Adaptive Eigenvalue Computation for Elliptic Operators

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A Model Eigenvalue Problem

Poisson eigenvalue problem on $\Omega \subset \mathbb{R}^d$ with Dirichlet bc.:

$$-\Delta u = \lambda u, \quad u|_{\partial\Omega} = 0.$$

Weak formulation on $H_0^1(\Omega)$:

$$\underbrace{\int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x}_{:=a(u,v)} = \lambda \underbrace{\int_\Omega u(x)v(x)\,\mathrm{d}x,}_{L_2(\Omega) \text{ inner product}} \quad \text{for all } v \in H_0^1.$$

Properties of bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$:

- $a(u,v) = a(v,u)$ symmetric
- $a(u,v) \lesssim \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$ bounded
- $a(u,u) \gtrsim \|u\|^2_{H_0^1(\Omega)}$ strongly positive

for all $u, v \in H_0^1(\Omega)$.

$a(\cdot,\cdot)$ induces energy norm on $H_0^1(\Omega)$ equivalent to $\|\cdot\|_{H_0^1(\Omega)}$.

Abstract Eigenvalue Problems

Gelfand-triple of Hilbert- $(H, (\cdot, \cdot), |\cdot|)$ and Banach-space $(V, \|\cdot\|)$

$$V \overset{d}{\hookrightarrow} H \cong H^* \overset{d}{\hookrightarrow} V^*,$$

with dual pairing $\langle \cdot, \cdot \rangle$ on $V^* \times V$.

In model problem: $V = H_0^1(\Omega)$, $H = L_2(\Omega)$

Operator formulation:

$$Au = \lambda Eu \quad \text{in } V^*$$

with operators $A : V \to V^*$ and $E : H \to H^*$ such that

$$\langle Au, v \rangle = a(u, v), \quad \langle Eu, v \rangle = (u, v),$$

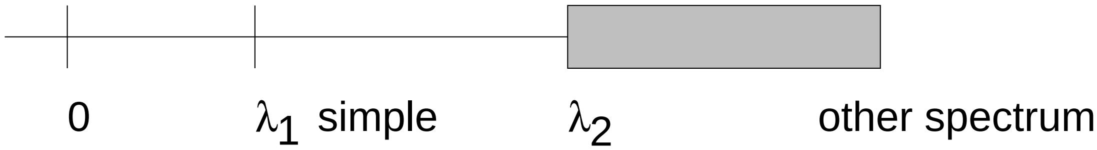
for given bounded, symmetric, strongly positive bilinear form $a : V \times V \to \mathbb{R}$.

In model problem: $A : H_0^1(\Omega) \to H^{-1}(\Omega)$.

Goal

Find lowest eigenpair (λ_1, u_1) of abstract eigenvalue problem

$$Au = \lambda Eu \quad \text{in } V^*.$$



Desired properties of algorithm:

- convergent with desired accuracy in finite number of steps,
- adaptive with quasi-optimal number of degrees of freedom,
- computational complexity linear in number of degrees of freedom.

Boundary value problems

Solution of boundary value problem

$$Au = f \quad \text{in } V^*$$

Optimal adaptive algorithm (Cohen/Dahmen/deVore 2001):

- Richardson-iteration with damping parameter γ^n:

$$u^{n+1} = u^n - \gamma^n B^{-1}(Au^n - f),$$

- stable wavelet-discretization for equivalent formulation in ℓ_2,
- approximation of operators (compressibility),
- nonlinear best N-term approximation.

Outline of the talk

(I) Iteration scheme and perturbations

(II) Numerical realization

(III) Optimality

(IV) (Preliminary) numerical results

Part I: Iteration scheme

Preconditioned inverse iteration

Steepest descent of Rayleigh-quotient

$$\mu(v) = \frac{\langle Av, v\rangle}{\langle Ev, v\rangle}, \quad \nabla\mu(v) \sim Av - \mu(v)Ev.$$

Recursion:

$$v' = v - \alpha B^{-1}(Av - \mu(v)Ev), \quad \text{(PINVIT)}$$

Preconditioner $B : V \to V^*$ with

$$\delta_0\langle Bv, v\rangle \leq \langle Av, v\rangle \leq \delta_1\langle Bv, v\rangle$$

Perfect preconditioner ($B = A$):

$$v' = \alpha\mu(v)A^{-1}Ev.$$

Convergence of PINVIT (I)

Thm 1 (D'yakonov/Orekhov, 1980). ***(matrix case)***
If $\mu(v) < \lambda_2$ *then the error in the Rayleigh-quotient decreases like*

$$\mu(v') - \lambda_1 \leq q \cdot (\mu(v) - \lambda_1), \quad q = q(\mu(v)) < 1,$$

from step to step, where the monotonically decreasing function $q(\mu)$ *depends asymptotically only on* δ_1/δ_0 *and* λ_2/λ_1.

