A Review of Robust A Posteriori Error Estimates

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Outline

Objectives of A Posteriori Error Estimation

- Derive upper error bounds (reliability)

$$\|e\| \leq c^* \eta.$$

- Derive lower error bounds (efficiency)

$$\eta \leq c_* \|e\|.$$

- $c_* c^*$ determines the quality of the error estimate.
- $c_* c^*$ should be uniformly bounded w.r.t. to any mesh-size or parameter of the pde (robustness).

Diffusion Dominated Model Problem

$$\begin{aligned} -\Delta u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma \end{aligned}$$

Variational Formulation

Find $u \in X$ such that

$$a(u, v) = \langle \ell, v \rangle$$

holds for all $v \in Y$.

$$\begin{aligned} X &= H_0^1(\Omega) \\ Y &= H_0^1(\Omega) \\ a(u, v) &= \int_\Omega \{\nabla u \cdot \nabla v + uv\} \\ \langle \ell, v \rangle &= \int_\Omega fv \end{aligned}$$

Energy norm

$$|||u|||^2 = a(u, u)$$

Meshes

- Affine equivalence: $\mathcal{T}$ consists of triangles or parallelograms, if $d = 2$, or tetrahedrons or parallelepipeds, if $d = 3$.
- Admissibility: Any two elements in $\mathcal{T}$ are either disjoint or share a vertex or a complete edge or – if $d = 3$ – a complete face.
- Shape regularity: For any element K, the ratio of its diameter h_K to the diameter ρ_K of the largest ball inscribed into K is bounded independently of K.
- Shape parameter:

$$c_{\mathcal{T}} = \max_{K \in \mathcal{T}} \frac{h_K}{\rho_K}.$$

Finite Element Spaces

$$R_k(K) = \begin{cases} \operatorname{span}\{x^\alpha : |\alpha|_1 \le k\} & \text{if } K \text{ is a triangle or} \\ & \text{a tetrahedron,} \\ \operatorname{span}\{x^\alpha : |\alpha|_\infty \le k\} & \text{if } K \text{ is a parallelogram or} \\ & \text{a parallelepiped.} \end{cases}$$

$$\begin{aligned} S^{k,-1}(\mathcal{T}) &= \{v : \Omega \to \mathbb{R} : v\big|_K \in R_k(K) \text{ for all } K \in \mathcal{T}\}, \\ S^{k,0}(\mathcal{T}) &= S^{k,-1}(\mathcal{T}) \cap C(\overline{\Omega}), \\ S_0^{k,0}(\mathcal{T}) &= S^{k,0}(\mathcal{T}) \cap H_0^1(\Omega) \\ &= \{v \in S^{k,0}(\mathcal{T}) : v = 0 \text{ on } \Gamma\}. \end{aligned}$$

Discrete Problem

Find $u_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$ such that

$$a(u_{\mathcal{T}}, v_{\mathcal{T}}) = \langle \ell, v_{\mathcal{T}} \rangle$$

holds for all $v_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$.

Equivalence of Error and Residual

Residual $R(u_{\mathcal{T}})$

$$\begin{aligned}\langle R(u_{\mathcal{T}}), v\rangle &= \langle \ell, v\rangle - a(u_{\mathcal{T}}, v)\\ &= a(e, v)\end{aligned}$$

Continuity of a

$$|||R(u_{\mathcal{T}})|||_* \le |||e|||$$

Coercivity of a

$$|||e||| \le |||R(u_{\mathcal{T}})|||_*$$

Galerkin Orthogonality

$$\langle R(u_{\mathcal{T}}), v_{\mathcal{T}}\rangle = 0$$

for all $v_{\mathcal{T}} \in S_0^{k,0}(\mathcal{T})$.

L^2-Representation of the Residual

Integration by parts element-wise

$$\begin{aligned}\langle R(u_{\mathcal{T}}), v\rangle = &\sum_{K\in\mathcal{T}} \int_K \underbrace{\{f + \Delta u_{\mathcal{T}} - u_{\mathcal{T}}\}}_{=R_K(u_{\mathcal{T}})} v\\ &+ \sum_{E\in\mathcal{E}_{\mathcal{T}}} \int_E \underbrace{-\mathbb{J}_E(n_E \cdot \nabla u_{\mathcal{T}})}_{=R_E(u_{\mathcal{T}})} v\end{aligned}$$

