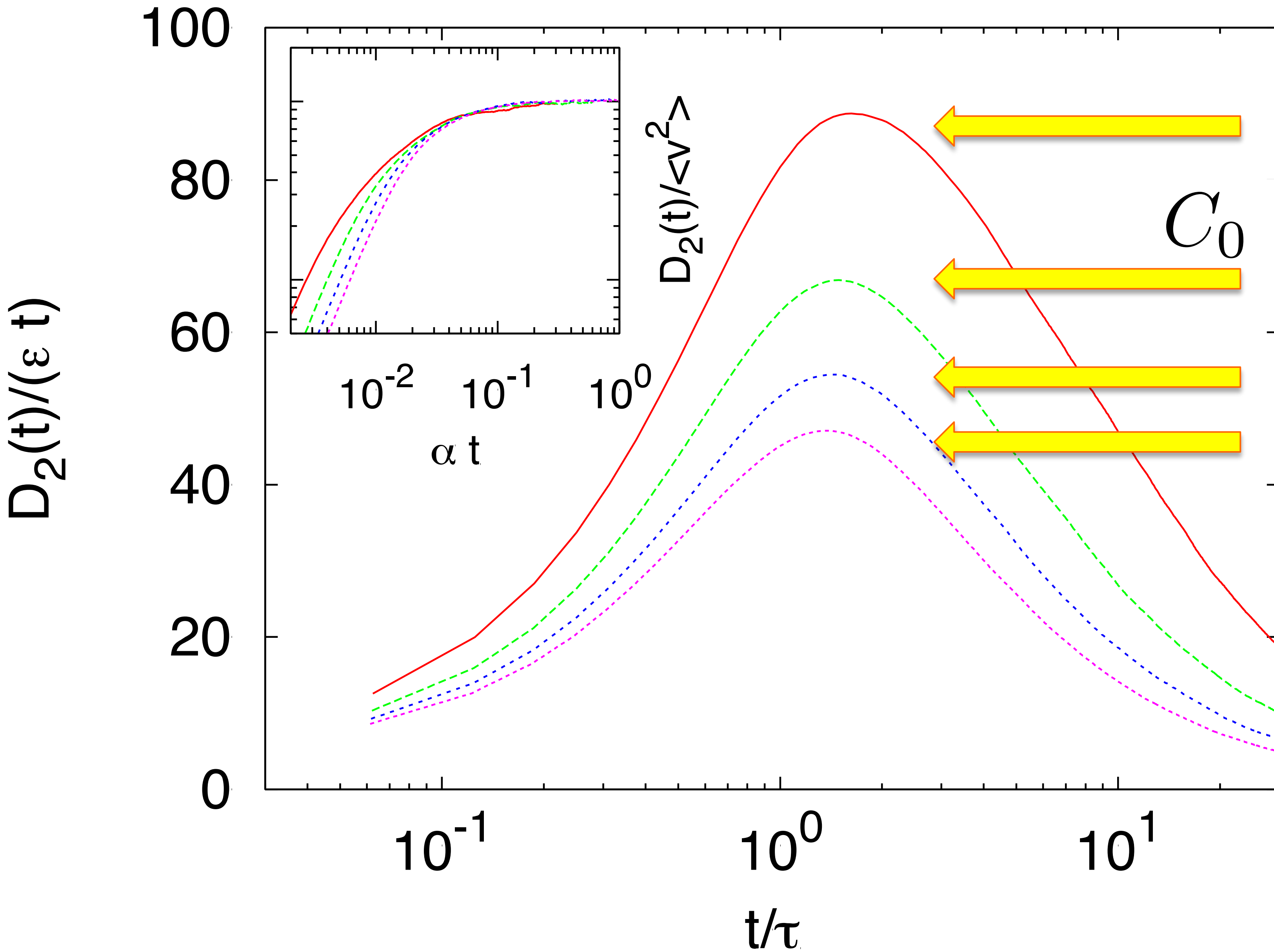


d=3

No scaling range observed from currently available DNS and experimental data.

$$D_2(\tau) \equiv \langle (\delta_\tau u)^2 \rangle \overset{?}{\sim} \epsilon\tau$$

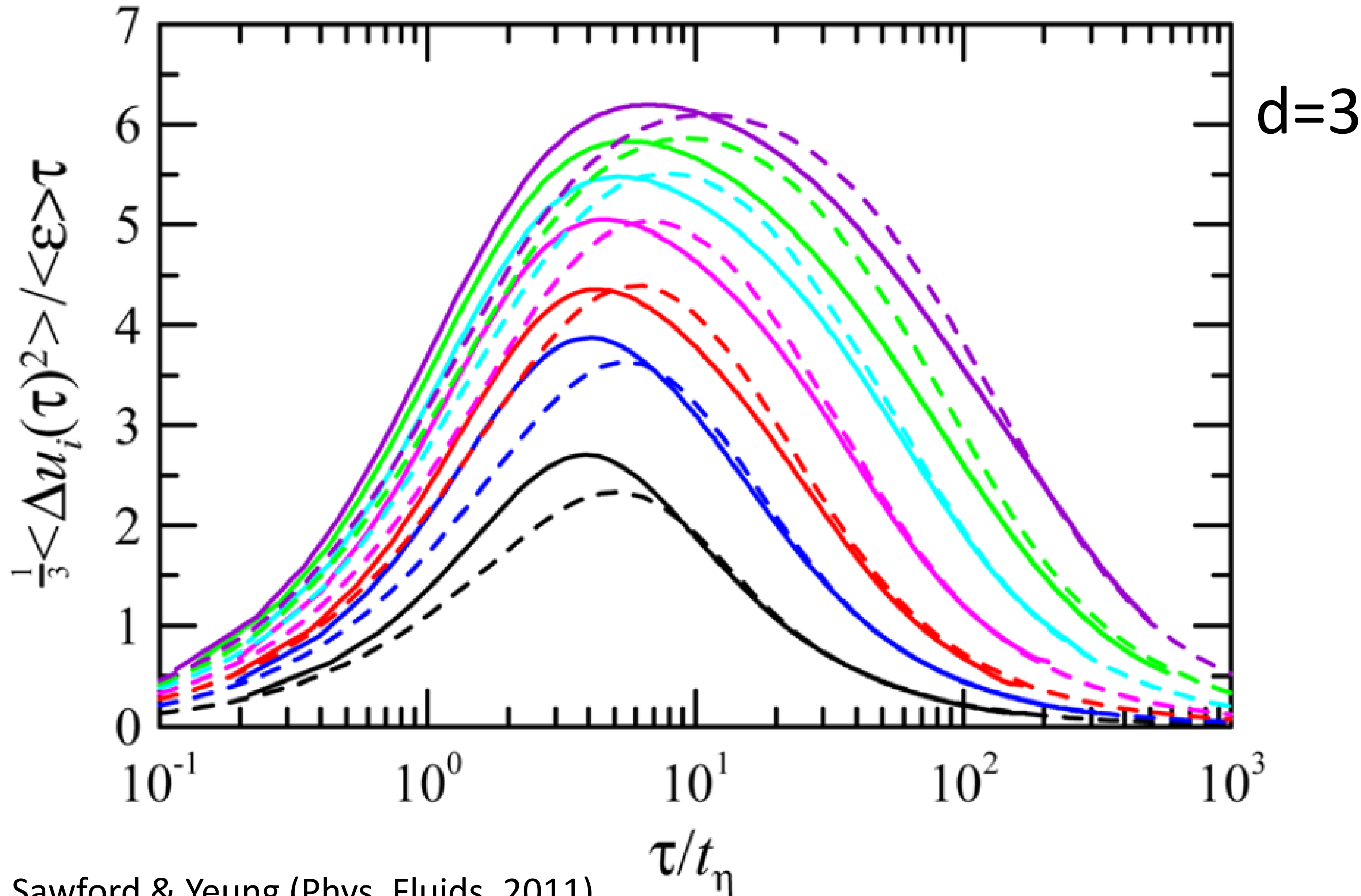
d=2



$$C_0 \sim 10^2$$

No scaling range observed from currently available DNS and experimental data.

$$D_2(\tau) \equiv \langle (\delta_\tau u)^2 \rangle \overset{?}{\sim} \epsilon\tau$$



Sawford & Yeung (Phys. Fluids, 2011)

No scaling range observed from currently available DNS and experimental data.

Acceleration auto-correlation

$$\frac{d}{d\tau}\langle(\delta_\tau u)^2\rangle = 2\langle a(\tau)[u(\tau) - u(0)]\rangle$$
$$= 2\int_0^\tau \langle a(0)a(t)\rangle dt$$

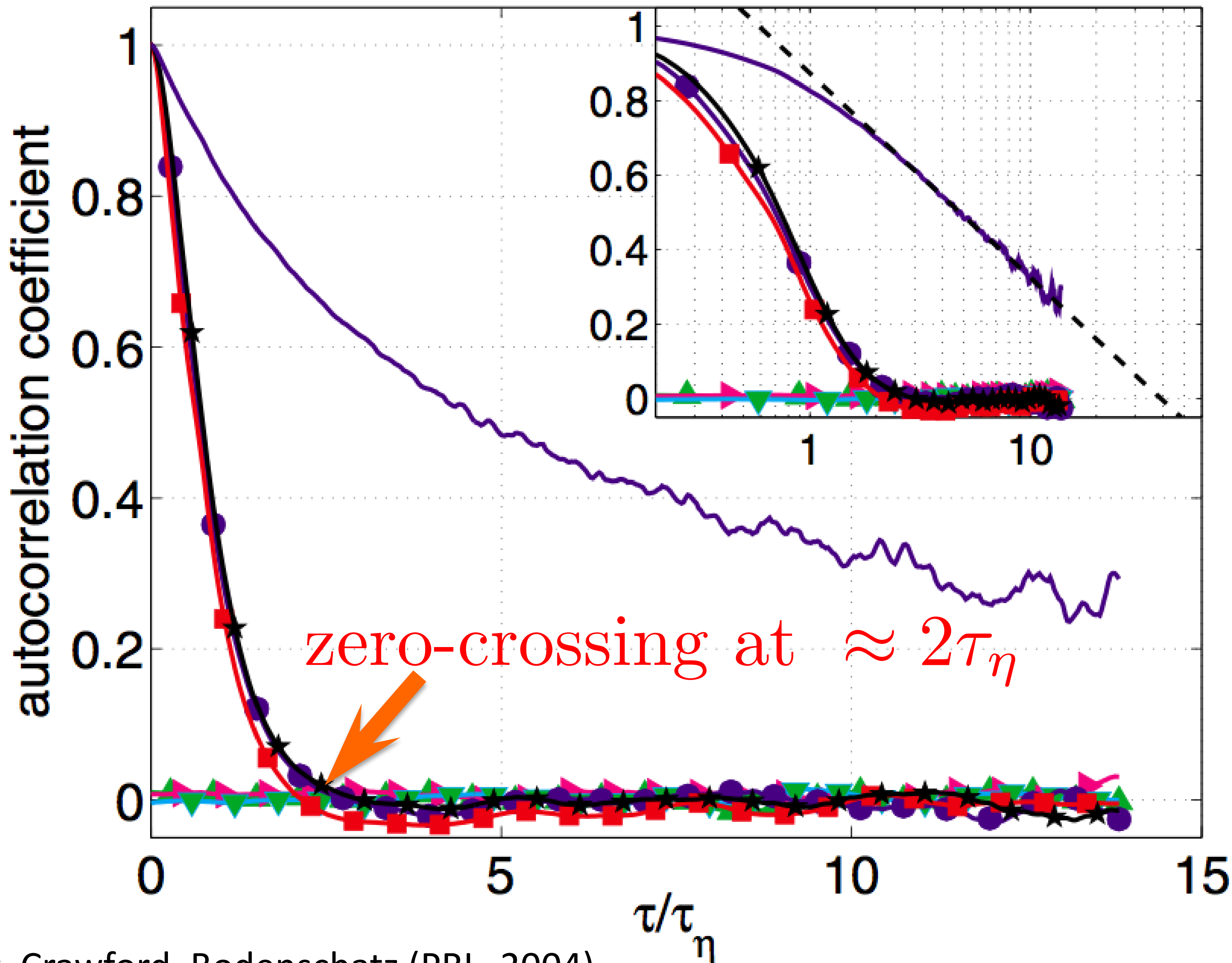
$(\delta_\tau u)^2 \sim \tau \Rightarrow$ acceleration is uncorrelated over time-lag τ