Improvements: Samokish (1958), Bramble/Knyacev/Pasciak (1996), Neymeyr (2001), Knyacev/Neymeyr (200x)

Convergence of PINVIT (II)

Thm 2 (Rohwedder/Schneider/Z., 2007). ***(operator case)***
If $\mu(v) < \lambda_2$ *then the error in the Rayleigh-quotient decreases like*

$$\mu(v') - \lambda_1 \leq q \cdot (\mu(v) - \lambda_1), \quad q = q(\mu(v)) < 1,$$

from step to step, where the monotonically decreasing function $q(\mu)$ *depends asymptotically only on* δ_1/δ_0 *and* λ_2/λ_1*.*

Sketch of proof.
Main estimate. Temple-Kato like inequality:

$$\|Av - \mu(v)Ev\|_{A^{-1}}^2 \geq \mu(v)\frac{(\mu(v) - \lambda_1)(\lambda_2 - \mu(v))}{\lambda_1\lambda_2}|v|^2,$$

through spectral resolution. Rest of the proof: algebraic reasoning, analogously to D'yakonov/Orekhov (1980). □

Perturbed PINVIT

For every $\epsilon > 0$ perturbed preconditioned inverse iteration

$$v'_\epsilon = v - \alpha B^{-1}(A_\epsilon(v) - \mu_\epsilon(v)E_\epsilon(v)), \quad \text{(PPINVIT).}$$

with nonlinear approximations

$$\|A_\epsilon(v) - Av\|_* \leq \epsilon\|v\|, \quad |E_\epsilon(v) - Ev|_* \leq \epsilon|v|$$

and perturbed Rayleigh quotient

$$\mu_\epsilon(v) = \frac{\langle A_\epsilon(v), v\rangle}{\langle E_\epsilon(v), v\rangle}.$$

Convergence of PPINVIT (I)

Thm 3 (Rohwedder/Schneider/Z., 2007). ***(perturbed scheme)***
If $\mu(v) < \lambda_2$ then the error in the Rayleigh-quotient decreases like

$$\mu(v'_\epsilon) - \lambda_1 \leq q \cdot (\mu(v) - \lambda_1) + C\epsilon, \quad q = q(\mu(v)) < 1,$$

from step to step, where the monotonically decreasing function $q(\mu)$ depends asymptotically only on δ_1/δ_0 and λ_2/λ_1. C can be bounded independently of v and ϵ.

Sketch of proof.

Perturbation argument, i.e. bound perturbations by multiple of ϵ, e.g.

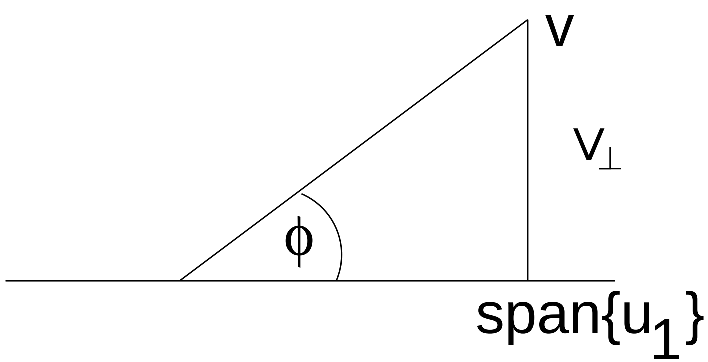
$$|\mu_\epsilon(v) - \mu(v)| \lesssim \epsilon, \quad \|v' - v'_\epsilon\| \lesssim \epsilon.$$

Resulting Rayleigh-quotient satisfies

$$|\mu(v') - \mu(v'_\epsilon)| \lesssim \epsilon,$$

proof by similar arguments as for matrices, see D'yakonov (1996). $\square$

Subspace convergence



Estimate for angle $\sin\phi \lesssim (\mu(v) - \lambda_1)^{1/2}$ (with respect to $\langle A\cdot, \cdot\rangle$).

For appropriately scaled vectors:

accuracy ϵ in $(V, \|\cdot\|) \sim$ application of A, E with tolerance ϵ^2

Convergence of PPINVIT (II)

Thm 4 (Dahmen/Rohwedder/Schneider/Z., 2007)**.** ***(improvement)***
If $\angle(v, u_1)$ is sufficiently small then the angle decreases like

$$\angle(v'_\epsilon, u_1) \leq \tilde{q} \cdot \angle(v, u_1) + \tilde{C}\epsilon, \quad \tilde{q} < 1$$

from step to step, and $\tilde{C}$ can be bounded independently of v and ϵ.

Sketch of proof.

Concentration on vector and its projection onto eigenspace.
Estimation of reduction perpendicular to eigenspace through ellipticity. □

For appropriately scaled vectors:

accuracy ϵ in $(V, ||\cdot||) \sim$ application of A, E with tolerance ϵ

A convergent algorithm

Starting vector v^0 and target accuracy τ

$\mathrm{ADAPTIVE}(v^0, \tau)$

Require: $\angle(v^0, u_1)$ sufficiently small

$\tau^0 \leftarrow \angle(v^0, u_1)$

$i \leftarrow 0$

while $\tau^i > \tau$ **do**

$v^{i+1} \leftarrow \mathrm{PPINVIT}(v^i, \alpha\tau^i, N)$ $\{N$ steps with tolerance $\alpha\tau^i\}$

$\tau^{i+1} \leftarrow \frac{1}{2}\tau^i$

$i \leftarrow i + 1$

end while

return v^i .

