Towards an adaptive scheme for convection-diffusion problems stabilized in a graph norm

Albert Cohen, Wolfgang Dahmen, Gerrit Welper

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Problem

The convection-diffusion equation is given by: find $u \in H_0^1(\Omega)$ s.t.

$$\epsilon(\nabla u, \nabla v) + (\mathbf{b}\nabla u, v) + (cu, v) = \langle f, v\rangle, \qquad \forall v \in H_0^1(\Omega)$$

Assumptions:

- $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$
- There are $c_0, c_b \geq 0$ s.t.

$$-\frac{1}{2}\text{div} b + c \geq c_0, \qquad \|c\|_{L^\infty} \leq c_b c_0$$

- $\epsilon \ll 1$

Problem:

- The problem is ill-conditioned $\rightsquigarrow$ non-physical oscillations.

Abstract setting and norms

Let $Y = X = H_0^1(\Omega)$ and $a : X \times Y \to \mathbb{R}$ be a coercive and continuous bilinear form. Define the bilinear forms

$$
\begin{aligned}
a_{sy}(u,v) &:= \frac{1}{2}\left(a(u,v) + a(v,u)\right) \\
a_{sk}(u,v) &:= \frac{1}{2}\left(a(u,v) - a(v,u)\right)
\end{aligned}
$$

and the norms

$$
\begin{aligned}
\|\cdot\|_Y^2 &:= a_{sy}(u,u) \\
\|\cdot\|_X^2 &:= \|\cdot\|_Y^2 + \|A_{sk}\cdot\|_{Y'}^2.
\end{aligned}
$$

see [Sangalli, 2005] and [Verfürth, 2005] . Then we have:

$$
a(u,v) \lesssim \|u\|_X \|v\|_Y, \qquad 1 \lesssim \inf_{u \in X} \sup_{v \in Y} \frac{a(u,v)}{\|u\|_X \|v\|_Y}
$$

where the constants are independent of ϵ and b.

Auxiliary Problem

- Find $u \in X$ such that

$$a(u, v) + \beta \langle Au, A_{\mathrm{sk}} v \rangle_{Y'} = \langle f, v \rangle + \beta \langle f, A_{\mathrm{sk}} v \rangle_{Y'} \quad \forall v \in X.$$

  where $\beta > 0$ is a parameter (similar to [Bertoluzza, Canuto, Tabacco, 2000]).

- For a numerical realization we have to evaluate the Y'-scalar product.
- Compare with the SUPG method: find $u \in X_h \subset H^1$ such that

$$\begin{aligned} a(u_h, v_h) + \sum_{K \in \mathcal{T}} \delta \left(Au_h, \frac{h_K}{|b|} A_{\mathrm{sk}} v_h \right)_K \\ = \quad \langle f_h, v_h \rangle + \sum_{K \in \mathcal{T}} \delta \left(f_h, \frac{h_K}{|b|} A_{\mathrm{sk}} v_h \right)_K \end{aligned}$$

  for all $v_h \in X_h$ where $\delta > 0$ is a parameter.

Equivalence and Mapping properties

Equivalence:

$$a(u,v) = \langle f, v \rangle, \qquad \forall v \in X$$

$$\Leftrightarrow$$

$$a(u,v) + \langle Au, A_{\mathsf{sk}} v \rangle_{Y'} = \langle f, v \rangle + \langle f, A_{\mathsf{sk}} v \rangle_{Y'}, \qquad \forall v \in X$$

Mapping Properties: For $u, v \in X$ we have:

$$\begin{aligned} a(u,v) + \langle Au, A_{\mathsf{sk}} v \rangle_{Y'} &\lesssim \|u\|_X \|v\|_X \\ a(u,u) + \langle Au, A_{\mathsf{sk}} u \rangle_{Y'} &\gtrsim \|u\|_X^2 \\ \|\langle f, \cdot \rangle + \langle f, A_{\mathsf{sk}} \cdot \rangle_{Y'}\|_{X'} &\sim \|f\|_{Y'} \end{aligned}$$

where all constants are independent of ϵ and b.