Kinematic constraint (Tennekes & Lumley (1972)):

$$\int_0^\infty \langle a(0)a(t)\rangle dt = 0$$

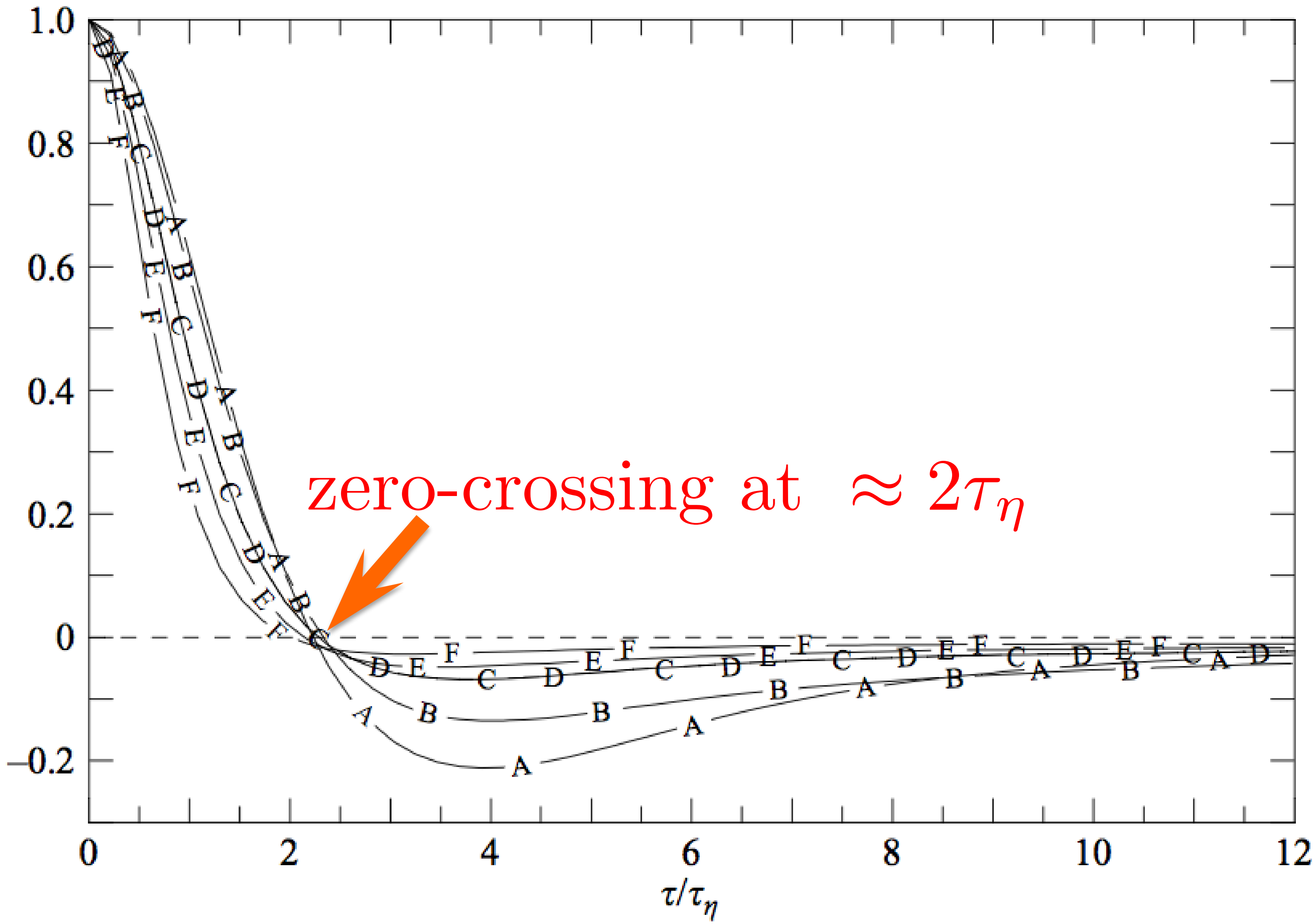
This restrict the shape of the acceleration auto-correlation.

Data from particle tracking measurements



Mordant, Crawford, Bodenschatz (PRL, 2004)

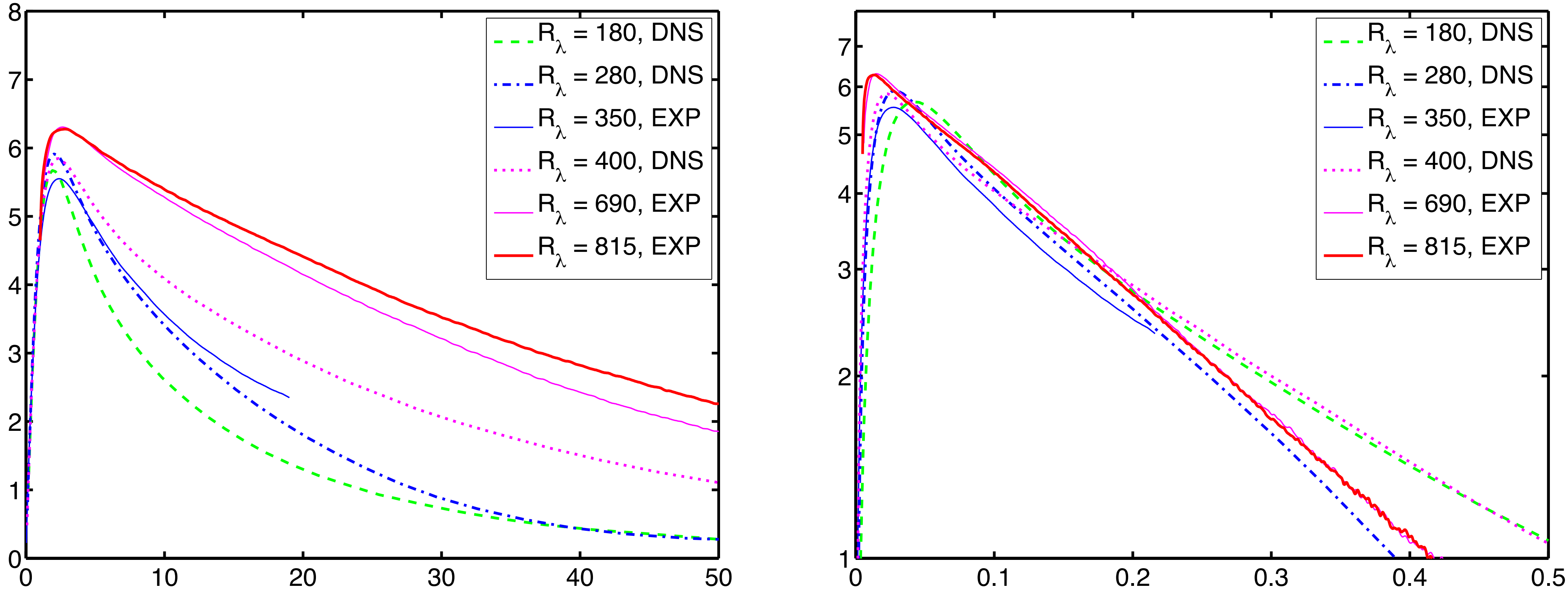
Data from DNS



Yeung et al. (JFM, 2007)

Zero-crossing of the acceleration auto-correlation gives the peak of

$$\frac{d\langle(\delta_\tau u)^2\rangle}{d\tau} = 2\langle a\delta_\tau u\rangle$$



$$\langle(\delta_\tau u)^2\rangle = C_0\epsilon\tau \quad \Rightarrow \quad 2\langle a\delta_\tau u\rangle = C_0\epsilon$$

However, experimental and DNS data suggest an exponential decay after the peak.

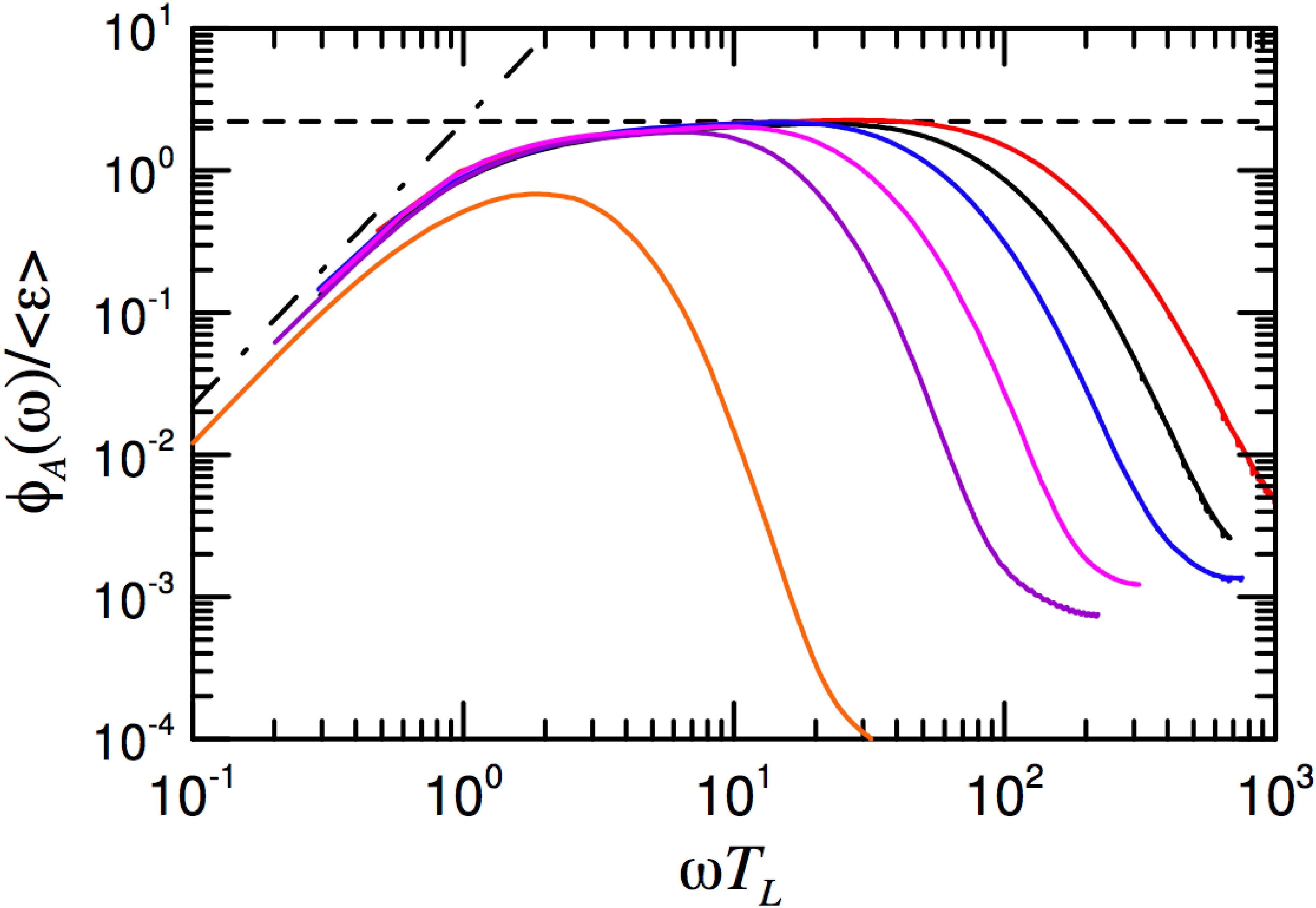
Acceleration spectra

Acceleration spectrum may show a wider scaling range than that of the velocity structure function (Lien & D'Asaro (Phys. Fluids, 2002), Sawford & Yeung (Phys. Fluids, 2011)).

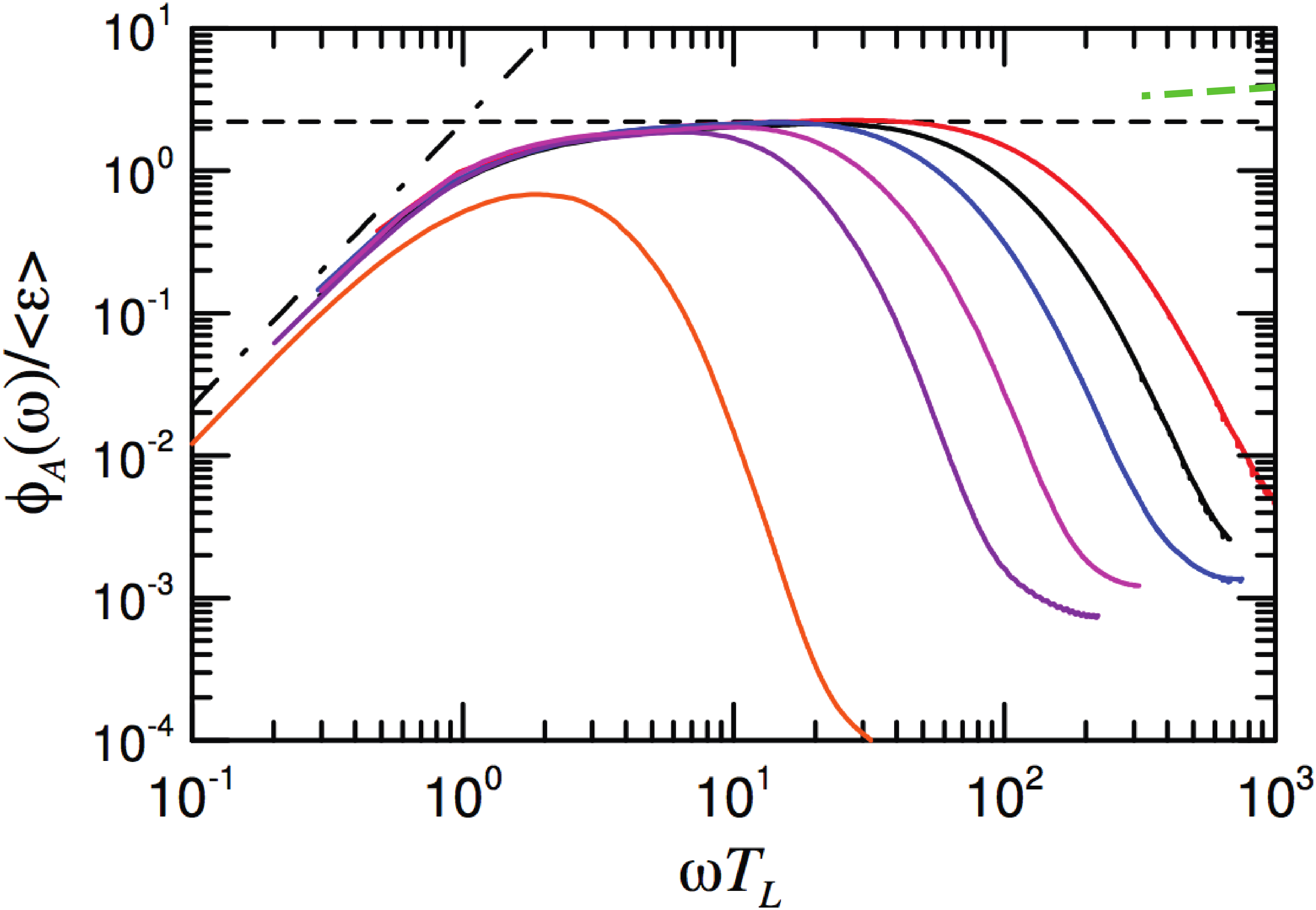
$$\langle(\delta_\tau u)^2\rangle \sim \tau \quad \Rightarrow \quad \Phi_A(\omega) \sim \omega^0, \quad (1/T_L \lesssim \omega \lesssim 1/\tau_\eta)$$

Remark: A flat acceleration spectrum implies δ-correlated acceleration.