N steps of PPINVIT to halve the error, independent of v^i and τ^i.

Part II: Numerical realization

Stable wavelet bases

For basis $(\psi_i)_{i\in\mathcal{I}}$ with index set $\mathcal{I}$

$$u = \sum_{i\in\mathcal{I}} u_i\psi_i \in V \Leftrightarrow \mathbf{u} = \begin{pmatrix} \vdots \\ u_i \\ \vdots \end{pmatrix} \in \ell_2(\mathcal{I}).$$

Stable bases (Riesz bases) for H and V:

$$\|\mathbf{u}\|_{\ell_2(\mathcal{I})} \sim \|u\|, \quad \|\mathbf{D}^{-1}\mathbf{u}\|_{\ell_2(\mathcal{I})} \sim |u|, \quad \mathbf{D} \text{ diagonal.}$$

Standard example:

$$V = H^t(\Omega),\ H = L_2(\Omega),\ \mathbf{D} = (\delta_{ij}2^{t|i|})_{i,j\in\mathcal{I}},$$

where $|i|$ level of ψ_i.

Equivalent ℓ_2 problem

With stable basis $(\psi_i)_{i\in\mathcal{I}}$

$$Au = \lambda E u \Leftrightarrow \mathbf{A}\mathbf{u} = \lambda \mathbf{E}\mathbf{u},$$

where

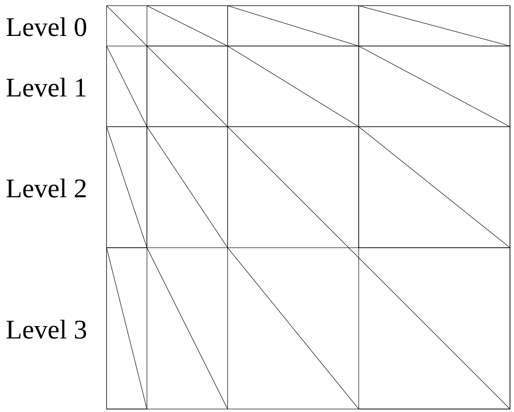
$$\mathbf{A} = (\langle A\psi_i, \psi_j\rangle)_{i,j\in\mathcal{I}}, \quad \mathbf{E} = (\langle E\psi_i, \psi_j\rangle)_{i,j\in\mathcal{I}}.$$

Built-in preconditioning:

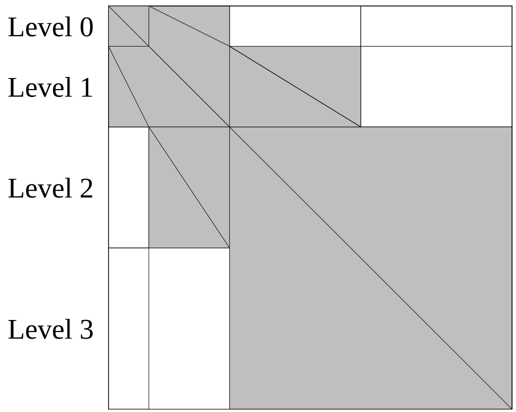
$$\langle \mathbf{A}\mathbf{u}, \mathbf{u}\rangle_{\ell_2} \sim \|\mathbf{u}\|_{\ell_2}^2$$

equivalent problem to $Au = \lambda E u$ through norm-equivalence

Quasi-sparsity



Original matrix $\mathbf{B}$.



Compressed matrix $\mathbf{B}_j$.

Controlled approximation: $\|\mathbf{B} - \mathbf{B}_j\|_{\ell_2} \approx 2^{-js}$, with $\approx 2^j$ entries per row/column (s^*-computable).

- wide class of discretized operators quasi-sparse
- simple approximation of $\mathbf{A}\mathbf{u}$ through

$$\mathbf{B}_\epsilon(\mathbf{u}) = \mathbf{B}_j \mathbf{u}, \quad j \text{ appropriate.}$$

- more advanced nonlinear scheme in Cohen/Dahmen/deVore (2001).

Exploding support

Approximate operator application:

$$\#\mathbf{v} < \infty \Rightarrow \#\mathbf{A}_\epsilon(\mathbf{v}), \#\mathbf{E}_\epsilon(\mathbf{v}) < \infty$$

with $\#\mathbf{v}$ number of non-zero entries.

A computable algorithm:

- Initial vector $\mathbf{v}^0$ with $\#\mathbf{v}^0 < \infty$
- Iterate

$$\mathbf{v}^{n+1} = \mathbf{v}^n - \alpha\left(\mathbf{A}_\epsilon(\mathbf{v}^n) - \mu_\epsilon(\mathbf{v}^n)\mathbf{E}_\epsilon(\mathbf{v}^n)\right)$$

Support of $\mathbf{v}^n$ stays finite but explodes:

$$\#\mathbf{v}^n \sim C^n, \quad C > 1.$$

Non-optimal behavior in number of degrees of freedom.

