Analysis and Convergent Adaptive Solution of the Einstein Constraint Equations

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Outline

The Einstein field equations

Existence results

AFEM for the Hamiltonian constraint

Gravity and General Relativity

Einstein's general theory of relativity states that spacetime can be modeled on a Lorentzian 4-manifold.

The metric and matter satisfy the *Einstein Equations*.

LIGO

LIGO (Laser Interferometer Gravitational-wave Observatory) is one of several recently constructed gravitational detectors, VIRGO, GEO600, TAMA300.

The design of LIGO is based on measuring distance changes between objects in perpendicular directions as the ripple in the metric tensor propagates through the device.

The three L-shaped LIGO observatories (in Washington and Louisiana), with legs at 2km and 4km, have phenomenal sensitivity, on the order of 10^{-15}m to 10^{-18}m. effective ranges (1.4Sol, 1:8 SNR): 7-15MPc

The Einstein field equations

The ten equations for the ten independent components of the symmetric spacetime metric tensor g_{ab} are the *Einstein Equations*:

$$G_{ab} = \kappa T_{ab}, \quad 0 \leqslant a \leqslant b \leqslant 3, \quad \kappa = 8\pi G/c^4,$$

- $R_{abc}{}^{d}$; Riemann (curvature) tensor
- $R_{ab} = R_{acb}{}^{c}$, $R = R_a{}^{a}$; Ricci tensor and scalar curvature
- $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$, T_{ab}; Einstein tensor and stress-energy tensor

Initial-value formulations well-posed [Choquet-Bruhat, Geroch, Friedrich, Chirstodoulou, Klainerman]; Various formalisms yield constrained (weakly/strongly/symmetric) hyperbolic evolution systems for a spatial 3-metric $\hat{h}_{ab}$, possibly also extrinsic curvature $\hat{k}_{ab} \sim \frac{d}{dt}\hat{h}_{ab}$.

The constraints and the conformal method

The twelve-component system for $(\hat{h}_{ab}, \hat{k}_{ab})$ is constrained by four coupled nonlinear equations which must be satisfied on any $\mathcal{S}(t)$, with $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab}$,

$$^3\hat{R} + \hat{\tau}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \qquad \hat{\nabla}^a\hat{\tau} - \hat{\nabla}_b\hat{k}^{ab} - \kappa\hat{j}^a = 0.$$

The *York conformal decomposition* splits initial data into 8 freely specifiable pieces plus 4 pieces determined by the constraints, through: $\hat{h}_{ab} = \phi^4 h_{ab}$, and $\hat{k}^{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab})] + \frac{1}{4}\phi^{-4}\tau h^{ab}$.

Results in a coupled elliptic system for conformal factor ϕ and a vector potential w^a:

$$\begin{aligned} -8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})(\sigma^{ab} + (\mathcal{L}w)^{ab})\phi^{-7} - 2\kappa\rho\phi^{-3} &= 0, \\ -\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6\nabla^b\tau + \kappa j^b &= 0. \end{aligned}$$

The differential structure on $\mathcal{M}$ is defined through a background 3-metric h_{ab}

$$(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab},$$

Coupled constraints

$(\mathcal{M}, h_{ab})$ Riemannian manifold.

$$\begin{aligned} -\Delta\phi + a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3} &= 0, \\ -\nabla_a(\mathcal{L}w)^{ab} + b_\tau^b\phi^6 + b_j^b &= 0, \end{aligned}$$

where

$$a_R = \frac{R}{8}, \quad a_\tau = \frac{\tau^2}{12}, \quad a_w = \frac{1}{8}[\sigma_{ab} + (\mathcal{L}w)_{ab}][\sigma^{ab} + (\mathcal{L}w)^{ab}], \quad a_\rho = \frac{\kappa\rho}{4},$$
$$b_\tau^a = \frac{2}{3}\nabla^a\tau, \quad b_j^a = \kappa j^a.$$

Notations:

$$\begin{aligned} L\phi + f(\phi, w) = 0 \quad &\Leftrightarrow \quad \phi = T(\phi, w), \\ \mathbf{L}w + \mathbf{f}(\phi) = 0 \quad &\Leftrightarrow \quad w = S(\phi). \end{aligned}$$

CMC and near-CMC existence

$\nabla^b \tau = 0$: Constant Mean Curvature (CMC) case: constraints de-couple.
Results: O'Murchadha-York (1973-74), Isenberg-Marsden (1982-83), Choquet-Bruhat-Isenberg-Moncrief (1992), Isenberg (1995), Maxwell (2004,2006), others.

$\nabla^b \tau \lesssim \inf |\tau|$: Near-Constant Mean Curvature (Near-CMC) case: constraints couple.
Isenberg-Moncrief (1996), Choquet-Bruhat-Isenberg-York (2001), Allen-Clausen-Isenberg (2007), and others; all based on Isenberg-Moncrief.