Quasi-Interpolation Operator

$$I_{\mathcal{T}} v = \sum_{x\in\mathcal{N}_{\mathcal{T}}} \lambda_x \pi_x v$$

with

$$\pi_x v = \frac{1}{\mu_d(\omega_x)} \int_{\omega_x} v$$

Local error estimates

$$\begin{aligned}\|v - I_{\mathcal{T}} v\|_{L^2(K)} &\le c_{A1} h_K \|\nabla v\|_{L^2(\widetilde{\omega}_K)}\\ \|v - I_{\mathcal{T}} v\|_{L^2(E)} &\le c_{A2} h_E^{\frac{1}{2}} \|\nabla v\|_{L^2(\widetilde{\omega}_E)}\end{aligned}$$

Upper Bound

Galerkin orthogonality, L^2-representation, Cauchy-Schwarz inequality, and local error estimates:

$$\begin{aligned}|||R(u_\mathcal{T})|||_* &\le c^*\Big\{\sum_{K\in\mathcal{T}} h_K^2\|R_K(u_\mathcal{T})\|^2_{L^2(K)} \\ &\qquad + \sum_{E\in\mathcal{E}_\mathcal{T}} h_E\|R_E(u_\mathcal{T})\|^2_{L^2(E)}\Big\}^{\frac{1}{2}} \\ &= c^*\Big\{\sum_{K\in\mathcal{T}}\eta_K^2\Big\}^{\frac{1}{2}}\end{aligned}$$

c^* only depends on c_{A1}, c_{A2} which in turn only depend on the shape parameter $c_\mathcal{T}$.

Local Cut-Off Functions

$$\psi_K = \alpha_K \prod_{x\in\mathcal{N}_K}\lambda_x$$
$$\psi_E = \beta_E \prod_{x\in\mathcal{N}_E}\lambda_x$$

Weights are determined by

$$\max_{x\in K}\psi_K(x) = 1$$
$$\max_{x\in E}\psi_E(x) = 1$$

Inverse Estimates

For all $v\in R_k(K)$, $\sigma\in R_k(E)$:

$$\begin{aligned}\|v\|_{L^2(K)} &\le c_{I1,k}\|\psi_K^{\frac{1}{2}}v\|_{L^2(K)} \\ \|\nabla(\psi_K v)\|_{L^2(K)} &\le c_{I2,k}h_K^{-1}\|v\|_{L^2(K)} \\ \|\sigma\|_{L^2(E)} &\le c_{I3,k}\|\psi_E^{\frac{1}{2}}\sigma\|_{L^2(E)} \\ \|\nabla(\psi_E\sigma)\|_{L^2(K)} &\le c_{I4,k}h_K^{-\frac{1}{2}}\|\sigma\|_{L^2(E)} \\ \|\psi_E\sigma\|_{L^2(K)} &\le c_{I5,k}h_K^{\frac{1}{2}}\|\sigma\|_{L^2(E)}\end{aligned}$$

Lower Bound

Insert functions $\psi_K R_K(u_\mathcal{T})$ and $\psi_E R_E(u_\mathcal{T})$ as test-functions in L^2-representation of residual:

$$\eta_K \le c_*|||e|||_{\omega_K}$$

c_* only depends on $c_{I1,k}$, …, $c_{I5,k}$ which in turn only depend on the polynomial degree k and the shape parameter $c_\mathcal{T}$.

Summary of Crucial Steps

- Equivalence of error and residual.
- Galerkin orthogonality.
- L^2-representation of residual.
- Local error estimates for quasi-interpolation operator.
- Inverse estimates for local cut-off functions.

Reaction Dominated Model Problem

$$\begin{aligned} -\varepsilon\Delta u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma \end{aligned}$$

with $0 < \varepsilon \ll 1$.
Change bilinear form a and energy norm correspondingly:

$$|||u|||^2 = \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}$$

Difficulties

- Choose standard H^1-norm:
  - Norm of a is ≈ 1.
  - Coercivity constant of a is $\approx \varepsilon$.
  - $c_* c^* \approx \varepsilon^{-1}$.
- Choose energy norm:
  - c_{A1}, c_{A2} depend on ε.
  - $c_{I1,k}$, ..., $c_{I5,k}$ depend on ε.
  - $c_* c^* \approx \varepsilon^{-1}$.
- Need a new idea!

Improved Interpolation Error Estimates

Elements

$$\begin{aligned} \|v - I_{\mathcal{T}} v\|_{L^2(K)} &\le c_{A1} h_K \|\nabla v\|_{L^2(\widetilde{\omega}_K)} \\ &\le c_{A1} \varepsilon^{-\frac{1}{2}} h_K |||v|||_{\widetilde{\omega}_K} \\ \|v - I_{\mathcal{T}} v\|_{L^2(K)} &\le c_{A3} \|v\|_{L^2(\widetilde{\omega}_K)} \\ &\le c_{A3} |||v|||_{\widetilde{\omega}_K} \\ \|v - I_{\mathcal{T}} v\|_{L^2(K)} &\le \widetilde{c}_{A1} \underbrace{\min\{\varepsilon^{-\frac{1}{2}} h_K, 1\}}_{=\alpha_K} |||v|||_{\widetilde{\omega}_K} \end{aligned}$$