Part III: Optimality

Adaptivity

Let $u \in V$

$$u = \sum_{i \in \mathcal{I}} u_i \psi_i \Rightarrow \mathbf{u} = \begin{pmatrix} \vdots \\ u_i \\ \vdots \end{pmatrix}.$$

For finite index set $\mathcal{I}_0$ with $u_i = 0$, $i \notin \mathcal{I}_0$:

$$u = \sum_{i \in \mathcal{I}_0} u_i \psi_i \quad \text{finite sum} \Leftrightarrow \mathbf{u} = \begin{pmatrix} \vdots \\ 0 \\ * \\ \vdots \end{pmatrix} \quad \text{finite number of entries.}$$

Non-zero entry $u_i \leftrightarrow$ basis function ψ_i needed in expansion.

Best-N-term approximation

Best-approximation in $\ell_2(\mathcal{I})$ with N non-zero elements:

$$\sigma_N(\mathbf{u}) = \inf\{\|\mathbf{u}-\mathbf{v}\|_{\ell_2}, \#\mathbf{v} \leq N\}.$$

Approximation space

$$\mathcal{A}^s = \{\mathbf{u} \in \ell_2(\mathcal{I}) : \|\mathbf{u}\|_{\ell_2} + |\mathbf{u}|_{\mathcal{A}^s} < \infty\}, \quad |\mathbf{u}|_{\mathcal{A}^s} = \sup_{N\geq 1} N^s \sigma_N(\mathbf{u}).$$

Approximation of $\mathbf{u} \in \mathcal{A}^s$

accuracy $\epsilon \leftrightarrow$ ***number of degrees of freedom*** $\epsilon^{-1/s}|\mathbf{u}|_{\mathcal{A}^s}$

Coarsening and approximation

Thm 5 (Cohen/Dahmen/deVore, 2001)**.**
Coarsening with appropriate tolerance of approximation $\mathbf{v}$ *with*

$$\|\mathbf{v} - \mathbf{u}\| \leq \epsilon, \quad \mathbf{u} \in \mathcal{A}^s$$

leads to quasi-optimal approximation $\mathbf{w}$*:*

$$\|\mathbf{w}\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}, \quad \#\mathbf{w} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s}^{1/s} \epsilon^{-1/s}, \quad \|\mathbf{u} - \mathbf{w}\| \lesssim \epsilon.$$

Time to time coarsening prevents exponential growth of the iterates.

Optimal adaptive algorithm

Starting vector v^0, target accuracy τ.

$\mathrm{MINIEIG}(v^0, \tau)$

Require: $\angle(v^0, u_1)$ sufficiently small

$\tau^0 \leftarrow \angle(v^0, u_1)$

$i \leftarrow 0$

while $\tau^i > \tau$ **do**

$\breve{v}^{i+1} \leftarrow \mathrm{PPINVIT}(v^i, \alpha_1 \tau^i, N)$ $\{\alpha_1$ fixed$\}$

$v^{i+1} \leftarrow \mathrm{APPROX}(\breve{v}^{i+1}, \alpha_2 \tau^i)$ $\{\alpha_2$ fixed, Coarsening step$\}$

$\tau^{i+1} \leftarrow \frac{1}{2}\tau^i$

$i \leftarrow i + 1$

end while

return v^i

Simplified version from Dahmen/Rohwedder/Schneider/Z. (2007).
Includes efficient and reliable error estimator.

Optimal convergence

Thm 6 (Dahmen/Rohwedder/Schneider/Z., 2007)**.**
For compressible operators $\mathbf{A}$ *and* $\mathbf{E}$*, the algorithm gives for each target accuracy* τ *a vector* $\mathbf{u}_\tau$*:*

$$\angle(\mathbf{u}_\tau, \mathbf{u}_1) \lesssim \tau, \quad |\mu_\tau(\mathbf{u}_\tau) - \lambda_1| \lesssim \tau.$$

If the eigenvector $\mathbf{u}_1 \in \mathcal{A}^s$ *then*

$$\#\mathbf{u}_\tau \lesssim \tau^{-1/s}$$

given sufficiently compressible operators. The floating point operations remain bounded linearly.