Eliminating the Y'-scalar product

- By the definition of the Y-norm we have:

$$\langle u, v \rangle_{Y'} = \langle u, A_{\mathrm{sy}}^{-1} v \rangle$$

- Define

$$\mathbb{X} := X \times Y \times Y$$

- Then an equivalent problem without the Y'-scalar product is: find $\mathbf{U} = [u, y, w] \in \mathbb{X}$ s.t.

$$\begin{array}{rcrcrcl} a(u,v) & - & \beta a_{\mathrm{sk}}(y,v) & + & \beta a_{\mathrm{sk}}(w,v) & = & \langle f, v \rangle \\ -\beta a(u,z) & + & \beta a_{\mathrm{sy}}(y,z) & & & = & 0 \\ & & & & \beta a_{\mathrm{sy}}(w,r) & = & \beta \langle f, r \rangle \end{array}$$

for all $\mathbf{V} = [v, z, r] \in \mathbb{X}$.

- This problem defines a bounded linear operator

$$\mathbf{A} : \mathbb{X} \to \mathbb{X}'$$

which fulfills an inf-sup condition.

Discretization

- Assume we have finite dimensional spaces $\mathbb{X}_h = X_h \times Y_h^y \times Y_h^w \subset \mathbb{X}$.
- Galerkin discretization: find $\mathbf{u}_h = [u_h, y_h, w_h] \in \mathbb{X}_h$ s.t.

$$\begin{array}{rcccccl} a(u_h, v_h,) & - & \beta a_{\rm sk}(y_h, v_h,) & + & \beta a_{\rm sk}(w_h, v_h,) & = & \langle f, v_h, \rangle \\ -\beta a(u_h, z_h,) & + & \beta a_{\rm sy}(y_h, z_h,) & & & = & 0 \\ & & & & \beta a_{\rm sy}(y_h, r_h,) & = & \beta \langle f, r_h, \rangle \end{array}$$

  for all $\mathbf{v}_h = [v_h, z_h, r_h] \in \mathbb{X}_h$.
- The operator $\mathbf{A}$ is not coercive $\rightsquigarrow$ what do we now about the quality of the solution?

The inf-sup condition

Assume we have an operator $P : Y_h \to Y_h$ s.t. for $v \in X_h$ we have

$$\begin{aligned} \|(I - P)S^{-1}A_{\mathrm{sk}}v\|_Y &\leq c\|S^{-1}A_{\mathrm{sk}}v\|_Y \\ \|PS^{-1}A_{\mathrm{sk}}v\|_Y &\lesssim \|S^{-1}A_{\mathrm{sk}}v\|_Y \end{aligned}$$

where the constant c of the first estimate and β are sufficiently small (independent of ϵ). Then we have:

$$\begin{aligned} &\left(\|u\|_X^2 + \|w\|_Y^2\right)^{\frac{1}{2}} \left(\|u\|_X^2 + \|w - PS^{-1}A_{\mathrm{sk}}u\|_Y^2\right)^{\frac{1}{2}} \\ \lesssim \quad &\langle \mathbf{A}[u, w], [u, w - PS^{-1}A_{\mathrm{sk}}u]\rangle. \end{aligned}$$

Furthermore $\mathbf{A} : X \times Y \to X' \times Y'$ is an isomorphism with condition number independent of ϵ and b.

Notations

Let $X_0 \subset X_1 \subset \cdots \subset X$ be a sequence of finite dimensional subspaces. Define

$$E_n(u)_X := \sup_{\phi \in X_n} \|u - \phi\|_X$$

$$\|u\|_{\mathcal{A}_2^s(X_n)_X} := \|u\|_X + \left(\sum_{n=1}^{\infty} [2^{sn} E_n(u)_X]^2 \right)^{\frac{1}{2}}$$

Discretization

- Approximate $u \in X$ from X_n and $w \in Y$ from Y_m
- Assume:

$$\|S^{-1}A_{\mathrm{sk}}\phi\|_{\mathcal{A}_2^s(Y_n)_Y} \lesssim 2^{sn}\|S^{-1}A_{\mathrm{sk}}\phi\|_Y, \qquad \text{for all } \phi \in X_n$$
$$\|S^{-1}A_{\mathrm{sy}}v\|_{\mathcal{A}_2^t(Y_n)_Y} \lesssim \|v\|_{\mathcal{A}_2^t(Y_n)_Y}, \qquad 0 < t < s$$

- Let $m \geq n + c$ with a suitable constant c independent of ϵ.
- Then for a usual Galerkin approximation (with given modified right hand side) we have:

$$\left(\|u - u_h\|_X^2 + \|y - y_h\|_Y^2\right)^{\frac{1}{2}} \lesssim 2^{-tn}\|u\|_{\mathcal{A}_2^t(X_n)_X}$$

- The requirements can be fulfilled for our example on the unit cube by wavelets when $b \in X_n^d$.