Trace Inequality

Reference element $\widehat{K}$, $\widehat{E}_i = \widehat{K} \cap \{x_i = 0\}$, $\widehat{v} \in H^1(\widehat{K})$ vanishes on edge or face opposite to $\widehat{E}_i$:

$$\|\widehat{v}\|_{L^2(\widehat{E}_i)} \leq \Big\{2\|\widehat{v}\|_{L^2(\widehat{K})}\|\frac{\partial \widehat{v}}{\partial x_i}\|_{L^2(\widehat{K})}\Big\}^{\frac{1}{2}}$$

General element K and edge or face E, $v \in H^1(K)$:

$$\|v\|_{L^2(E)} \leq c\Big\{h_K^{-1}\|v\|^2_{L^2(K)} + \|v\|_{L^2(K)}\|\nabla v\|_{L^2(K)}\Big\}^{\frac{1}{2}}$$

c only depends on shape parameter $c_{\mathcal{T}}$ and number of vertices of E.

Improved Interpolation Error Estimates

Edges or Faces

$$\begin{aligned}\|v - I_{\mathcal{T}}v\|_{L^2(E)} &\leq c\Big\{h_K^{-1}\alpha_K^2 + \alpha_K\varepsilon^{-\frac{1}{2}}\Big\}^{\frac{1}{2}}|||v|||_{\widetilde{\omega}_E} \\ &\leq \widetilde{c}_{A2}\varepsilon^{-\frac{1}{4}}\alpha_E^{\frac{1}{2}}|||v|||_{\widetilde{\omega}_E}\end{aligned}$$

Upper Bound

Galerkin orthogonality, L^2-representation, Cauchy-Schwarz inequality, and improved interpolation error estimates:

$$\begin{aligned}|||R(u_{\mathcal{T}})|||_* &\leq c^*\Big\{\sum_{K\in\mathcal{T}}\alpha_K^2\|R_K(u_{\mathcal{T}})\|^2_{L^2(K)} \\ &\qquad + \sum_{E\in\mathcal{E}_{\mathcal{T}}}\varepsilon^{-\frac{1}{2}}\alpha_E\|R_E(u_{\mathcal{T}})\|^2_{L^2(E)}\Big\}^{\frac{1}{2}} \\ &= c^*\Big\{\sum_{K\in\mathcal{T}}\eta_K^2\Big\}^{\frac{1}{2}}\end{aligned}$$

c^* only depends on $\widetilde{c}_{A1}$, $\widetilde{c}_{A2}$ which in turn only depend on the shape parameter $c_{\mathcal{T}}$.

Inverse Estimates

Elements

$$\begin{aligned}|||\psi_K v|||_K &\leq \Big\{\varepsilon c_{I2,k}^2 h_K^{-2} + 1\Big\}^{\frac{1}{2}}\|v\|_{L^2(K)} \\ &\leq \widetilde{c}_{I2,k}\alpha_K^{-1}\|v\|_{L^2(K)}\end{aligned}$$

Inverse Estimates

Edges or Faces

$$
\begin{aligned}
|||\psi_E v|||_K &\le \left\{\varepsilon c_{I4,k}^2 h_E^{-1} + c_{I5,k} h_E\right\}^{\frac{1}{2}} \|v\|_{L^2(E)} \\
&\le \widetilde{c}_{I4,k} h_E^{\frac{1}{2}} \alpha_K^{-1} \|v\|_{L^2(E)} \\
\|\psi_E v\|_{L^2(K)} &\le c_{I5,k} h_E^{\frac{1}{2}} \|v\|_{L^2(E)}
\end{aligned}
$$

but we need

$$
\begin{aligned}
|||\psi_E v|||_K &\le c\varepsilon^{\frac{1}{4}} \alpha_E^{-\frac{1}{2}} \|v\|_{L^2(E)} \\
\|\psi_E v\|_{L^2(K)} &\le c\varepsilon^{\frac{1}{4}} \alpha_E^{\frac{1}{2}} \|v\|_{L^2(E)}
\end{aligned}
$$