DNS results from Sawford & Yeung (Phys. Fluids, 2011)



DNS results from Sawford & Yeung (Phys. Fluids, 2011)



Acceleration spectra suggest anomalous scaling for velocity increments:

$$\Phi_A(\omega) \sim \omega^{\mu} \quad \Rightarrow \quad \langle (\delta_\tau u)^2 \rangle \sim t^{1-\mu}$$

Moreover, acceleration variance is:

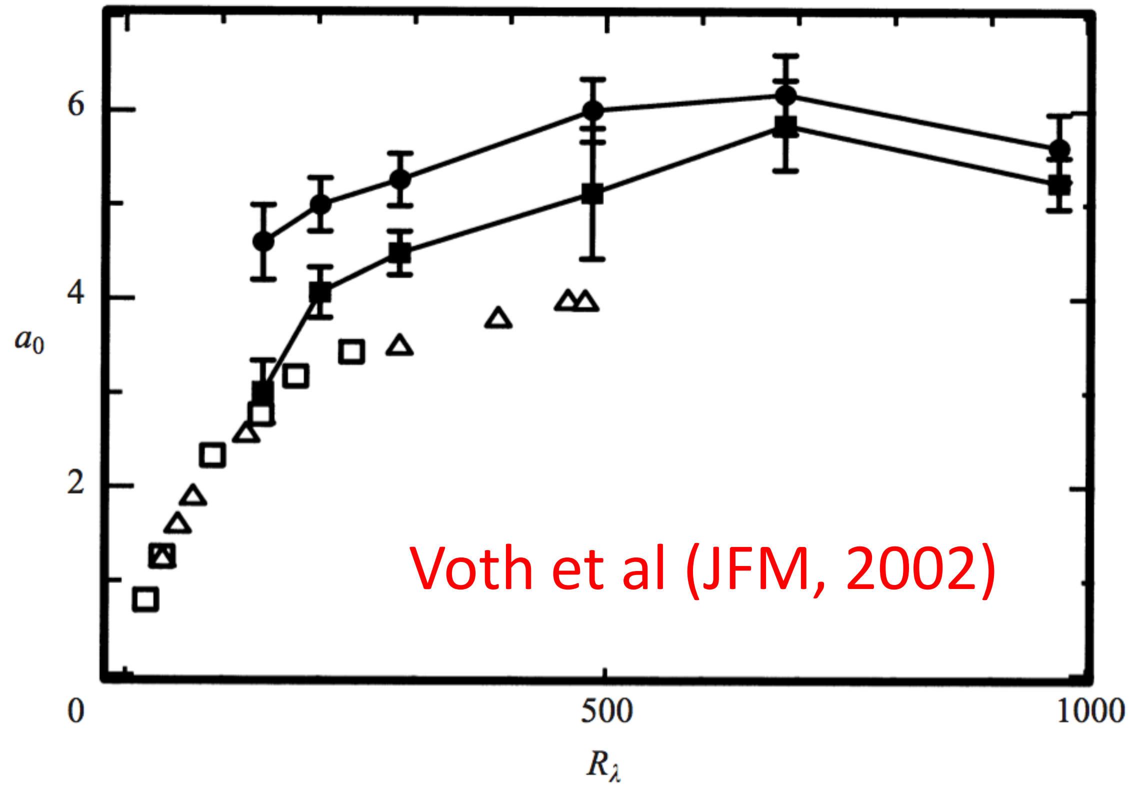
$$\langle a^2 \rangle = \int_0^\infty d\omega \Phi_A(\omega) \approx \int_{1/T_L}^{1/\tau_\eta} A_0 \epsilon \omega^\mu d\omega$$

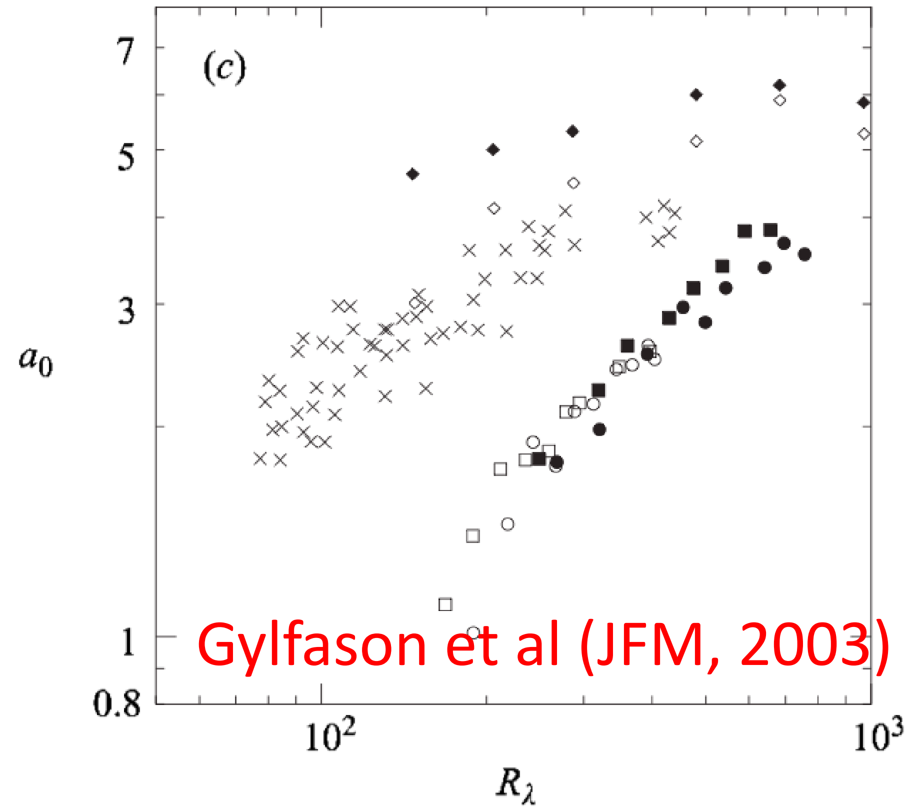
$$\sim \epsilon \tau_\eta^{-(1+\mu)} \sim \frac{\epsilon^{3/2}}{\nu^{1/2}} R_\lambda^\mu$$

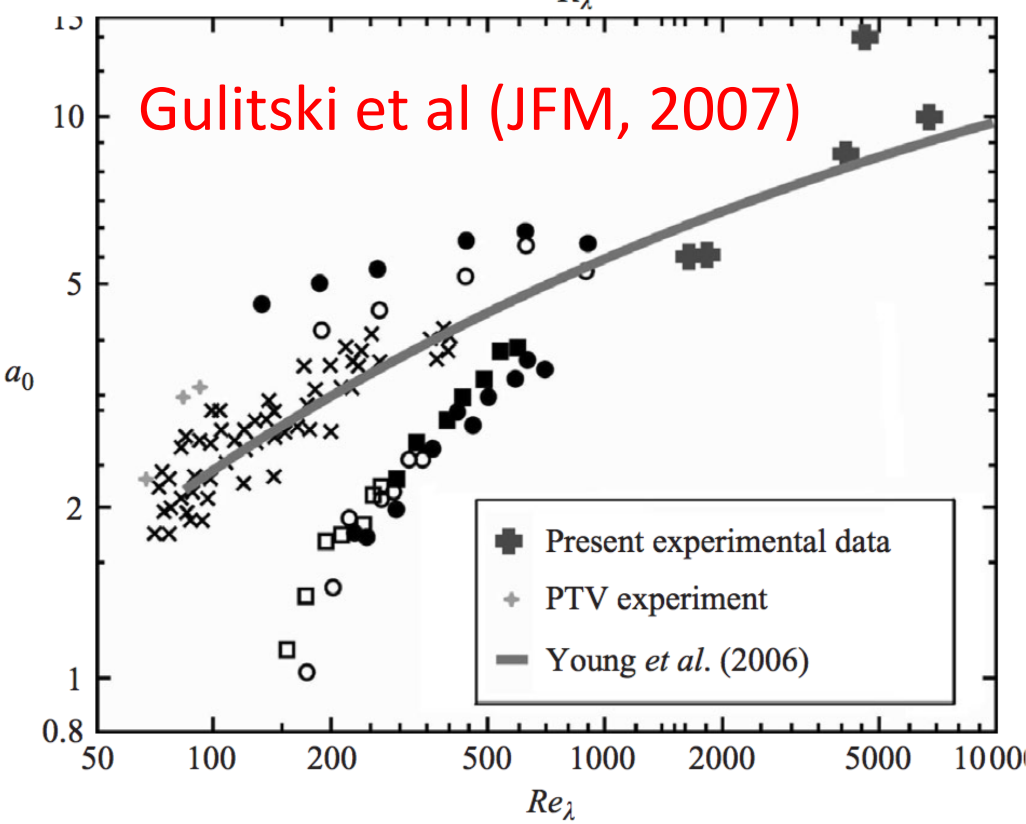
Which implies:

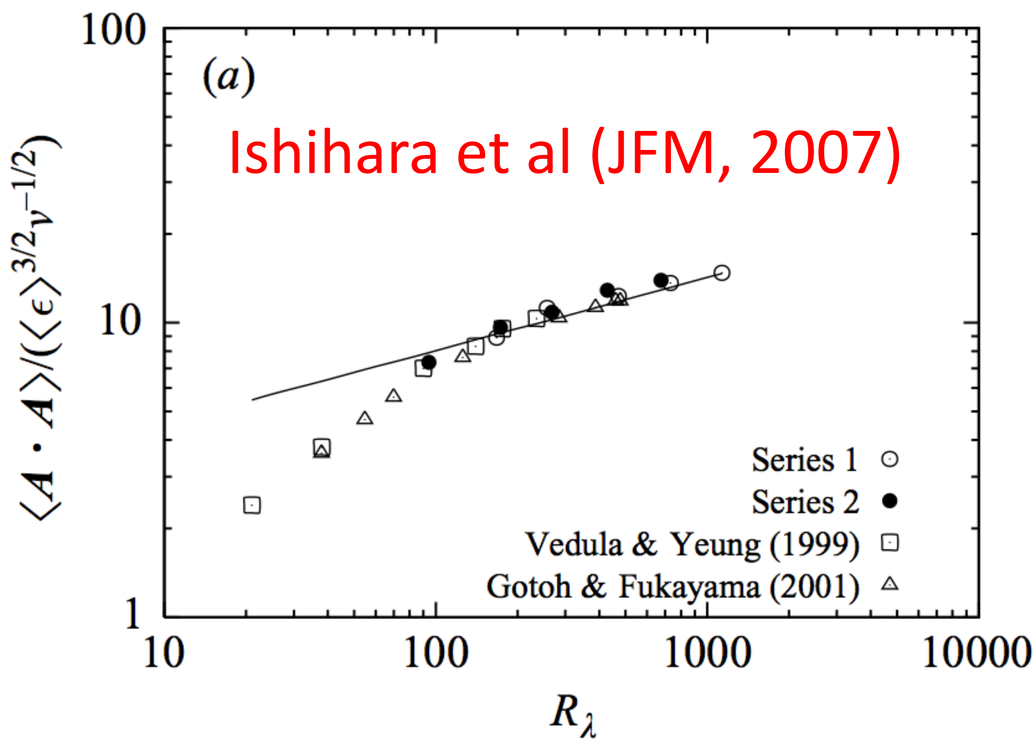
$$a_0 \equiv \frac{\langle a^2 \rangle \nu^{1/2}}{\epsilon^{3/2}} \sim R_\lambda^\mu$$

Consistent with observations from experiments and DNS.


Voth et al (JFM, 2002)
a_0
R_λ
0
2
4
6
500
1000


(c)
Gylfason et al (JFM, 2003)
a_0
R_λ
0.8
1
3
5
7
10^2
10^3


Gulitski et al (JFM, 2007)
a_0
Re_λ
Present experimental data
PTV experiment
Young et al. (2006)
0.8
1
2
5
10
50
100
200
500
1000
2000
5000


(a)
Ishihara et al (JFM, 2007)
$\langle A \cdot A \rangle / (\langle \epsilon \rangle^{3/2} \nu^{-1/2})$
R_λ
Series 1
Series 2
Vedula & Yeung (1999)
Gotoh & Fukayama (2001)
1
10
100
1000
10000

Summary for part 1

Dimensional scaling for Lagrangian velocity structure functions is not consistent with either theoretical considerations or experimental/numerical data.

Using extended-self-similarity to the study of Lagrangian velocity structure functions is questionable.

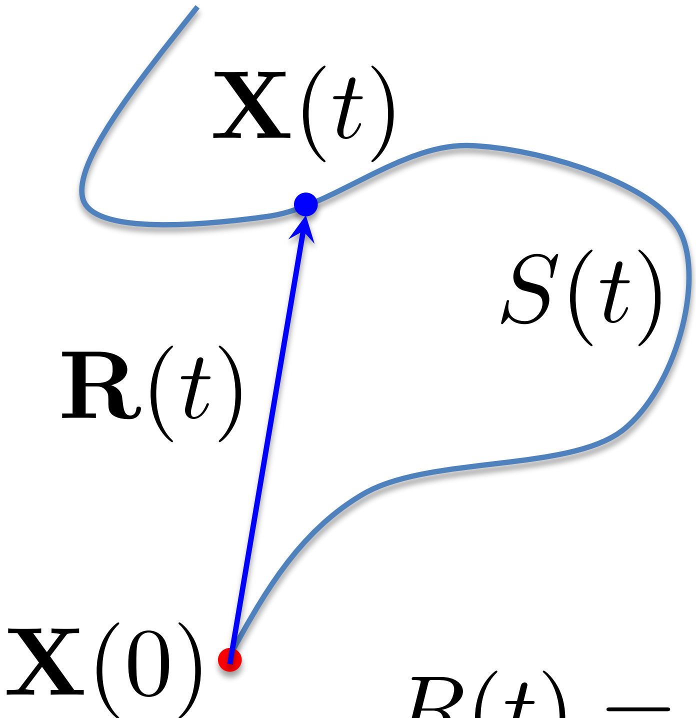
Interpolation schemes that bridges viscous, inertial, and large scales can only be used with caution, as the scaling relations are built-in while constructing the scheme.

Future work is on the study of the relation between Lagrangian statistics and energy flux, to which multi-point, multi-time statistics is of great interest.