Perturbed coercivity

- Assume that

$$\|y(u_h) - y_n\|_Y + \|w - w_n\|_Y \le c\|y_n - w_n\|_Y \qquad (1)$$

  for a sufficiently small constant $c > 0$. Then we have:

$$\|u - u_n\|_X \sim \|y_n - w_n\|_Y$$

  and

$$\|u - u_n\|_X \lesssim \inf_{\phi \in X_n} \|u - \phi\|_X$$

- The terms on the left hand side of (1) can be estimated by known a posteriori error estimators e.g. by Verfürth.
- The terms on the right hand side of (1) can easily be computed.
- $\rightsquigarrow$ We can test a-posteriori if it makes sense to refine the grid of the variable u or the grids of the auxiliary variables y and w.

A-posteriori error estimates 1

In the following slides we construct an a-posteriori error estimator analogous to the ones of Verfürth for convection diffusion problems.

Let T be a cell and κ be an edge of the triangulation. Define:

$$R_{T,u} := (f + \epsilon\Delta u_h - A_{\mathrm{sk}} u_h - c u_h)|_T \qquad J(u_h)_\kappa := \epsilon \left[\frac{\partial u_h}{\partial n}\right]_\kappa$$

$$R_{T,y} := (\epsilon\Delta(y_h - u_h) + A_{\mathrm{sk}} u_h + c(u_h - y_n))|_T \qquad J(u_h)_\kappa := \epsilon \left[\frac{\partial y_h}{\partial n}\right]_\kappa$$

$$R_{T,w} := (f + \epsilon\Delta w_h - c u_h)|_T \qquad J(u_h)_\kappa := \epsilon \left[\frac{\partial w_h}{\partial n}\right]_\kappa$$

A-posteriori error estimates 2

Define:

$$\alpha_S := \min\{\epsilon^{-1/2} h_S, c_0^{-1/2}\}, \quad S \in \{T, \kappa\}, \quad h_S := \operatorname{diam} S$$

and

$$\eta_{T,\Box} := \alpha_T^2 \|R_{T,\Box}\|_{L_2(T)}^2 + \frac{1}{2} \sum_{\kappa \in \partial T} \epsilon^{-1/2} \alpha_\kappa \|J_\kappa(\Box_h)\|_{L_2(\kappa)}^2$$

where $\Box \in \{u, y, w\}$.

A-posteriori error estimates 3

Now we can define the following error estimator:

$$
\begin{aligned}
R_h^2 \quad := \quad & \sum_{T \in \mathcal{T}_u} \eta_{T,u}^2 + \|y_h - w_h\|_Y^2 \\
& + \sum_{T \in \mathcal{T}_y} \eta_{T,y}^2 \\
& + \sum_{T \in \mathcal{T}_w} \eta_{T,w}^2
\end{aligned}
$$

A-posteriori error estimates 4

With the given definitions we get the following estimates:

$$\|U - U_h\|_{\mathbb{X}} \lesssim R_h^2 + \text{data errors}$$

and

$$R_h^2 \lesssim \|U - U_h\|_{\mathbb{X}} + \text{data errors}$$

A-posteriori error estimates 4

With the given definitions we get the following estimates:

$$\|U - U_h\|_{\mathbb{X}} \lesssim R_h^2 + \text{data errors}$$

and

$$R_h^2 \lesssim \|U - U_h\|_{\mathbb{X}} + \text{data errors}$$

Outline of a solution method

One might find a solution $\mathbf{U}_h = [u_h, y_h, w_h]$ of the auxiliary system with accuracy $\|\mathbf{U} - \mathbf{U}_h\|_{\mathbb{X}} \lesssim \delta$ by the following algorithm:

while $R_h > \delta$ **do**
{
  compute error estimators of u_h
  refine u_h
  solve the discrete system
  while not (1) **do**
  {
    compute error estimators of y_h and w_h
    refine y_h and w_h
    solve the discrete system
  }
}

Goal

- Adaptive finite element schemes are usually of the form

$$\text{estimate} \longrightarrow \text{mark} \longrightarrow \text{refine} \longrightarrow \text{solve}$$

- Let u_H, u_h be two consecutive solutions in such a scheme.
- A typical result of the convergence analysis is the error reduction

$$\|u - u_h\| \leq \theta \|u - u_H\|$$

with $\theta < 1$.

Example

- We treat the problem

$$-10^{-5}u'' + u' + u = 1$$

  with zero boundary conditions on uniform grids.