Modified Cut-Off Functions

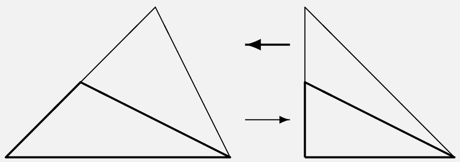
- Transform K, E to reference element $\widehat{K}$ and edge or face $\widehat{E}_d$.
- Apply transformation

$$
(x_1, \dots, x_{d-1}, x_d) \mapsto (x_1, \dots, x_{d-1}, \delta x_d).
$$

- Transform back to K, E and obtain "squeezed" element K_δ.
- Denote by $\lambda_{x,\delta}$ the nodal shape functions of K_δ and set

$$
\psi_{E,\delta} = \beta_E \prod_{x \in \mathcal{N}_E} \lambda_{x,\delta}.
$$

Squeezing Elements



Inverse Estimates for Modified Cut-Off Functions

$$
\begin{aligned}
\|\psi_{E,\delta} v\|_{L^2(K)} &\le c\delta^{\frac{1}{2}} h_E^{\frac{1}{2}} \|v\|_{L^2(E)} \\
\|\nabla(\psi_{E,\delta} v)\|_{L^2(K)} &\le c\delta^{-\frac{1}{2}} h_E^{-\frac{1}{2}} \|v\|_{L^2(E)}
\end{aligned}
$$

choose

$$
\begin{aligned}
\delta &= \varepsilon^{\frac{1}{2}} h_E^{-1} \alpha_E \\
&= \min\{1, \varepsilon^{\frac{1}{2}} h_E^{-1}\} \\
\|\psi_{E,\delta} v\|_{L^2(K)} &\le c\varepsilon^{\frac{1}{4}} \alpha_E^{\frac{1}{2}} \|v\|_{L^2(E)} \\
|||\psi_{E,\delta} v|||_K &\le c\varepsilon^{\frac{1}{4}} \alpha_E^{-\frac{1}{2}} \|v\|_{L^2(E)}
\end{aligned}
$$

Lower Bound

Insert functions $\psi_K R_K(u_\mathcal{T})$ and $\psi_{E,\delta} R_E(u_\mathcal{T})$ as test-functions in L^2-representation of residual:

$$\eta_K \le c_* |||e|||_{\omega_K}$$

c_* only depends on the polynomial degree k and the shape parameter $c_\mathcal{T}$.

Summary of Crucial Steps

- Modified weights for element and edge or face residuals.
- Improved interpolation error estimates.
- Modified local cut-off functions.

Convection Dominated Model Problem

$$\begin{aligned} -\varepsilon \Delta u + \underline{a} \cdot \nabla u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma \end{aligned}$$

with $0 < \varepsilon \ll 1$, $\underline{a} \in \mathbb{R}^d$ and $|\underline{a}| = 1$.
Change bilinear form a correspondingly; energy norm remains unchanged.

SUPG-Discretization

Find $u_\mathcal{T} \in S_0^{k,0}(\mathcal{T})$ such that

$$\begin{aligned} & a(u_\mathcal{T}, v_\mathcal{T}) \\ & \quad + \sum_{K \in \mathcal{T}} \delta_K \int_K \{-\varepsilon \Delta u_\mathcal{T} + \underline{a} \cdot \nabla u_\mathcal{T} + u_\mathcal{T}\} \underline{a} \cdot \nabla v_\mathcal{T} \\ = & \langle \ell, v_\mathcal{T} \rangle \\ & \quad + \sum_{K \in \mathcal{T}} \delta_K \int_K f \underline{a} \cdot \nabla v_\mathcal{T} \end{aligned}$$

holds for all $v_\mathcal{T} \in S_0^{k,0}(\mathcal{T})$.

Difficulties

- Missing Galerkin orthogonality due to stabilization terms.
  - Must bound the consistency error.
- a is coercive w.r.t. the energy norm, but

$$a(u,v) \leq |||u||| |||v||| + \|\underline{a}\cdot\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.$$

  - Error and residual are not fully equivalent

$$|||e||| \leq |||R(u_\mathcal{T})|||_*$$
$$|||R(u_\mathcal{T})|||_* \leq \left\{1+\varepsilon^{-\frac{1}{2}}\right\}|||e|||.$$

  - More detailed analysis yields $c^* \approx 1$ and $c_* \approx 1+\varepsilon^{-\frac{1}{2}}h$.

Estimation of the Consistency Error

$$\begin{aligned}\langle R(u_\mathcal{T}), v_\mathcal{T}\rangle &= \sum_{K\in\mathcal{T}} \delta_K \int_K \{f+\varepsilon\Delta u_\mathcal{T} - \underline{a}\cdot\nabla u_\mathcal{T} - u_\mathcal{T}\}\underline{a}\cdot\nabla v_\mathcal{T} \\ &= \sum_{K\in\mathcal{T}} \delta_K \int_K R_K(u_\mathcal{T})\underline{a}\cdot\nabla v_\mathcal{T} \\ &\leq \sum_{K\in\mathcal{T}} \delta_K \|R_K(u_\mathcal{T})\|_{L^2(K)}\|\underline{a}\cdot\nabla v_\mathcal{T}\|_{L^2(K)} \\ &\leq \sum_{K\in\mathcal{T}} \delta_K h_K^{-1}\alpha_K \|R_K(u_\mathcal{T})\|_{L^2(K)} |||v_\mathcal{T}|||_K\end{aligned}$$

Assume that $\delta_K \leq c h_K$. This holds for all schemes used in practice.