Path Length Statistics

N. T. Ouellette, EB, H. Xu

(J. Stat. Phys. 145:93, 2011)



Displacement:

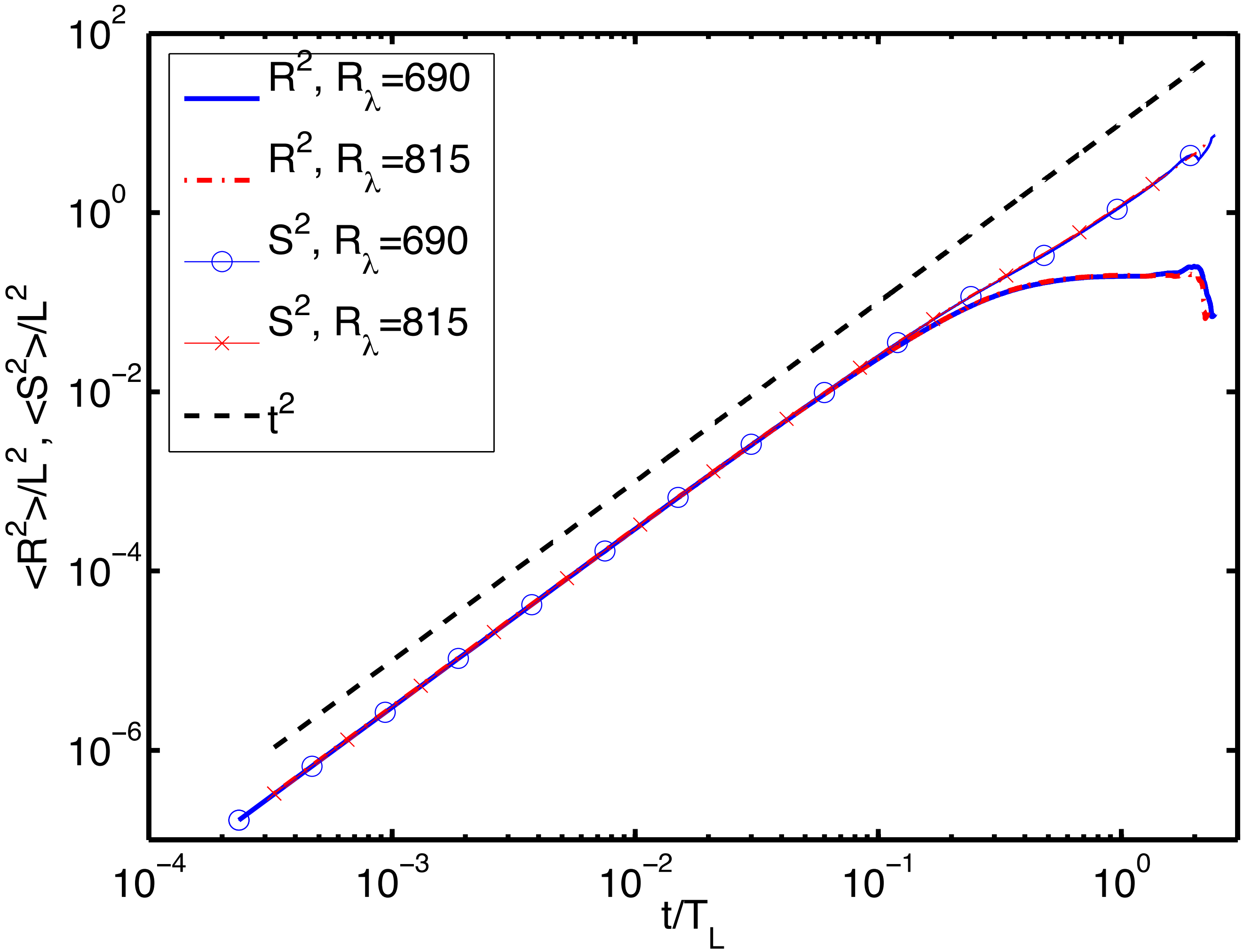
$$\mathbf{R}(t) \equiv \mathbf{X}(t) - \mathbf{X}(0)$$

Taylor (1922):

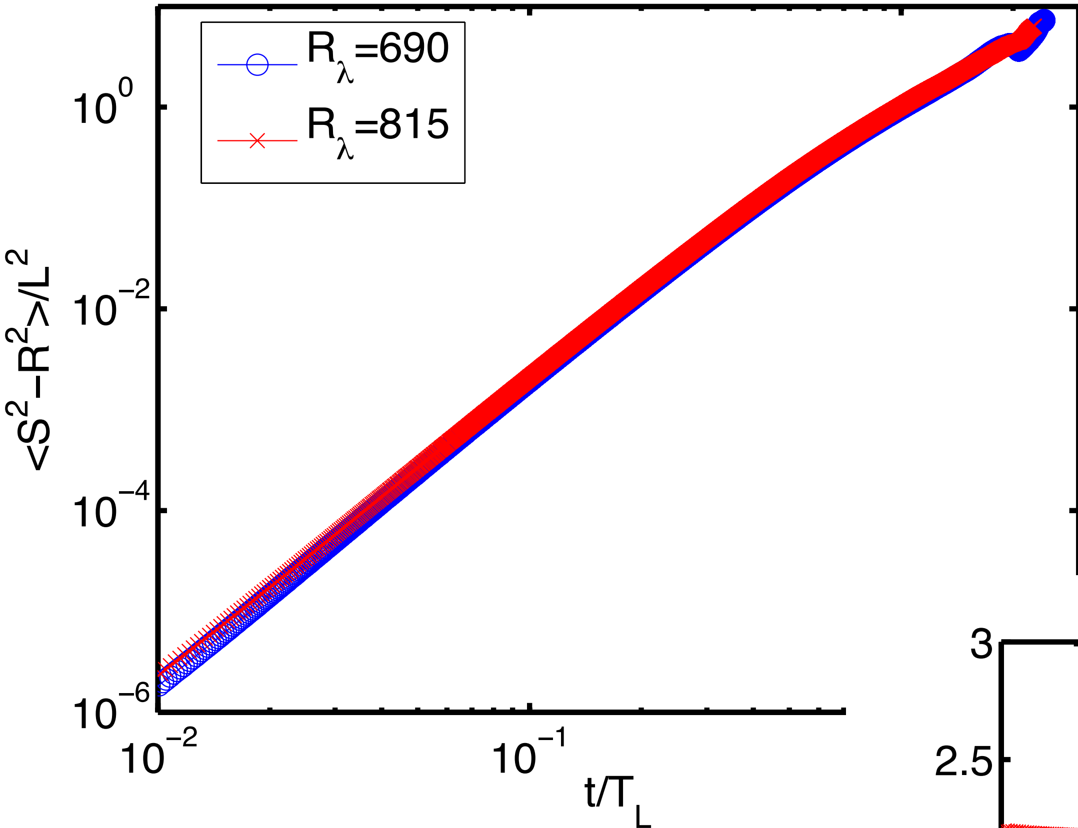
$$R(t) = |\mathbf{R}(t)| = \begin{cases} u'^2 t^2 & \text{if } t \ll T_L \\ u'^2 T_L t & \text{if } t \gg T_L \end{cases}$$

What about path length?

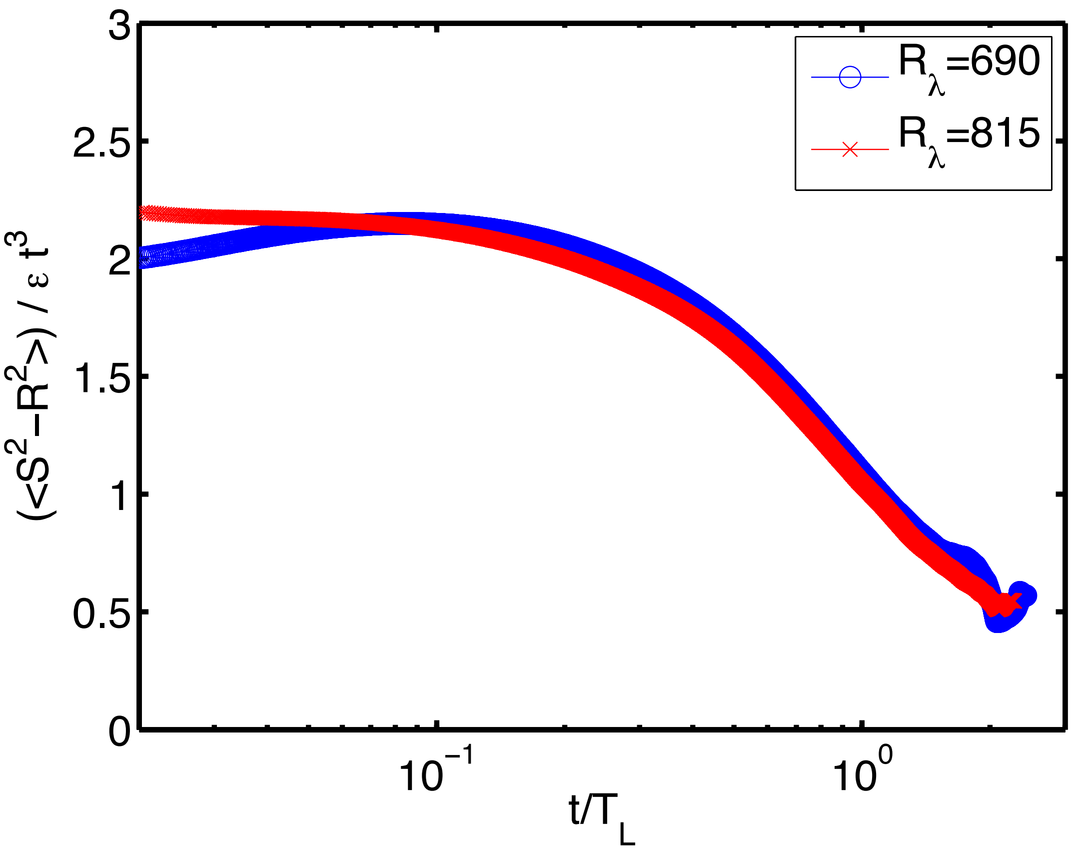
$$S(t) \equiv \int_0^t |\mathbf{u}(t')| dt'$$



$$\langle S^2 \rangle \sim t^2$$

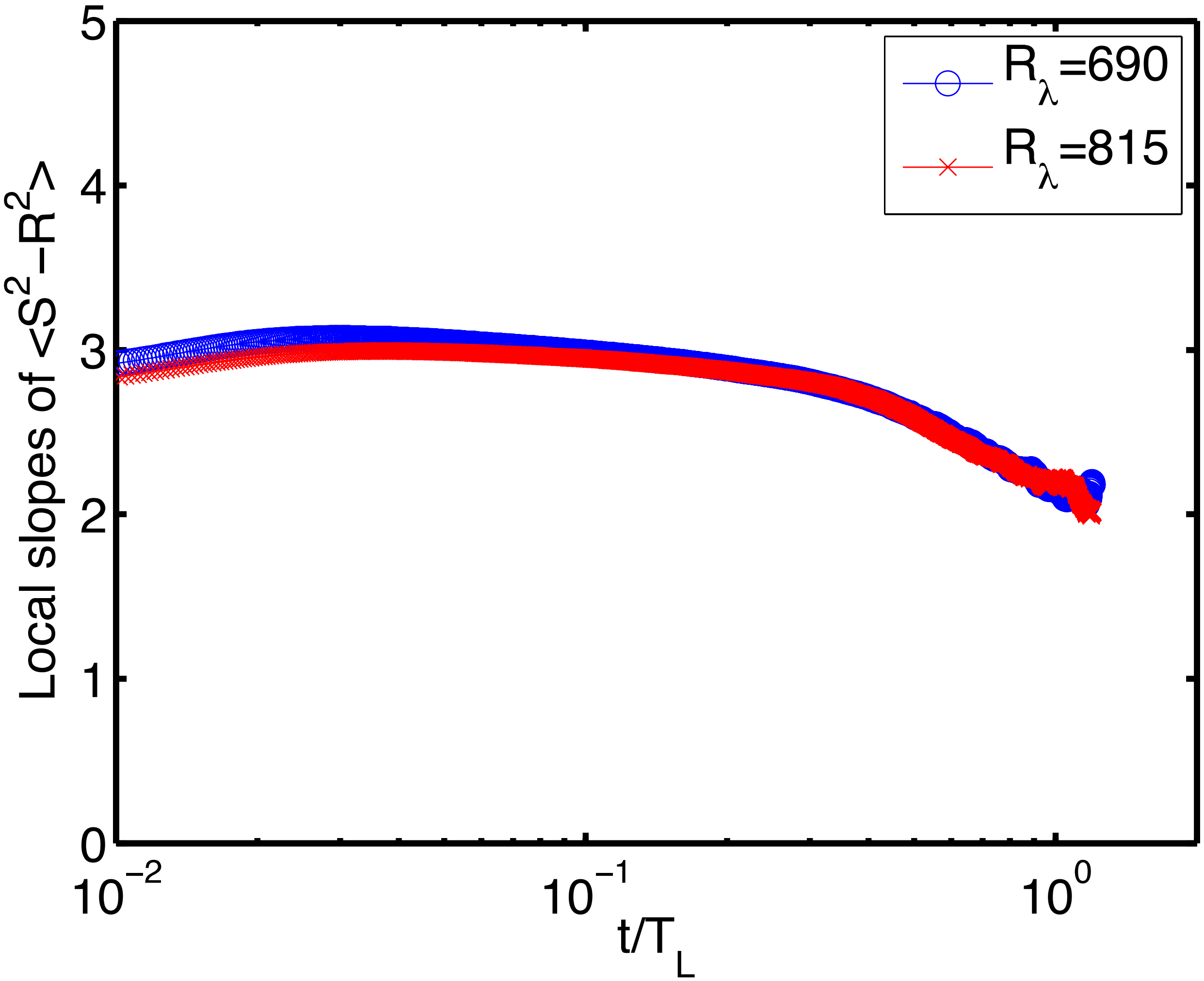


$$\langle S^2 \rangle - \langle R^2 \rangle \sim ?$$



$$\langle S^2 \rangle - \langle R^2 \rangle \sim t^3$$

Local slope $\dfrac{d\log\langle S^2 - R^2\rangle}{d\log t}$



Q: Which physics determines the difference S^2-R^2?
Vortical structures?