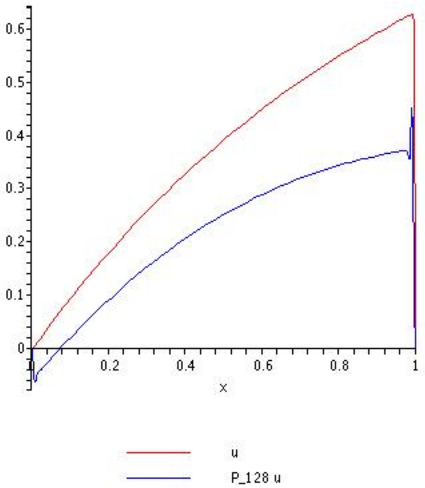
- Here for our proposed schemes the Y'-scalar product and the error the X-norm are computed exactly.

#cells	$\|u - u_h\|_X$	$\|u - u_h\|_X / \|u - u_H\|_X$
4	0.959861	
8	0.957969	0.998029
16	0.957373	0.999378
32	0.956982	0.999592
64	0.956269	0.999255
128	0.954335	0.997978
256	0.949497	0.994931
512	0.939050	0.988997

$\rightsquigarrow$ we cannot expect an error reduction

Example

$$-10^{-3}u'' + u' + u = 1, \qquad u(0) = u(1) = 0$$



Possible Problems

- The error estimator contains the term $\|y_n - w_n\|_Y$ which possibly cannot be treated with standard arguments
- The proof of the lower bounds of the error estimator contains anisotropic bubble functions $\rightsquigarrow$ problems with the error reduction.
- A reason for the shifts might be the following heuristically argument:

$$\begin{aligned} \langle u - u_h, 1\rangle &= \langle A_x(u-u_h), A_x^{-1}1\rangle \\ &= \langle A_x(u-u_h), A_x^{-1}1-\phi\rangle \\ &\leq \|u-u_h\|_X \|A_x^{-1}1-\phi\|_X \end{aligned}$$

where $\phi \in X_h$. But $\|A_x^{-1}1 - \phi\|_X$ will be large.

Stabilization in the Y-norm

- Solve the normal equations in the Y-norm: find $u \in Y$ s.t.

$$\langle Au, Av \rangle_{X'} = \langle f, Av \rangle_{X'}$$

  for all $v \in Y$.

- Define

$$\mathbb{Y} := Y \times X \times Y$$

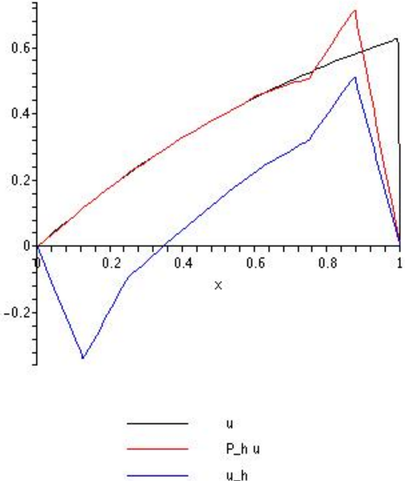
- we get the equivalent problem: find $\mathbf{U} = [u, y, w] \in \mathbb{Y}$ s.t.

$$\begin{array}{rcrcrcl} a_{\mathrm{sy}}(u,v) & - & a(v,y) & & & = & 0 \\ & & a_{\mathrm{sy}}(y,z) & - & a_{\mathrm{sk}}(w,z) & = & \langle f, r \rangle \\ & - & a_{\mathrm{sk}}(y,r) & + & a_{\mathrm{sy}}(w,r) & = & 0 \end{array}$$

  for all $\mathbf{V} = [v, z, r] \in \mathbb{Y}$.

Example

$$\frac{\|u - u_h\|_Y}{\|u - P_h u\|_Y} = 1.1501 \qquad \epsilon = 10^{-3} \qquad \sharp\text{cells} = 8, 16, 32$$



Example

- We treat the problem

$$-10^{-5}u'' + u' + u = 1$$

with zero boundary conditions on uniform grids.

- Error reduction for the best approximation in the Y-norm on uniform grids.

#cells	$\|u - u_h\|_X$	$\|u - u_h\|_X / \|u - u_H\|_X$
4	0.478322	
8	0.462815	0.967580
16	0.454909	0.982918
32	0.450869	0.991119
64	0.448741	0.995280
128	0.447464	0.997154
256	0.446381	0.997580

$\rightsquigarrow$ we cannot expect an error reduction

Further reading

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