Robust Bounds for the Residual

$$|||R(u_\mathcal{T})|||_* \leq c^*\left\{\sum_{K\in\mathcal{T}}\eta_K^2\right\}^{\frac{1}{2}}$$
$$\left\{\sum_{K\in\mathcal{T}}\eta_K^2\right\}^{\frac{1}{2}} \leq c_*|||R(u_\mathcal{T})|||_*$$

c_*c^* is independent of ε.
Need robust estimates of the error in terms of the residual.

Modified Norm

Equip $X = H_0^1(\Omega)$ with norm

$$\|u\|_X = |||u||| + |||\underline{a}\cdot\nabla u|||_*$$

and $Y = H_0^1(\Omega)$ with the energy norm

$$\|v\|_Y = |||v|||.$$

a is continuous with norm 1

$$a(u,v) \leq \|u\|_X\|v\|_Y.$$

a is uniformly stable

$$\inf_{u\in X}\sup_{v\in Y}\frac{a(u,v)}{\|u\|_X\|v\|_Y} \geq \frac{1}{3}.$$

Uniform Stability

Choose $\theta \in (0,1)$ and $w_\theta \in Y$ with

$$|||w_\theta||| = 1$$

and

$$\int_\Omega \underline{a} \cdot \nabla u w_\theta \geq \theta |||\underline{a} \cdot \nabla u|||_*.$$

Set

$$v_\theta = u + \frac{1}{2}|||u|||w_\theta.$$

Uniform Stability

Then

$$|||v_\theta||| \leq \frac{3}{2}|||u|||$$

and

$$\begin{aligned} a(u, v_\theta) &= a(u,u) + \frac{1}{2}|||u|||a(u,w_\theta) \\ &\geq |||u|||^2 + \frac{1}{2}|||u|||a(u,w_\theta) \\ &\geq |||u|||^2 + \frac{1}{2}|||u|||\Big\{\theta|||\underline{a}\cdot\nabla u|||_* - |||u|||\Big\} \\ &= \frac{1}{2}|||u|||\Big\{|||u||| + \theta|||\underline{a}\cdot\nabla u|||_*\Big\} \end{aligned}$$

Robust Error Estimate

$$|||e||| + |||\underline{a}\cdot\nabla e|||_* \leq c^*\Big\{\sum_{K\in\mathcal{T}} \eta_K^2\Big\}^{\frac{1}{2}}$$

$$\Big\{\sum_{K\in\mathcal{T}} \eta_K^2\Big\}^{\frac{1}{2}} \leq c_*\Big\{|||e||| + |||\underline{a}\cdot\nabla e|||_*\Big\}$$

$c_* c^*$ only depends on the polynomial degree k and the shape parameter $c_\mathcal{T}$.

SUPG-Norm

Mesh-dependent SUPG-norm:

$$|||v||| + |||\underline{a}\cdot\nabla v|||_\mathcal{T}$$

with

$$|||\underline{a}\cdot\nabla v|||_\mathcal{T}^2 = \sum_{K\in\mathcal{T}} \delta_K \|\underline{a}\cdot\nabla v\|_{L^2(K)}^2.$$

Relation to modified norm:

$$|||\underline{a}\cdot\nabla v|||_* \leq \max_{K\in\mathcal{T}} \delta_K^{-\frac{1}{2}} |||\underline{a}\cdot\nabla v|||_\mathcal{T}$$

$$|||\underline{a}\cdot\nabla v_\mathcal{T}|||_\mathcal{T} \leq \max_{K\in\mathcal{T}} \delta_K^{\frac{1}{2}} \alpha_K^{-1} |||\underline{a}\cdot\nabla v_\mathcal{T}|||_*.$$

Relation of both norms depends on mesh Péclet number.

Interpretation of Results

- Error estimator is robust w.r.t. the modified norm.
- Error estimator may over-estimate the energy norm.
- Over-estimation of the energy norm indicates an unresolved layer.

Summary of Crucial Steps

- Estimation of the consistency error.
- Problem-adapted norm for error estimation.