Can we understand the scaling of $\langle S^2 - R^2 \rangle$?
Dimensional argument:

$$\langle S^2 - R^2 \rangle \sim \epsilon t^3$$

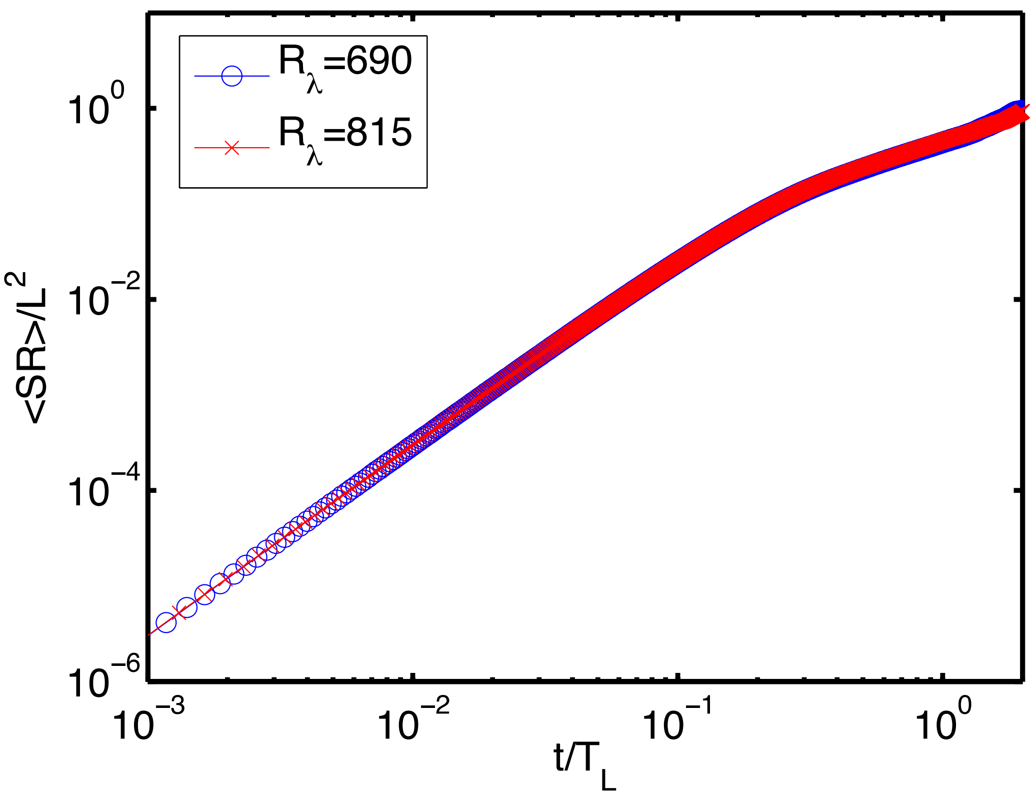
What about other terms like:

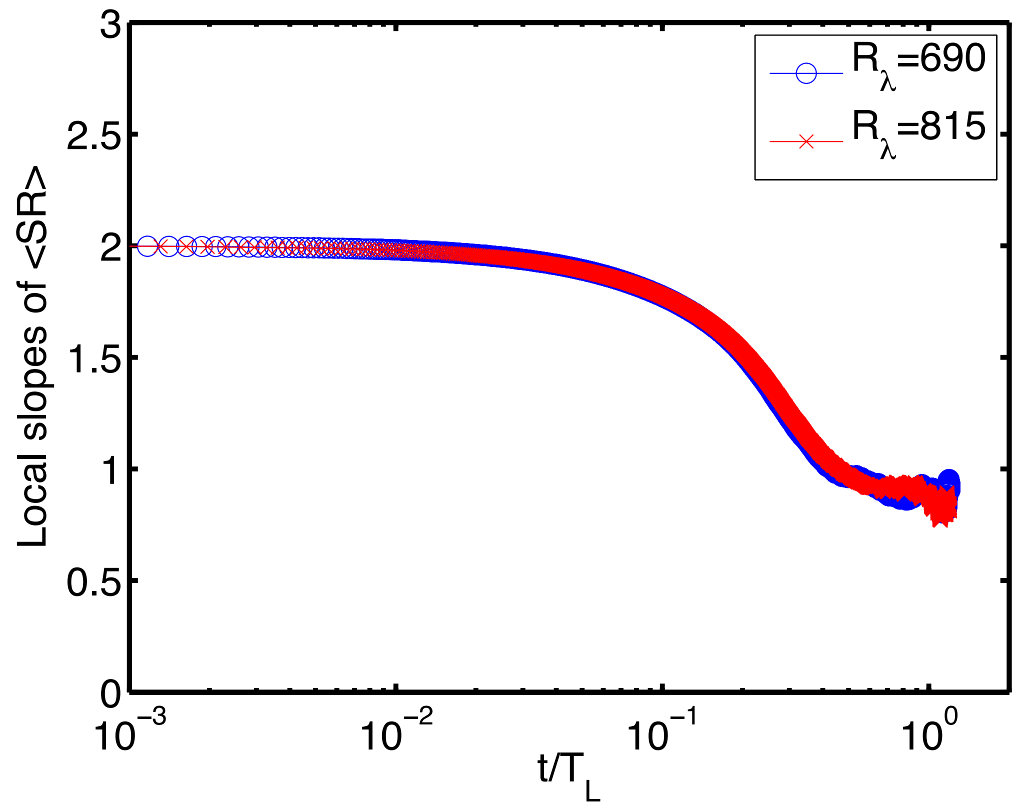
$$\langle (S-R)^2 \rangle \qquad \langle SR \rangle$$

Note:

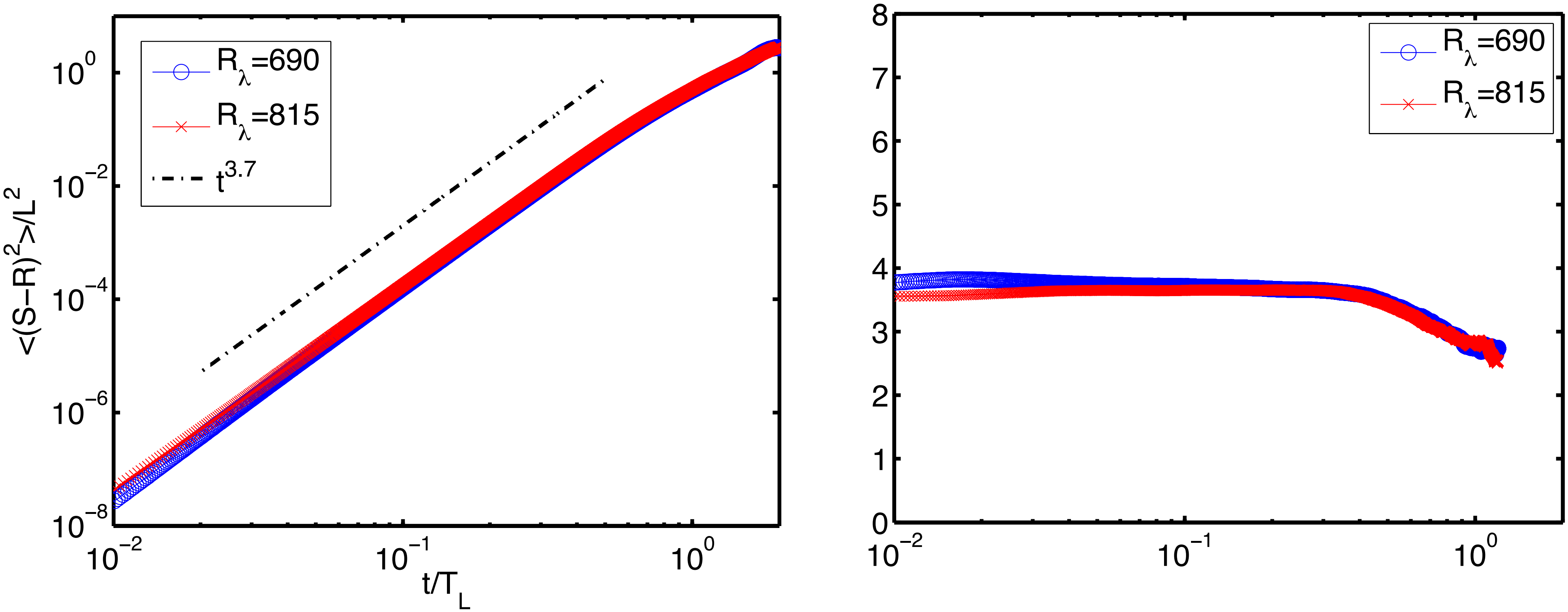
$$\langle S^2 - R^2 \rangle = \langle (S-R)^2 \rangle + 2\langle SR \rangle - 2\langle R^2 \rangle$$

$$\langle SR \rangle \sim t^2 \quad \text{for } t \ll T_L$$





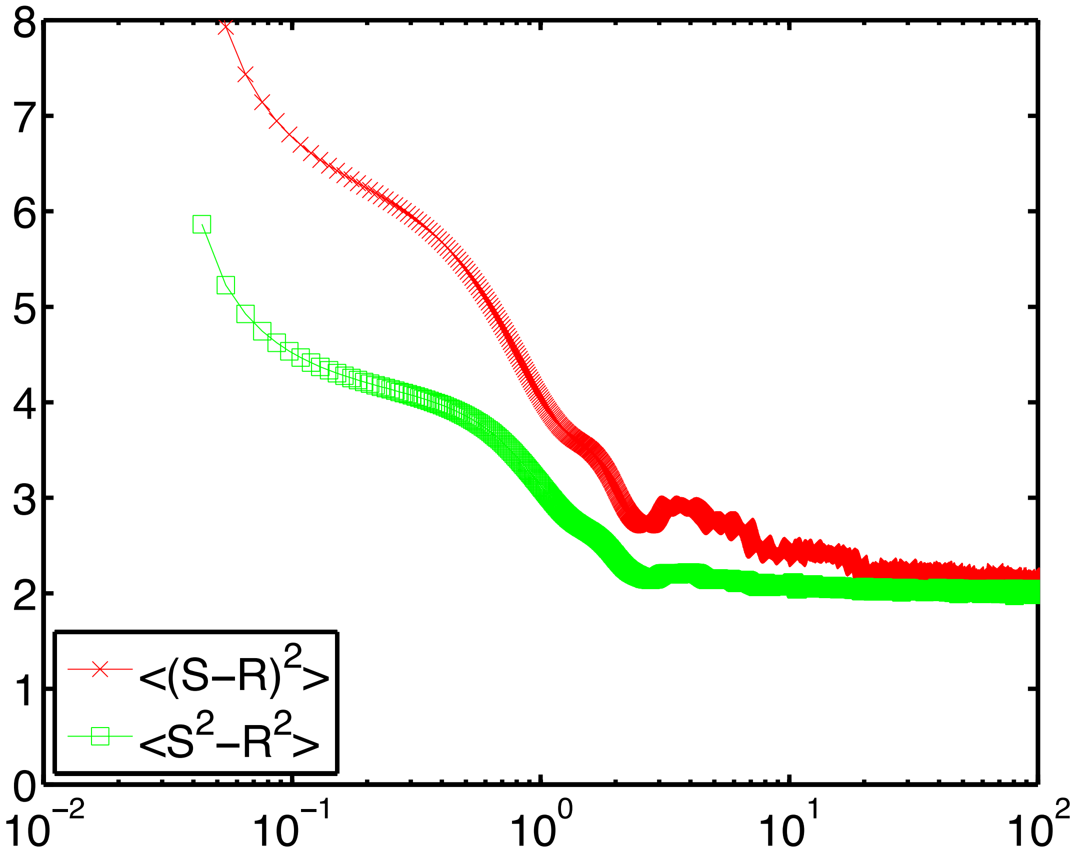
$$\langle (S-R)^2 \rangle \sim t^{3.7} \quad \text{for } t \ll T_L$$



Are these special features of turbulence or some generic kinematic relations that are widely applicable?

Test with two other flows: a synthetic ABC flow and a Lagrangian stochastic model for single particle trajectories in turbulence.

Steady Arnold-Beltrami-Childress (ABC) flow:

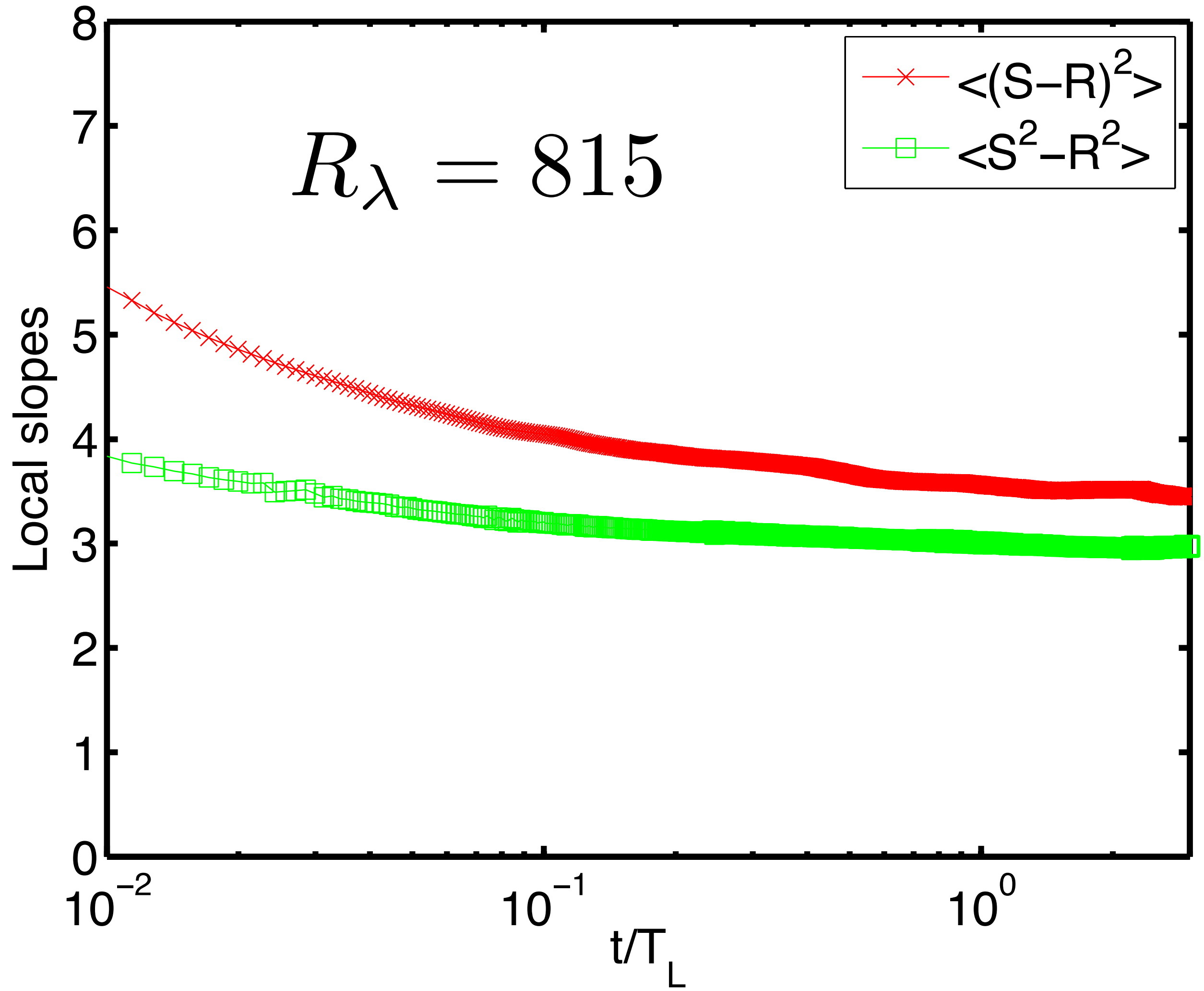


$$\mathbf{u}(x,y,z) = (A\sin kz + C\cos ky)\hat{\mathbf{e}}_x + (B\sin kx + A\cos kz)\hat{\mathbf{e}}_y + (C\sin ky + B\cos kx)\hat{\mathbf{e}}_z$$

$$T_{ABC} = \frac{2\pi/k}{(A^2+B^2+C^2)^{1/2}}$$

No scaling range at short times.