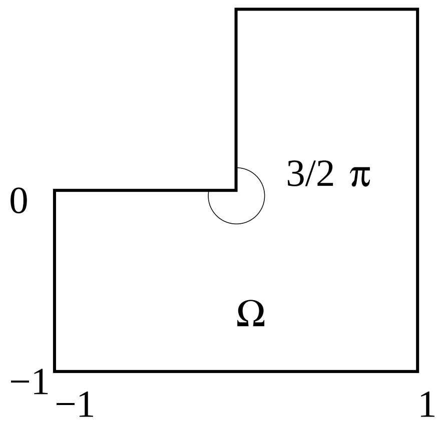
Sketch of proof.
Convergence of $\mathrm{PPINVIT}$, estimate perturbation due to coarsening.
Bookkeeping of $\|\cdot\|_{\mathcal{A}^s}$, support sizes and flops. $\square$

Part IV: (Preliminary) numerical results

A model problem

Poisson eigenvalue problem in $\mathbb{R}^2$:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{on } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$



- Reduced Sobolev regularity due to corner. Classical theory:

$$u_1 \in H^s, \quad s < 1 + \frac{\pi}{\alpha} = \frac{5}{3}$$

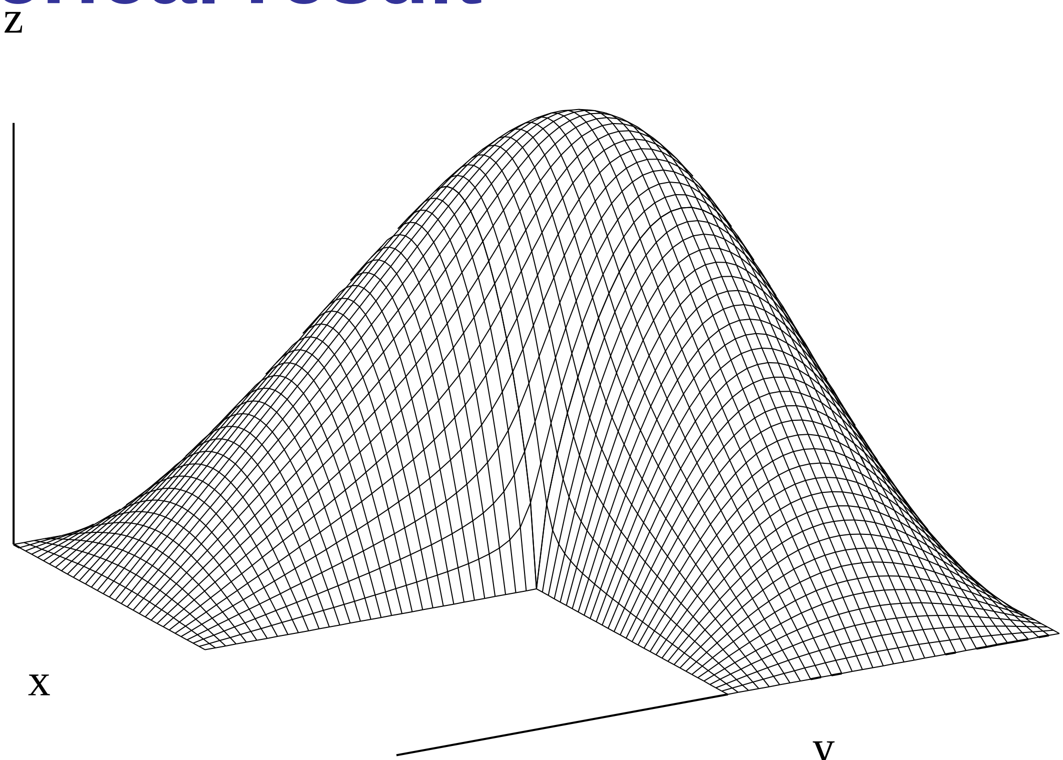
rates: $\|u_1 - u_h\| \lesssim N^{-1/3}$, $|\lambda_1 - \mu(u_h)| \lesssim N^{-2/3}$.

- Higher Besov regularity (Dahlke 1999)

$$u_1 \in B^{2s+1}_{\tau,\tau}, \quad \frac{1}{\tau} = s + \frac{1}{2}, \quad 0 < s < \frac{13}{12}.$$

rates: $\|u_1 - u_h\| \lesssim N^{-1/2}$, $|\lambda_1 - \mu(u_h)| \lesssim N^{-1}$
due to linear wavelets. Likely not optimal (bootstrapping).