Non-Stationary Model Problem

$$\begin{aligned} \partial_t u - \varepsilon \Delta u + \underline{a} \cdot \nabla u + u &= f && \text{in } \Omega \times (0,T] \\ u &= 0 && \text{on } \Gamma \times (0,T] \\ u &= u_0 && \text{in } \Omega \end{aligned}$$

Space and Norm

$$\begin{aligned} X(a,b) = \Big\{ & u \in L^2(a,b;H_0^1(\Omega)) \cap L^\infty(a,b;L^2(\Omega)) : \\ & \partial_t u + \underline{a} \cdot \nabla u \in L^2(a,b;H^{-1}(\Omega)) \Big\}, \\ \|u\|_{X(a,b)} = \Big\{ & \operatorname*{ess.sup}_{a<t<b} \|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_a^b |||u(\cdot,t)|||^2 dt \\ & + \int_a^b |||(\partial_t u + \underline{a} \cdot \nabla u)(\cdot,t)|||_*^2 dt \Big\}^{\frac{1}{2}}. \end{aligned}$$

Space-Time Elements

Temporal partition:

$$\mathcal{I} = \{[t_{n-1}, t_n] : 1 \le n \le N_{\mathcal{I}}\}.$$

Time-steps:

$$\tau_n = t_n - t_{n-1}.$$

Spatial partitions:

$$\mathcal{T}_0, \dots, \mathcal{T}_{N_{\mathcal{I}}}.$$

Finite element spaces:

$$X_0, \dots, X_{N_{\mathcal{I}}}.$$

Conditions

- The partitions $\mathcal{T}_n$ are affine equivalent, admissible, and shape regular.
- Transition condition: For every n there is an affine equivalent, admissible, and shape regular partition $\widetilde{\mathcal{T}}_n$ such that it is a refinement of both $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ and such that

$$c_{\mathcal{I}} = \sup_{1 \le n \le N_{\mathcal{I}}} \sup_{K \in \widetilde{\mathcal{T}}_n} \sup_{\substack{K' \in \mathcal{T}_n \\ K \subset K'}} \frac{h_{K'}}{h_K} < \infty$$

  uniformly with respect to all partitions $\mathcal{I}$.
- Degree condition: Each X_n consists of continuous functions which are piecewise polynomials, the degrees being at least one and being bounded uniformly with respect to all partitions $\mathcal{T}_n$ and $\mathcal{I}$.

Discrete Problem

Find $u^n_{\mathcal{T}_n} \in X_n$, $0 \le n \le N_{\mathcal{I}}$, such that

$$u^0_{\mathcal{T}_0} = \pi_0 u_0$$

and, for $n = 1, \dots, N_{\mathcal{I}}$,

$$\int_\Omega \frac{u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}}{\tau_n} v_{\mathcal{T}_n} + a(\theta \nabla u^n_{\mathcal{T}_n} + (1-\theta)\nabla u^{n-1}_{\mathcal{T}_{n-1}}, v_{\mathcal{T}_n})$$
$$+ \sum_{K \in \widetilde{\mathcal{T}}_n} \delta_K \int_K \Big(\frac{u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}}{\tau_n} + L(\theta u^n_{\mathcal{T}_n} + (1-\theta) u^{n-1}_{\mathcal{T}_{n-1}}) \Big) \underline{a} \cdot \nabla v_{\mathcal{T}_n}$$
$$= \langle \ell, v_{\mathcal{T}_n} \rangle + \sum_{K \in \widetilde{\mathcal{T}}_n} \delta_K \int_K f \underline{a} \cdot \nabla v_{\mathcal{T}_n}$$

for all $v_{\mathcal{T}_n} \in X_n$.

Notation

$u_{\mathcal{I}}$ is piecewise affine w.r.t. time and equals $u^n_{\mathcal{T}_n}$ at time t_n:

$$u_{\mathcal{I}}(\cdot, t) = \frac{1}{\tau_n}\big((t_n - t) u^{n-1}_{\mathcal{T}_{n-1}} + (t - t_{n-1}) u^n_{\mathcal{T}_n}\big) \quad \text{on } [t_{n-1}, t_n]$$
$$\partial_t u_{\mathcal{I}} = \frac{1}{\tau_n}(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}) \quad \text{on } [t_{n-1}, t_n].$$

Residual

Residual $R(u_{\mathcal{I}})$:

$$\langle R(u_{\mathcal{I}}), v\rangle = \langle \ell, v\rangle - \int_\Omega \partial_t u_{\mathcal{I}} v - a(u_{\mathcal{I}}, v)$$
$$= \int_\Omega \partial_t e v + a(e, v).$$

Equivalence of Error and Residual

Insert any function $w \in L^2(0,T;H_0^1(\Omega))$ into the residual equation:

$$\|R(u_{\mathcal{I}})\|_{L^2(0,T;H^{-1}(\Omega))} \le \sqrt{2}\|e\|_{X(0,T)}.$$

Insert error e into residual equation (parabolic energy estimate):

$$\|e\|_{X(0,t_n)} \le \Big\{4\|u_0 - \pi_0 u_0\|^2_{L^2(\Omega)} + 6\|R(u_{\mathcal{I}})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}\Big\}^{\frac{1}{2}}.$$