Results from stochastic model (Sawford, Phys. Fluids, 1991):



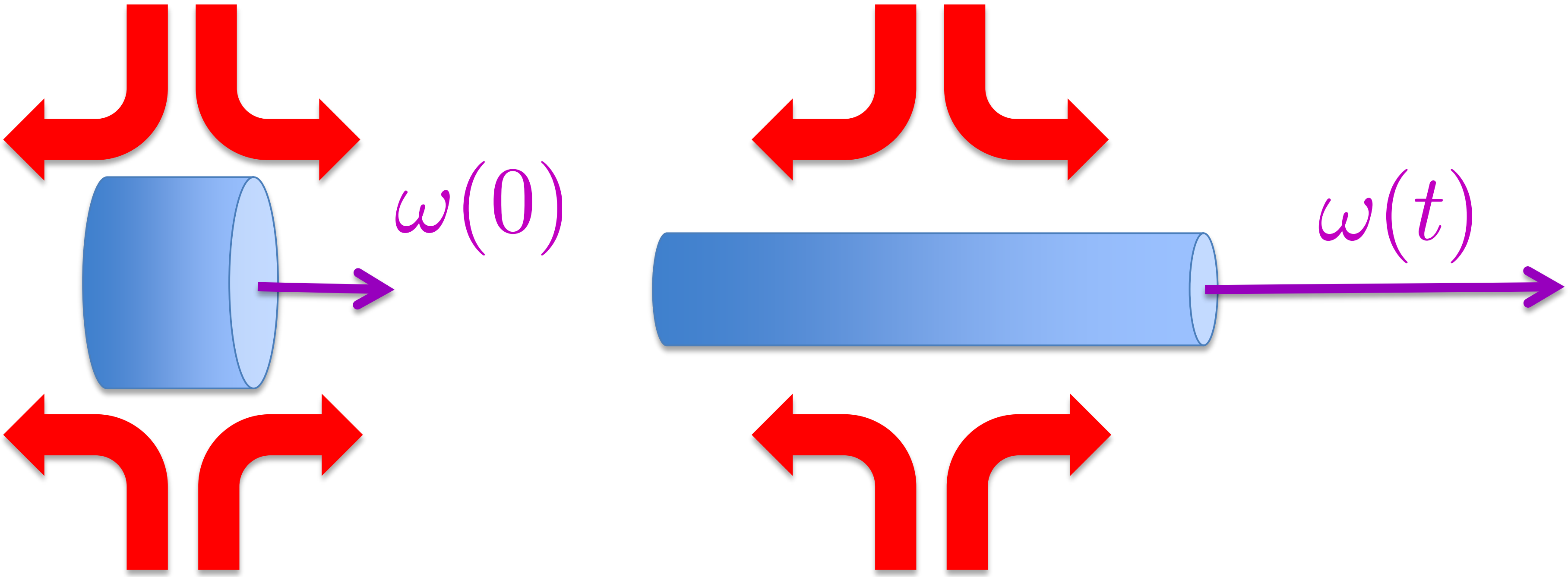
Similar results as in experiments, but with smaller inertial range.

Summary for part 2

For Lagrangian trajectories, path length and displacement scale similarly at short times.

The difference between path length and the displacement has interesting power-law scaling in the inertial range, which might be related to the Lagrangian structures in the turbulent flow.

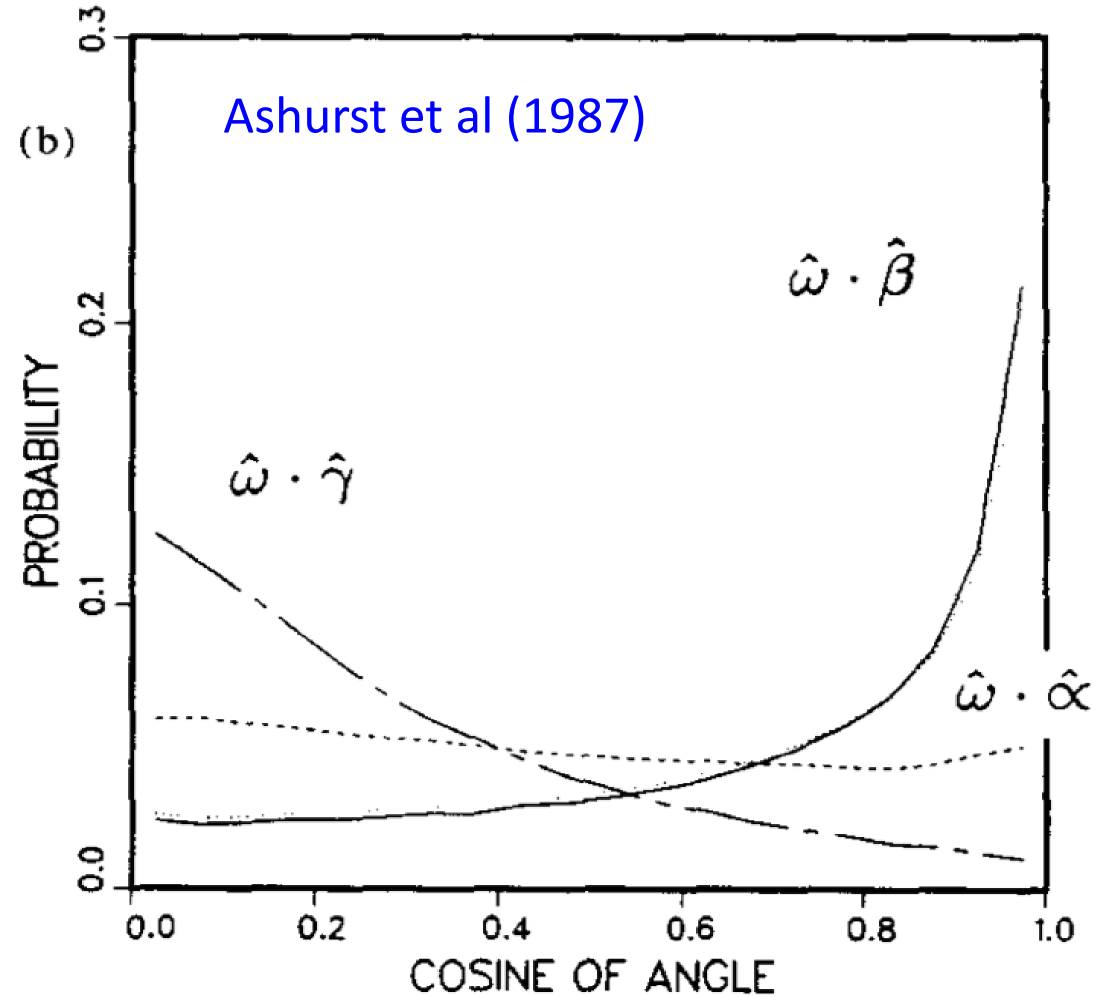
Vortex stretching in turbulence



Expect alignment of vorticity with the strongest eigenvalue of the rate o
strain tensor.

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = S_{ij}\omega_j + \nu\nabla^2\omega_i$$

Instantaneous vorticity aligns with the intermediate eigenvalue of rate of strain tensor

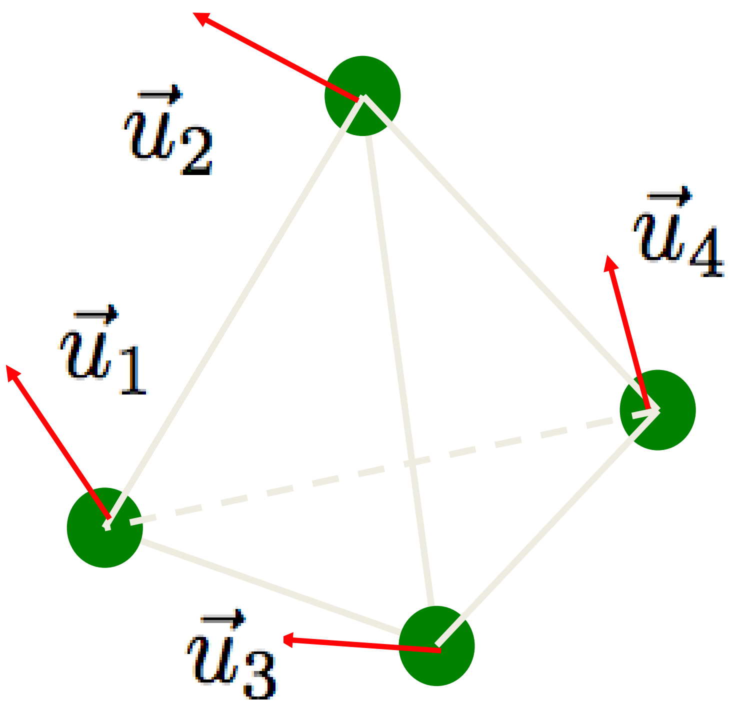


Siggia (1981), Ashurst et al (1987), She et al (1991), Majda (1991), Jimenez (1992), Tsinober et al (1992), Zeff et al (2003), Lüthi et al (2005), Hamlington et al (2008), Tsinober (2009)

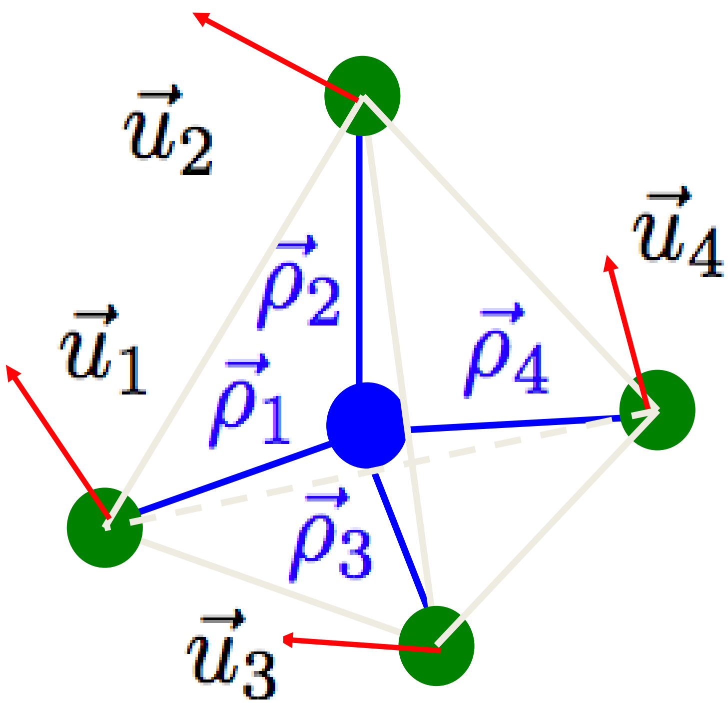
- How does vorticity evolve in time when following the flow in response to the initial stretching?