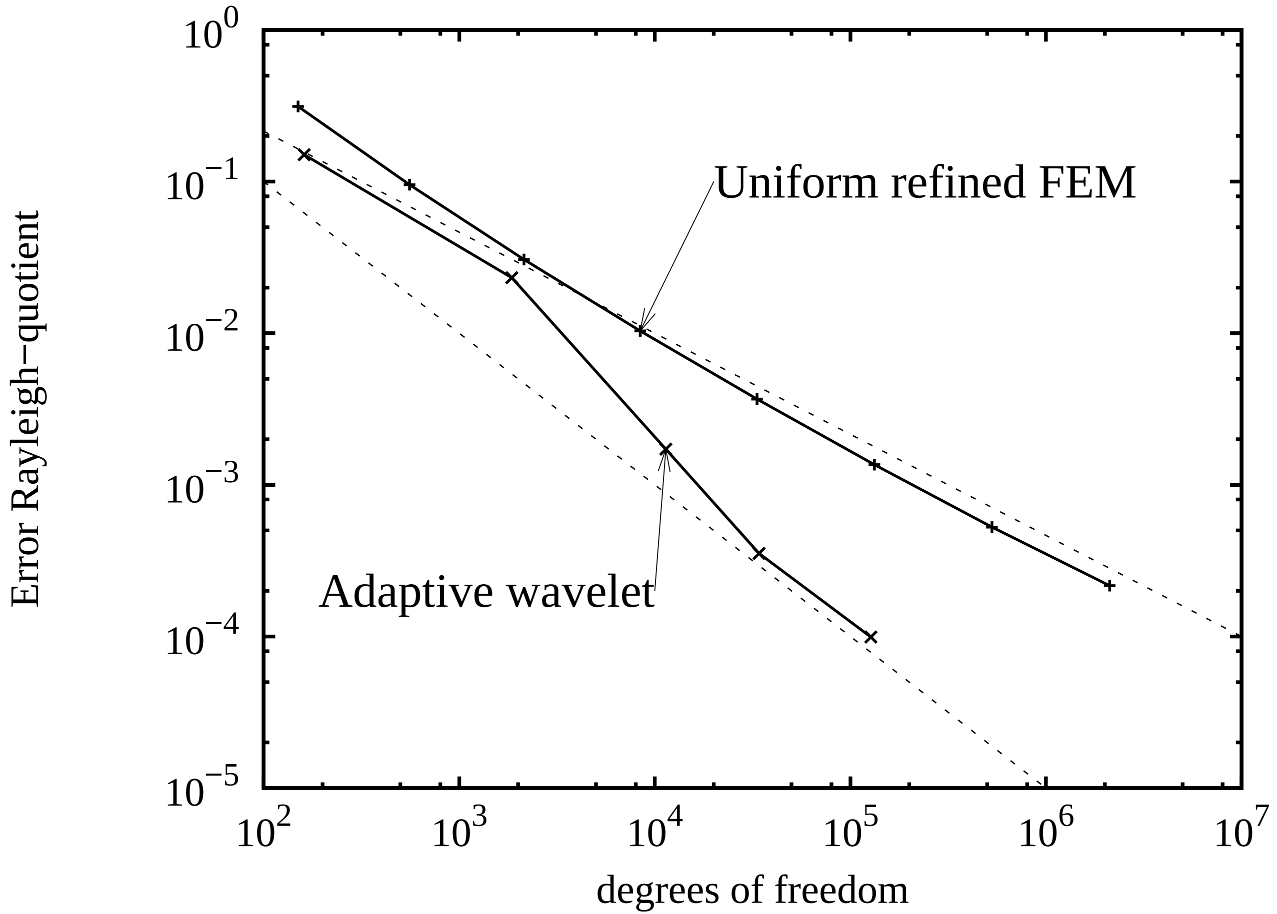
Numerical result



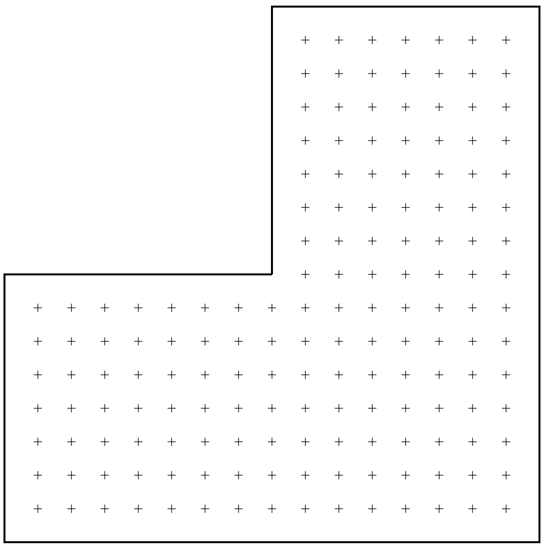
Joint work with Jürgen Vorloepper (RWTH Aachen).

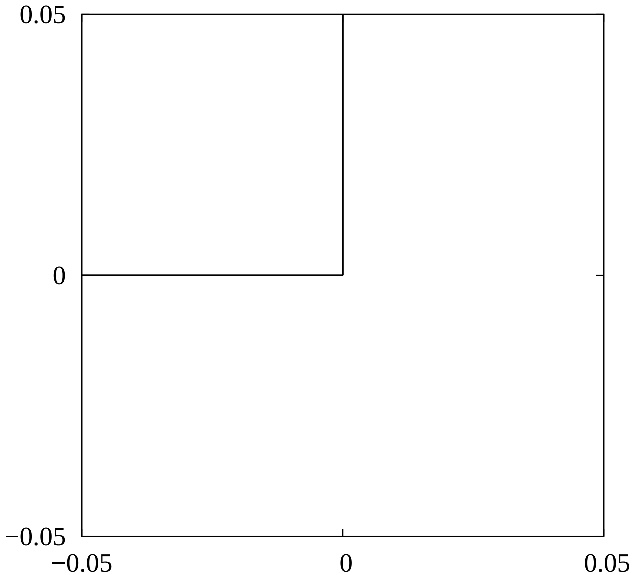
- Linear wavelets with tree-structured index sets
- Exact application of operators for given index set
- Galerkin-solution with frozen index sets