Isenberg-Moncrief: $w^{(k)} = S(\phi^{(k-1)})$, $\phi^{(k)} = T(\phi^{(k)}, w^{(k)})$
For case $R = -1$ on a closed manifold, under strong smoothness assumptions, and under the near-CMC condition, Isenberg-Moncrief establish this defines a contraction mapping in Hölder spaces:

$$[\phi^{(k+1)}, w^{(k+1)}] = G([\phi^{(k)}, w^{(k)}]).$$

Proof Outline: *Maximum principles, sub- and super-solutions, Banach algebra properties, together with a contraction-mapping argument.* $\square$

The near-CMC condition and fixed-point theorems

To establish a contraction property for coupled PDE systems often produces strong restrictions on the data; in the case of the Einstein constraints, a restriction that results is the near-CMC condition:

$$\|\nabla \tau\|_Y < C \inf_{\mathcal{M}} |\tau|,$$

where $\|\cdot\|_Y$ is an appropriate norm (e.g. $Y = L^\infty$). This condition appears in two distinct places in Isenberg-Moncrief:

(1) Construction of the contraction G,

(2) Construction of the set U on which G is a contraction (using barriers: sub- and super-solutions).

The mappings S and T

We will attempt to build a different fixed-point argument that avoids the near-CMC condition in both places. It is useful now to make precise the particular choices we will make for the mappings S and T for our fixed-point argument.

To deal with the non-trivial kernel that exists for L on closed manifolds, fix an arbitrary positive *shift* $s > 0$. Introduce the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ as

$$S(\phi)^a := -[\mathbf{L}^{-1}\mathbf{f}(\phi)]^a, \tag{1}$$

$$T(\phi, w) := -(L + sI)^{-1}[f(\phi, w) - s\phi]. \tag{2}$$

Both maps are well-defined when $s > 0$ ($L + sI$ is invertible) and when there are no conformal Killing vectors ($\mathbf{L}$ is invertible).

The Schauder Theorem

Theorem (Schauder)

Let Z be a Banach space, and let $U \subset Z$ be a non-empty, convex, closed, bounded subset. If $G : U \to U$ is a compact operator, then there exists a fixed-point $u \in U$ such that $u = G(u)$.

Here is a simple consequence tuned for the constraints.

Theorem

Let X, Y, and Z be Banach spaces, with compact embedding $i : X \hookrightarrow Z$. Let $U \subset Z$ be non-empty, convex, closed, bounded, and let $S : U \to \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \to U \cap X$ be continuous maps. Then, there exist $w \in \mathcal{R}(S)$ and $\phi \in U \cap X$ such that

$$\phi = T(\phi, w) \quad \textit{and} \quad w = S(\phi).$$

Proof Outline: *Compactness of $\phi \mapsto i\,T(\phi, S(\phi)) : U \subset Z \to U \subset Z$ and Schauder.* $\square$

Another fixed point theorem

Identifying $U \subset X$ on which T acts invariantly requires stronger assumptions on T than desirable.

Theorem

Let X and Y be reflexive Banach spaces, and let Z be a Banach space with compact embedding $X \hookrightarrow Z$. Let $U \subset Z$ be nonempty closed, and let $S : U \to \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \to X$ be uniformly bounded and uniformly Lipschitz maps. Assume T also satisfies: For any $w \in \mathcal{R}(S)$, there exists $\phi_w \in U \cap X$ such that

$$\phi_w = T(\phi_w, w). \tag{3}$$

Then, there exist $w \in \mathcal{R}(S)$ and $\phi \in U \cap X$ such that

$$\phi = T(\phi, w) \quad \textit{and} \quad w = S(\phi). \tag{4}$$

Proof Outline: *Compactness arguments directly rather than through Schauder.* □

Sub- and super-solutions

Although we no longer need the near-CMC condition for the fixed-point argument since we do not build k-contractions, we still need to worry about constructing compatible global barriers that are free of the near-CMC condition.

Sub- and super-solutions, or barriers, to the Hamiltonian constraint:

$$L\phi_- + f(\phi_-, w) \leqslant 0, \tag{5}$$

$$L\phi_+ + f(\phi_+, w) \geqslant 0. \tag{6}$$

It will be critical to construct compatible barriers: $0 < \phi_- \leqslant \phi_+ < \infty$, which are also global barriers: Barriers for the Hamiltonian constraint which hold for any solution w to the momentum constraint with source $\phi \in [\phi_-, \phi_+]$.

Momentum constraint

Under the assumption that any $\phi \in L^\infty$ appearing as the source in the momentum constraint equation satisfies

$$\phi \in [\phi_-, \phi_+] \subset L^\infty,$$

then one can establish the required boundedness and Lipschitz properties for the mapping S

$$\|S(\phi)\|_Y \leqslant C_{SB}, \qquad \|S(\phi_1) - S(\phi_2)\|_Y \leqslant C_{SL}\|\phi_1 - \phi_2\|_Z,$$

$$Y = W^{2,p}, \qquad Z = L^\infty.$$

For $p > 3$ we have

$$a_{\mathbf{w}} \leqslant K_1 \|\phi\|_\infty^{12} + K_2$$

Note that the near-CMC condition is not required for these results.