- what is the behavior for dissipative and inertial scales?

perceived velocity gradients scale r_0

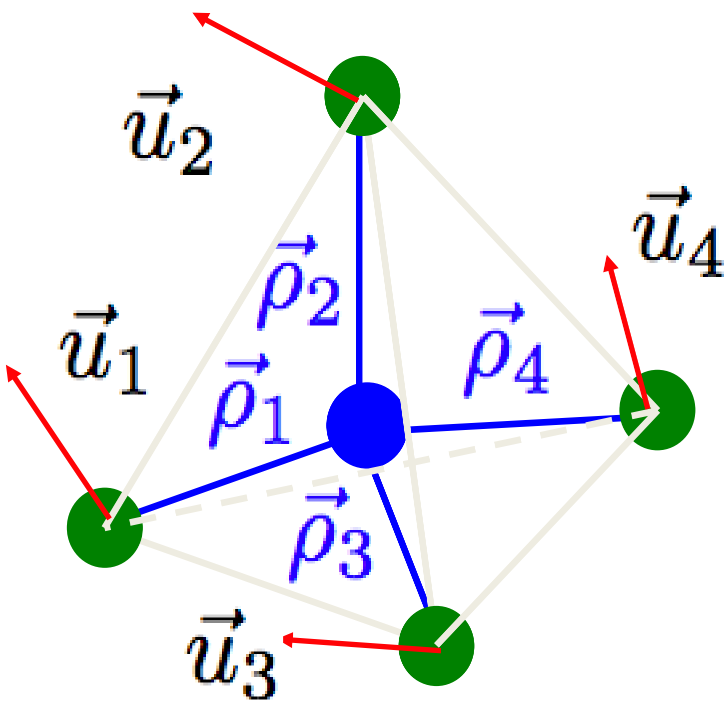


perceived velocity gradients scale r_0



$$\mathbf{M}: \ \mathbf{M}\vec{\rho}_i = \vec{u}_i - \frac{1}{4}\sum_i \vec{u}_i$$

perceived rate of strain and vorticity at scale r_0



perceived rate of strain:

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{M} + \mathbf{M}^T\right)$$

perceived vorticity:

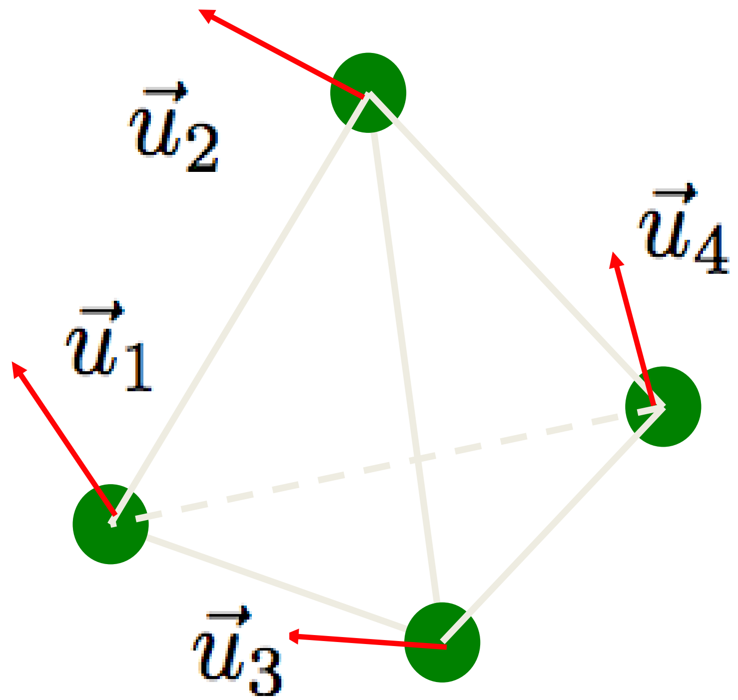
$$\mathbf{\Omega} = \frac{1}{2}\left(\mathbf{M} - \mathbf{M}^T\right)$$

Chertkov, Pumir, Shraiman (**Phys. Fluids**, 1999)

direct numerical simulation

Spectral code, up to 384^3

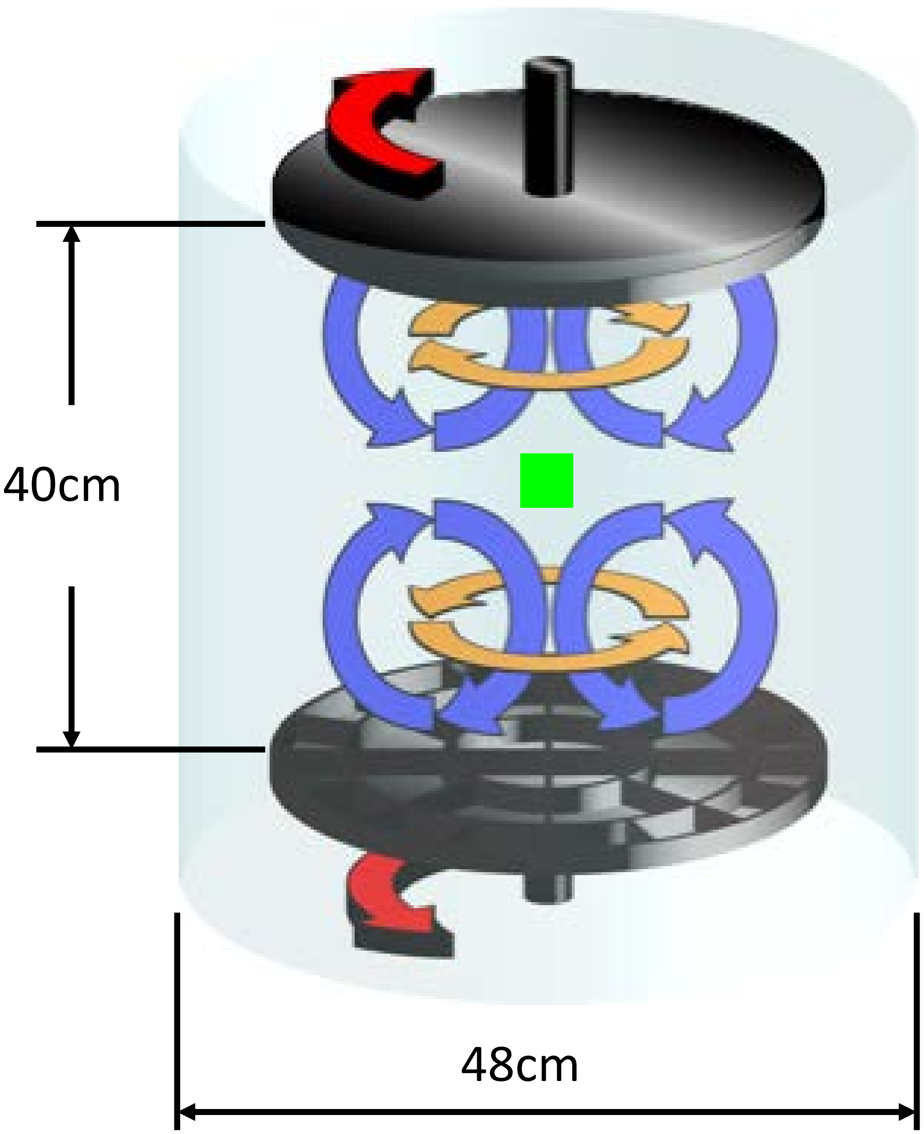
R_λ up to 170, $L_{\text{int}}/\eta \approx 300$



Seed particles that form initially isotropic tetrahedra and follow their motion

Pumir, Shraiman, Chertkov (**Phys. Rev. Lett**, 2000)

experiment



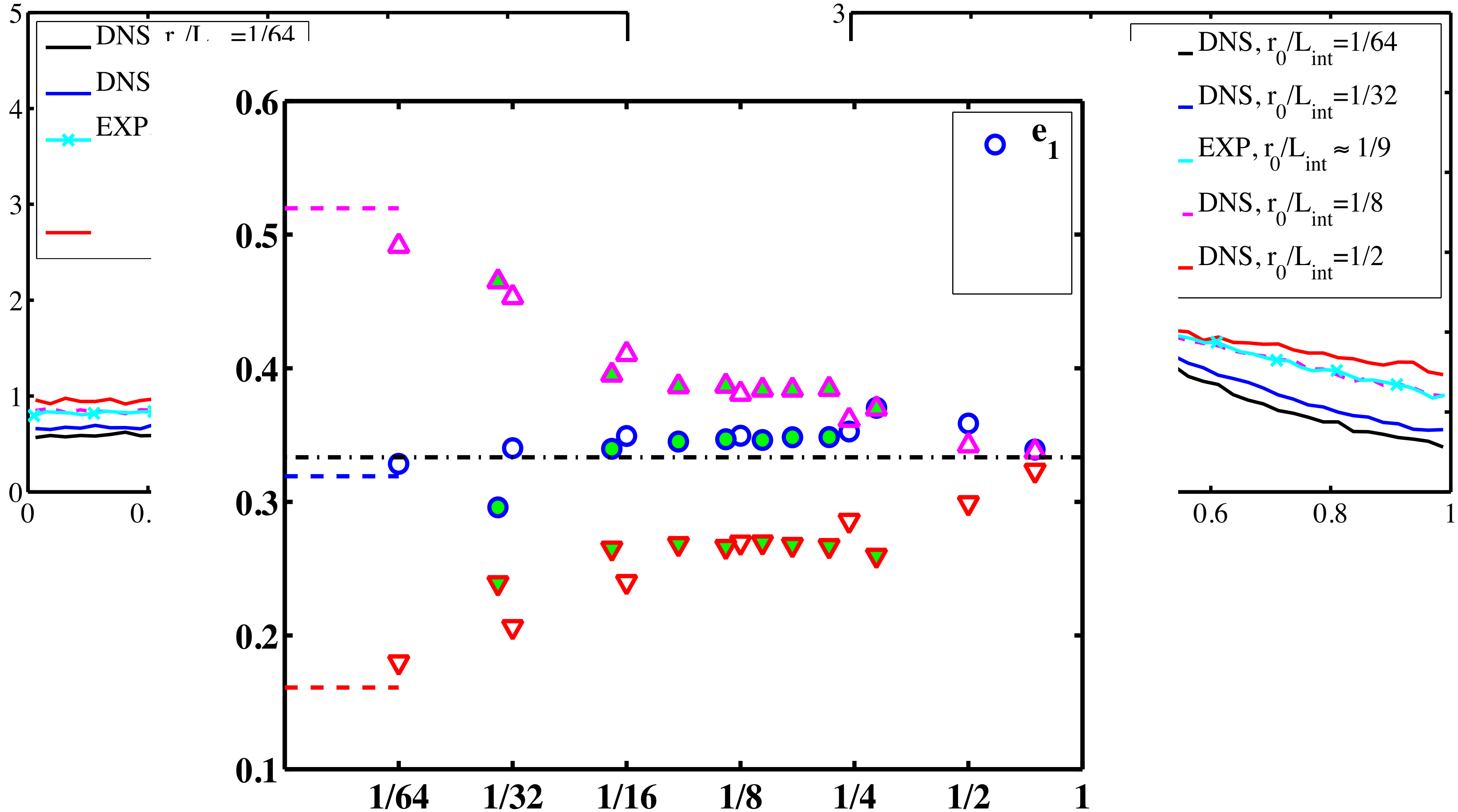
$$R_\lambda = 350 - 815$$
$$L_{\text{int}} \approx 7 \text{ cm}$$

$$L_{\text{int}}/\eta \approx 830 \text{ @ } R_\lambda = 350$$

Particle tracking ~ 100 particles.

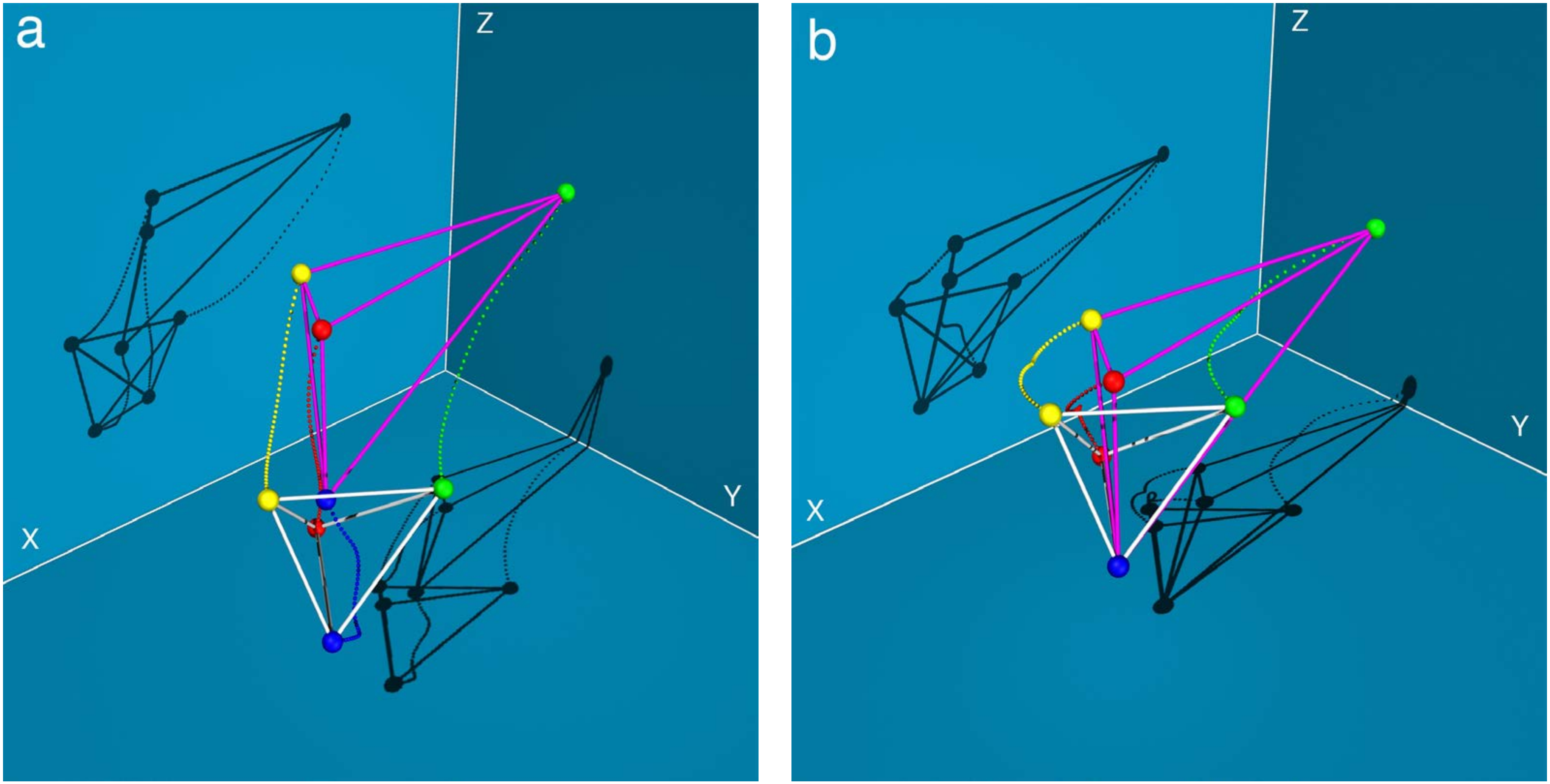
Isotropic tetrads:
4 particles within $(1 \pm 0.1) r_0$.