Convergence rates



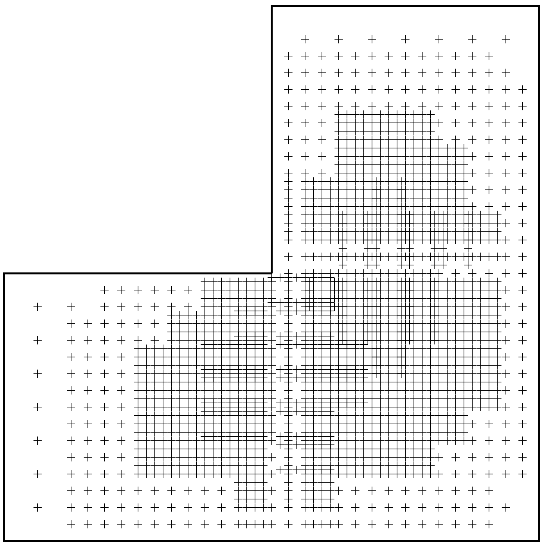
Index sets - Initial iterate

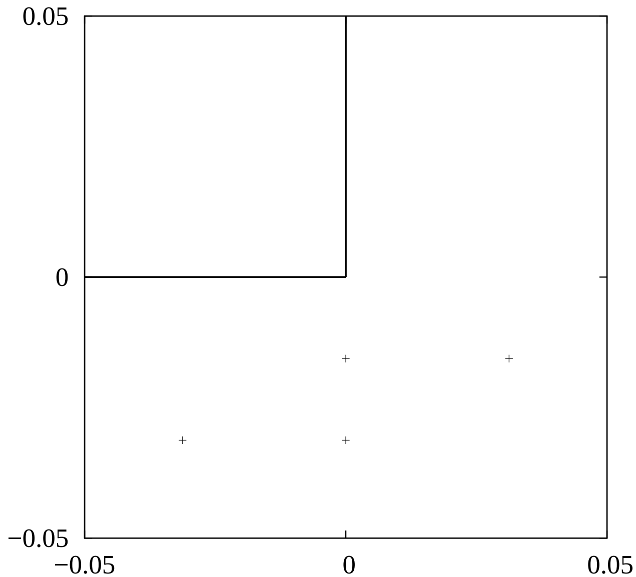




161 degrees of freedom

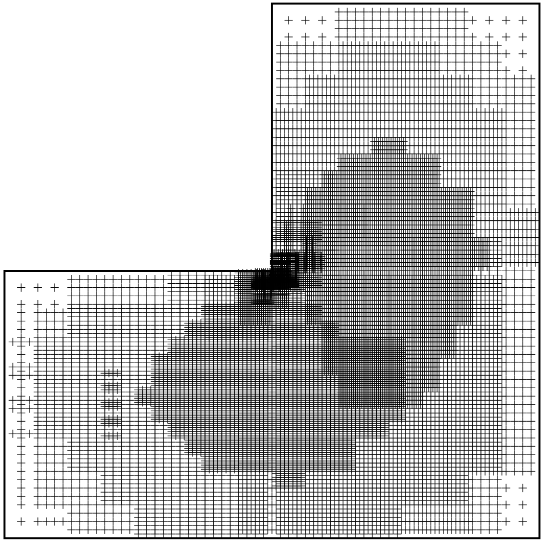
Index sets - 1st iterate

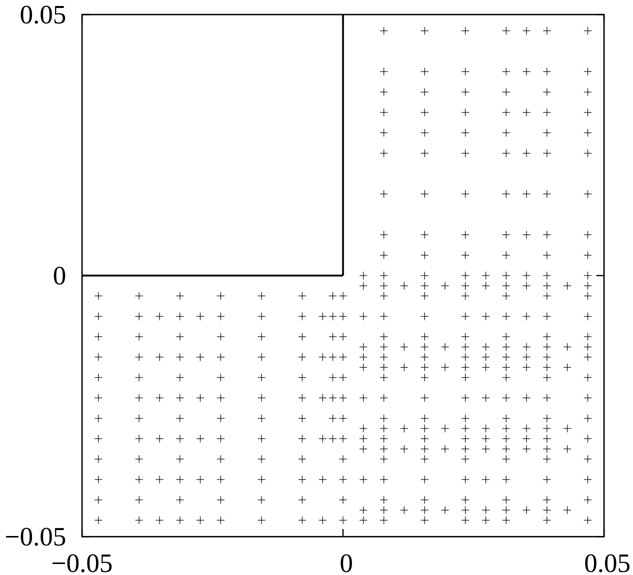




1859 degrees of freedom

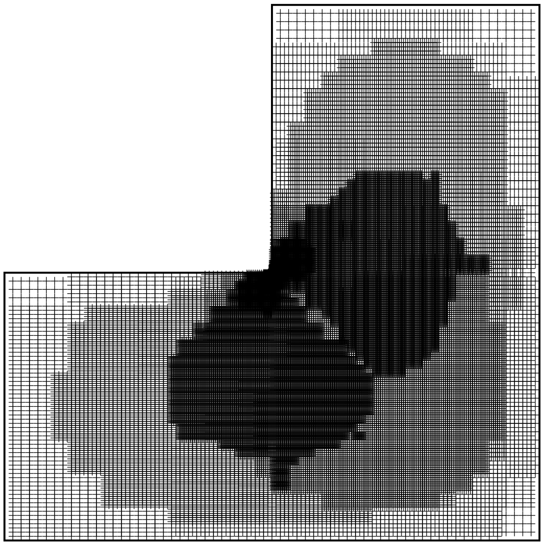
Index sets - 2nd iterate

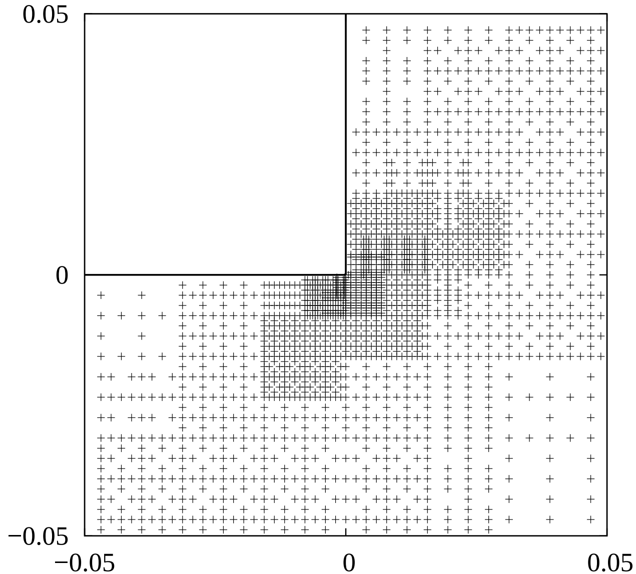




11393 degrees of freedom

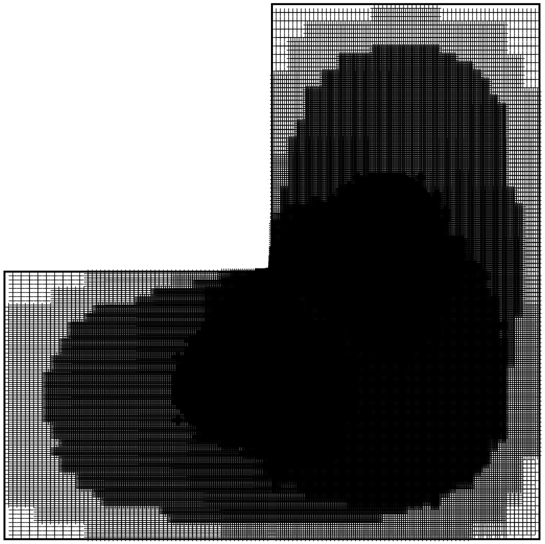
Index sets - 3rd iterate

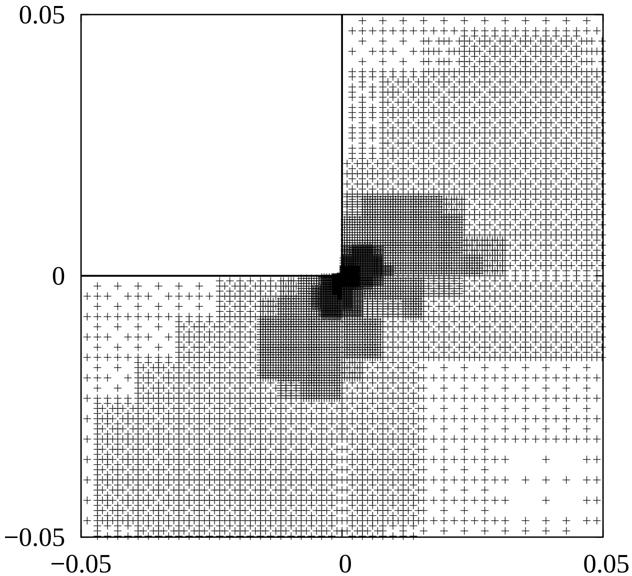




34124 degrees of freedom

Index sets - 4th iterate





127355 degrees of freedom

Summary

- Preconditioned inverse iteration as abstract iteration scheme
- Robustness of PINVIT against perturbations
- Numerical realization through stable wavelet bases
- Optimality through coarsening
- First (preliminary) numerical realization

References

[1] T. Rohwedder, R. Schneider, A. Zeiser, *Perturbed preconditioned inverse iteration for operator eigenvalue problems with applications to adaptive wavelet discretizations*, submitted 2007.

[2] W. Dahmen, T. Rohwedder, R. Schneider, A. Zeiser, *Adaptive Eigenvalue Computation – Complexity Estimates*, submitted 2007.

available at `http://www.math.tu-berlin.de/~zeiser`.

Thank you for your attention.