Decomposition of the Residual

Temporal residual $R_\tau(u_{\mathcal{I}})$:

$$\langle R_\tau(u_{\mathcal{I}}), v\rangle = a([\theta u^n_{\mathcal{I}_n} + (1-\theta)u^{n-1}_{\mathcal{I}_{n-1}} - u_{\mathcal{I}}], v).$$

Spatial residual $R_h(u_{\mathcal{I}})$:

$$\langle R_h(u_{\mathcal{I}}), v\rangle = \langle \ell, v\rangle - \int_\Omega \partial_t u_{\mathcal{I}} v - a([\theta \nabla u^n_{\mathcal{I}_n} + (1-\theta)\nabla u^{n-1}_{\mathcal{I}_{n-1}}], v).$$

Decomposition:

$$R(u_{\mathcal{I}}) = R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}}).$$

Spatial Residual

Spatial error indicator $\eta^n_{\mathcal{I}_n}$:

$$\eta^n_{\mathcal{I}_n} = \Big\{\sum_{K\in\widetilde{\mathcal{T}}_n} \alpha_K^2 \|R_K\|^2_{L^2(K)} + \sum_{E\in\mathcal{E}_{\widetilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}}\alpha_E \|R_E\|^2_{L^2(E)}\Big\}^{\frac{1}{2}}.$$

Standard arguments for stationary problems yield:

$$|||R_h(u_{\mathcal{I}})|||_* \le c^\dagger \eta^n_{\mathcal{I}_n},$$
$$\eta^n_{\mathcal{I}_n} \le c_\dagger |||R_h(u_{\mathcal{I}})|||_*.$$

$c^\dagger c_\dagger$ only depend on the polynomial degrees and on the shape parameters $c_{\widetilde{\mathcal{T}}_n}$ and $c_{\mathcal{I}}$.

Temporal Residual

Representation of the temporal residual, the stability and continuity of the bilinear form a, and integration on $[t_{n-1}, t_n]$ yield:

$$\left\{\int_{t_{n-1}}^{t_n} |||R_\tau(u_\mathcal{I})(\cdot,s)|||_*^2 ds\right\}^{\frac{1}{2}} \leq \sqrt{\frac{2}{3}}\tau_n^{\frac{1}{2}}\left\{|||u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1}|||^2 + |||\underline{a}\cdot\nabla(u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1})|||_*^2\right\}^{\frac{1}{2}},$$

$$\left\{\int_{t_{n-1}}^{t_n} |||R_\tau(u_\mathcal{I})(\cdot,s)|||_*^2 ds\right\}^{\frac{1}{2}} \geq \frac{1}{24}\sqrt{\frac{3}{8}}\tau_n^{\frac{1}{2}}\left\{|||u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1}|||^2 + |||\underline{a}\cdot\nabla(u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1})|||_*^2\right\}^{\frac{1}{2}}.$$

Preliminary Error Estimator

Define a preliminary space-time error estimator by:

$$\widehat{\eta}_n = \tau_n^{\frac{1}{2}}\left[\left(\eta_{\mathcal{I}_n}^n\right)^2 + |||u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1}|||^2 + |||\underline{a}\cdot\nabla(u_{\mathcal{I}_n}^n - u_{\mathcal{I}_{n-1}}^{n-1})|||_*^2\right]^{\frac{1}{2}}.$$

Upper Error Bound

Equivalence of error and residual, the decomposition of the residual, the triangle inequality and the upper bounds for the spatial and temporal residuals yield:

$$\|e\|_{X(0,T)} \leq \widehat{c}^*\left\{\|u_0 - \pi_0 u_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N_\mathcal{I}} \widehat{\eta}_n^2\right\}^{\frac{1}{2}}.$$

Lower Error Bound

Equivalence of error and residual, the decomposition of the residual, the triangle inequality, the lower bounds for the spatial and temporal residuals, and a test-function of the form

$$w_{n,\tau} + (\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^\alpha w_{n,h}$$

with a suitable α yield:

$$\widehat{\eta}_n \leq \widehat{c}_*\|e\|_{X(t_{n-1},t_n)}.$$

$\widehat{c}_*\widehat{c}^*$ only depends on the polynomial degrees, the shape parameters $c_{\widetilde{\mathcal{T}}_n}$ and the constant in the transition condition.