instantaneous alignment



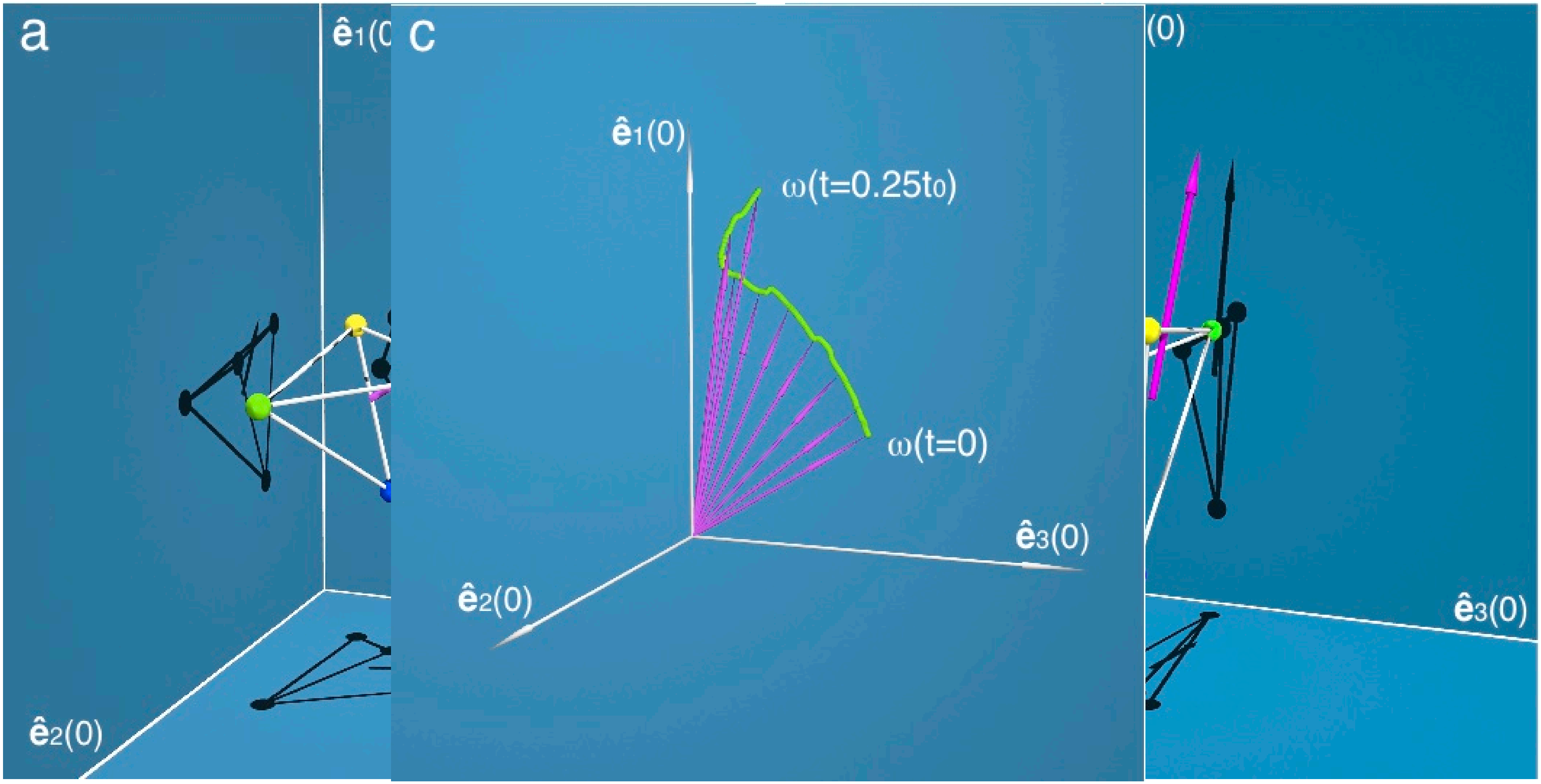
Vorticity aligns with intermediate eigenvalue of the strain

alignment dynamics



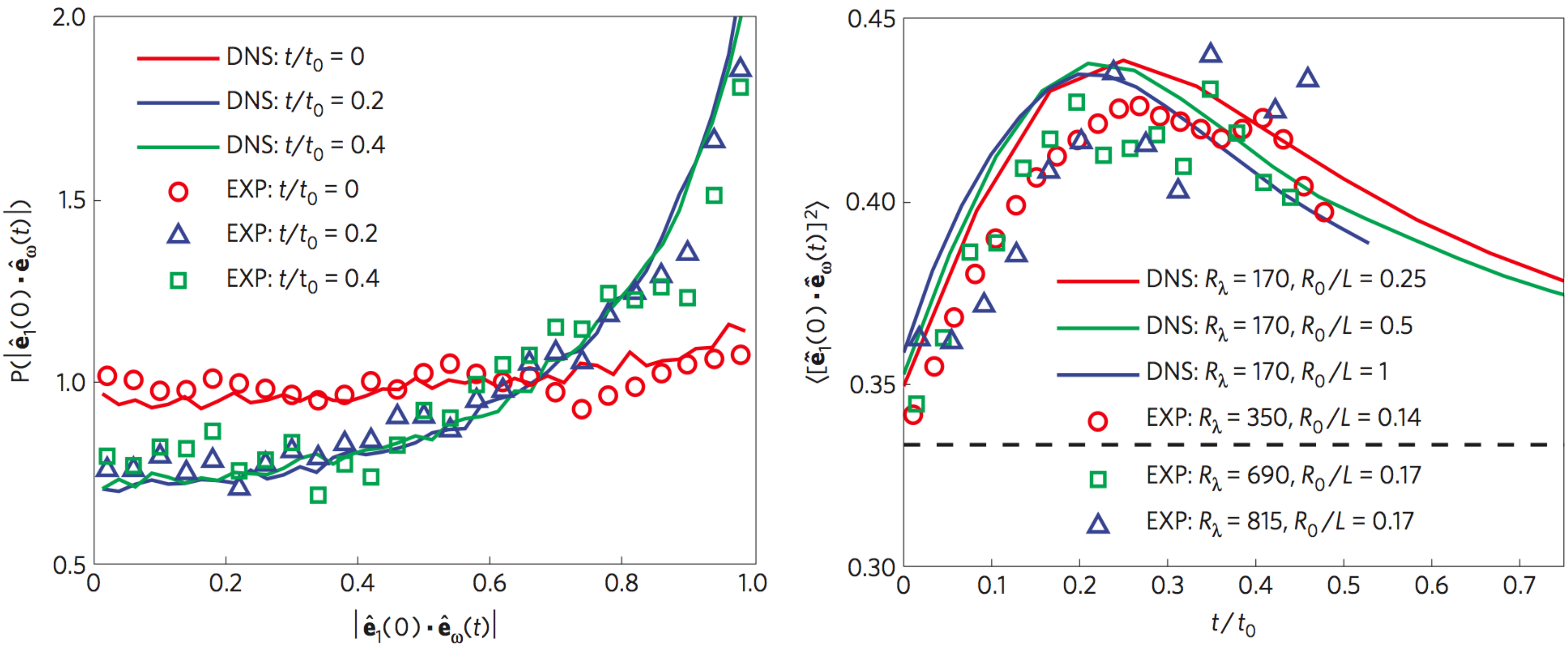
alignment dynamics

In the coordinates formed by the eigenvectors of strain $\hat{\mathrm{e}}_i(0)$

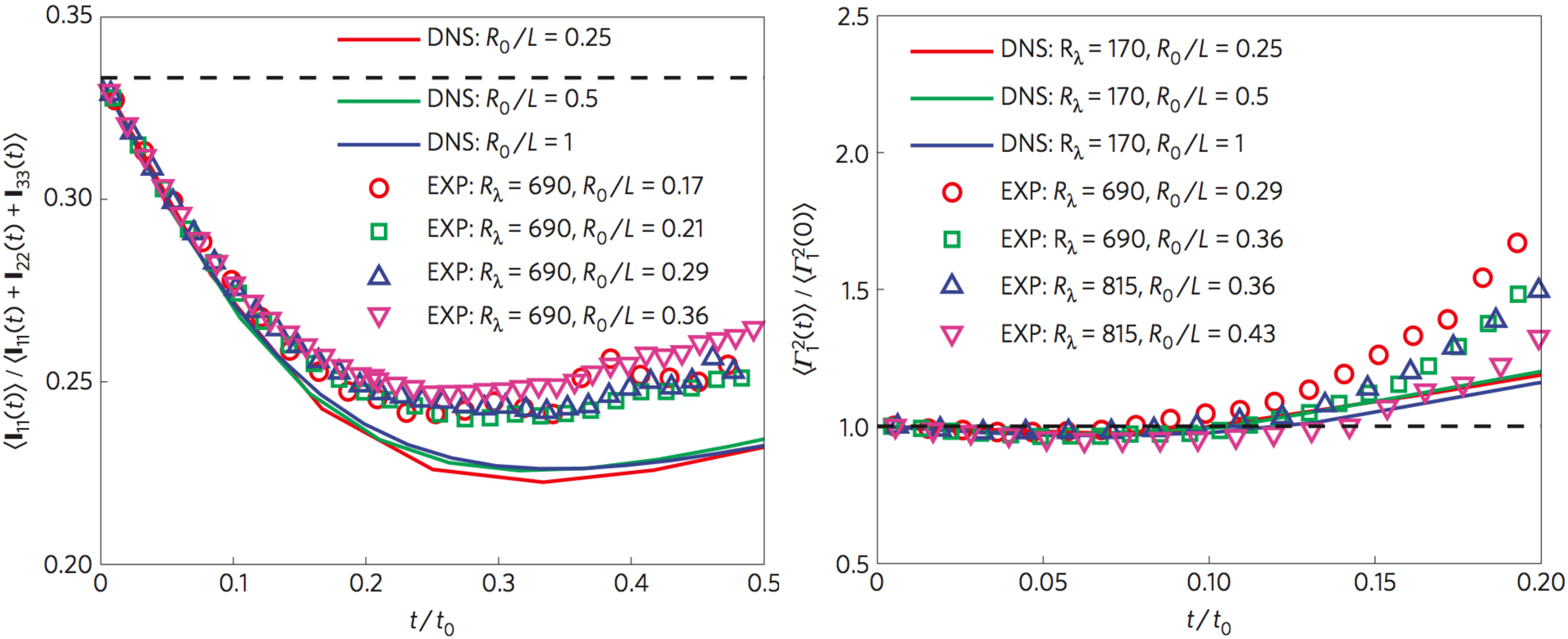


$$t_0 \equiv (r_0^2/\epsilon)^{1/3}$$

alignment dynamics



alignment dynamics



increase in vorticity and decrease in moment of inertia.

derivation from restricted Euler model

Vorticity dynamics:

$$\frac{\mathrm{d}\omega_i}{\mathrm{d}t} = (1-\alpha)S_{ij}\omega_j + \zeta\omega_i$$

Euler equation: $\alpha = 0 \quad \text{and} \quad \zeta = 0$

Tetrad model : $0 < \alpha < 1$

In the frame of $\hat{\mathbf{e}}_i(0)$, the eigenvectors of S_{ij}

$$\left.\frac{\mathrm{d}\omega_i}{\mathrm{d}t}\right|_{t=0} = (1-\alpha)\lambda_i\omega_i + \zeta\omega_i$$

Let $c_i(t) \equiv \hat{\mathbf{e}}_\omega(t) \cdot \hat{\mathbf{e}}_i(0)$

Note $\omega_i = \omega c_i \qquad \sum_i c_i^2 = 1$

$$\left.\frac{\mathrm{d}c_i^2}{\mathrm{d}t}\right|_{t=0} = 2(1-\alpha)c_i^2\left(\lambda_i - \sum_j \lambda_j c_j^2\right)$$

In component form:

$$\left.\frac{\mathrm{d}c_1^2}{\mathrm{d}t}\right|_{t=0} = 2(1-\alpha)c_1^2[c_2^2\overset{\geq 0}{\boxed{(\lambda_1-\lambda_2)}} + c_3^2\overset{\geq 0}{\boxed{(\lambda_1-\lambda_3)}}] \qquad \geq 0$$

$$\left.\frac{\mathrm{d}c_2^2}{\mathrm{d}t}\right|_{t=0} = 2(1-\alpha)c_2^2[c_1^2\underset{\leq 0}{\boxed{(\lambda_2-\lambda_1)}} + c_3^2\overset{\geq 0}{\boxed{(\lambda_2-\lambda_3)}}] \qquad \approx 0$$

$$\left.\frac{\mathrm{d}c_3^2}{\mathrm{d}t}\right|_{t=0} = 2(1-\alpha)c_3^2[c_1^2\underset{\leq 0}{\boxed{(\lambda_3-\lambda_1)}} + c_2^2\underset{\leq 0}{\boxed{(\lambda_3-\lambda_2)}}] \qquad \leq 0$$

Note that $\lambda_1 \geq \lambda_2 \geq \lambda_3$

derivation from extended Euler model

Vorticity dynamics:

$$\frac{\mathrm{d}\omega_i}{\mathrm{d}t} = (1-\alpha)S_{ij}\omega_j + \zeta\omega_i$$

Euler equation: $\alpha = 0 \quad \text{and} \quad \zeta = 0$

Tetrad model : $0 < \alpha < 1$

In the frame of $\hat{\mathbf{e}}_i(0)$, the eigenvectors of S_{ij}

$$\left.\frac{\mathrm{d}\omega_i}{\mathrm{d}t}\right|_{t=0} = (1-\alpha)\lambda_i\omega_i + \zeta\omega_i$$

Let $c_i(t) \equiv \hat{\mathbf{e}}_\omega(t) \cdot \hat{\mathbf{e}}_i(0)$

Note $\omega_i = \omega c_i \qquad \sum_i c_i^2 = 1$

summary

- perceived velocity gradient based on tetrads (4 points) in inertial scales.
- instantaneous alignment between vorticity and the intermediate stretching direction – also in inertial scales
- dynamic alignment between vorticity and the strongest initial stretching direction
- observed for inertial and dissipation scales.
- time scale is given $t_0 \equiv (r_0^2/\epsilon)^{1/3}$
- angular momentum is conserved for 0.1 $t_0(r_0, \varepsilon)$.

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