Difficulty

- Term $|||\underline{a}\cdot\nabla(u^n_{\mathcal{T}_n}-u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ in $\widehat{\eta}_n$ is not computable.
- Inverse estimate spoils the robustness.
- Idea:
  - $|||\cdot|||_*$ is the dual norm of a reaction diffusion problem.
  - Solve an auxiliary discrete reaction diffusion problem.
  - Replace $|||\underline{a}\cdot\nabla(u^n_{\mathcal{T}_n}-u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ by the energy norm of the solution of the auxiliary problem and a residual error estimator for the auxiliary problem.

Auxiliary Problem

Find $\widetilde{u}^n_{\mathcal{T}_n}\in S_0^{1,0}(\widetilde{\mathcal{T}}_n)$ such that

$$\int_\Omega\{\varepsilon\nabla\widetilde{u}^n_{\mathcal{T}_n}\cdot\nabla v_{\mathcal{T}_n}+\widetilde{u}^n_{\mathcal{T}_n}v_{\mathcal{T}_n}\}=\int_\Omega\underline{a}\cdot\nabla(u^n_{\mathcal{T}_n}-u^{n-1}_{\mathcal{T}_{n-1}})v_{\mathcal{T}_n}$$

holds for all $v_{\mathcal{T}_n}\in S_0^{1,0}(\widetilde{\mathcal{T}}_n)$.

Estimation of Dual Norm

Set

$$\widetilde{\eta}^n_{\mathcal{T}_n}=\Big\{\sum_{K\in\widetilde{\mathcal{T}}_n}\alpha_K^2\|\underline{a}\cdot\nabla(u^n_{\mathcal{T}_n}-u^{n-1}_{\mathcal{T}_{n-1}})+\varepsilon\Delta\widetilde{u}^n_{\mathcal{T}_n}-\widetilde{u}^n_{\mathcal{T}_n}\|^2_{L^2(K)}$$
$$+\sum_{E\in\mathcal{E}_{\widetilde{\mathcal{T}}_n}}\varepsilon^{-\frac12}\alpha_E\|\mathbb{J}_E(n_E\cdot\nabla\widetilde{u}^n_{\mathcal{T}_n})\|^2_{L^2(E)}\Big\}^{\frac12}.$$

Then

$$\widetilde{c}_\dagger\Big\{|||\widetilde{u}^n_{\mathcal{T}_n}|||+\widetilde{\eta}^n_{\mathcal{T}_n}\Big\}\le|||\underline{a}\cdot\nabla(u^n_{\mathcal{T}_n}-u^{n-1}_{\mathcal{T}_{n-1}})|||_*$$
$$\le\widetilde{c}^\dagger\Big\{|||\widetilde{u}^n_{\mathcal{T}_n}|||+\widetilde{\eta}^n_{\mathcal{T}_n}\Big\}.$$

Final Error Estimate

Define the final space-time error estimator by:

$$\eta_n=\tau_n^{\frac12}\Big[\Big(\eta^n_{\mathcal{T}_n}\Big)^2$$
$$+|||u^n_{\mathcal{T}_n}-u^{n-1}_{\mathcal{T}_{n-1}}|||^2+|||\widetilde{u}^n_{\mathcal{T}_n}|||^2+\Big(\widetilde{\eta}^n_{\mathcal{T}_n}\Big)^2\Big]^{\frac12}.$$

Then

$$\|e\|_{X(0,T)}\le c^*\Big\{\|u_0-\pi_0u_0\|^2_{L^2(\Omega)}+\sum_{n=1}^{N_\mathcal{I}}\eta_n^2\Big\}^{\frac12},$$
$$\eta_n\le c_*\|e\|_{X(t_{n-1},t_n)}.$$

c_*c^* only depends on the polynomial degrees and the shape parameters $c_{\widetilde{\mathcal{T}}_n}$ and $c_\mathcal{I}$.

Summary of Crucial Steps

- Parabolic energy estimate for the equivalence of error and residual.
- Decomposition of the residual into a spatial and a temporal contribution.
- Standard stationary results for the robust estimation of the spatial residual.
- Robust estimation of the temporal residual.
- Judicious choice of test functions for the estimation of the combined residuals.
- Auxiliary discrete reaction diffusion problem and corresponding residual error estimate for handling the dual energy norm in the error estimator.

Extensions

- Variable coefficients and general source terms introduce additional data errors.
- General convection and reaction terms require

$$-\frac{1}{2}\operatorname{div}\underline{a} + b \geq \beta \geq 0$$
$$\|b\|_{L^\infty(\Omega)} \leq c_b\beta.$$

- Anisotropic meshes are handled by using suitable geometric quantities.
- Discontinuities of the diffusion should be caught by the coarsest mesh.
- Anisotropic diffusion can be handled by measuring the length in a diffusion-dependent norm.
- Non-linear problems are handled via the implicit function theorem.

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