Hamiltonian constraint

Theorem

Let $(\mathcal{M}, h_{ab})$ be a closed Riemannian manifold. Let the free data τ^2, σ^2 and ρ be in L^p, with $p \geqslant 2$. Let ϕ_- and ϕ_+ be barriers for a vector $w^a \in W^{1,2p}$. Then, there exists a solution $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ of the Hamiltonian constraint.

This result, together with the results on the momentum constraint above and the results on barriers below, lead to the required boundedness and Lipschitz properties for the mapping T

$$\|T(\phi, w)\|_X \leqslant C_{TB}, \qquad \|T(\phi_1, w) - T(\phi_2, w)\|_X \leqslant C_{TL}\|\phi_1 - \phi_2\|_Z,$$
$$\|T(\phi, w_1) - T(\phi, w_2)\|_X \leqslant C_{TLW}\|w_1 - w_2\|_Y,$$

$$X = W^{2,p}, Y = W^{2,p}, \qquad Z = L^\infty.$$

Construction of U

The remaining assumptions:

(1) There exists a fixed-point $\phi_w = T(\phi_w, w)$ for any $w \in \mathcal{R}(S)$.

(2) The subset $U \subset Z$ be nonempty and closed in Z,

The first of these assumptions holds by the theorem above; note that the fixed-point framework allows us to establish existence of ϕ_w using any technique, including variational methods, giving existence under weakest possible coefficient assumptions. The second assumption will hold if we can construct a pair of compatible global barriers (addressed next), due to

Lemma

For $1 \leqslant p \leqslant \infty$ and $\phi_-, \phi_+ \in L^p$, the set

$$U = [\phi_-, \phi_+] = \{\phi \in L^p : \phi_- \leqslant \phi \leqslant \phi_+\} \subset L^p$$

is closed convex and bounded.

The Yamabe problem

Given a compact Riemannian manifold $(\mathcal{M}, g)$ of dimension $n \geqslant 3$, find a metric conformal to g with constant scalar curvature.

$$-4\frac{n-1}{n-2}\Delta u + Ru = \lambda u^{\frac{2n}{n-2}-1}$$

- Yamabe '60: Claim
- Trudinger '68: Found error in Yamabe's proof, repaired for some cases
- Aubin '74: $n \geqslant 6$
- Schoen '84: $n \leqslant 5$
- Lee and Parker '87: unified expository

Three Yamabe classes: $\mathcal{Y}^+$, $\mathcal{Y}^-$, $\mathcal{Y}^0$

Near-CMC-free global barrier

Establishing boundedness and Lipschitz properties for S and T without near-CMC conditions rests critically on establishing the existence of compatible global barriers $0 < \phi_- \leqslant \phi_+ < \infty$ to define the nonempty topologically closed subset $U = [\phi_-, \phi_+]$.

Lemma

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in the positive Yamabe class with no conformal Killing vectors. Let u be a smooth positive solution of the Yamabe problem

$$-\Delta u + a_R u = u^5,$$

and let $k = u^\wedge / u^\vee$. If the function τ is non-constant and the rescaled matter sources j^a, ρ, and traceless transverse tensor σ^{ab} are sufficiently small, then

$$\phi_+ = \epsilon u,$$

is a global super-solution for any sufficiently small $\epsilon > 0$.

Proof outline

Proof Outline: Using the notation

$$E(\phi_+) = -\Delta\phi_+ + a_R\phi_+ + a_\tau\phi_+^5 - a_w\phi_+^{-7} - a_\rho\phi_+^{-3},$$

we have to show $E(\phi_+) \geqslant 0$. The definition of $\phi_+ = \epsilon u$ implies $-\Delta\phi_+ + a_R\phi_+ = \epsilon\, u^5$. We have then

$$E(\phi_+) \geqslant -\Delta\phi_+ + a_R\phi_+ - \frac{K_1(\phi_+^\wedge)^{12} + K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3}$$
$$\geqslant \epsilon\, u^5 - K_1\left[\frac{\phi_+^\wedge}{\phi_+^\vee}\right]^{12}\phi_+^5 - \frac{K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3}.$$

Notice that $\phi_+^\wedge/\phi_+^\vee = u^\wedge/u^\vee = k$, therefore we have

$$E(\phi_+) \geqslant \epsilon u^5 - K_1\, k^{12}\,(\epsilon u)^5 - \frac{K_2}{(\epsilon u)^7} - \frac{a_\rho^\wedge}{(\epsilon u)^3}$$
$$\geqslant \epsilon u^5\left[1 - K_1\, k^{12}\epsilon^4 - \frac{K_2}{\epsilon^8 u^{12}} - \frac{a_\rho^\wedge}{\epsilon^4 u^8}\right].$$

Non-constant Mean Curvature Weak Solutions

Theorem

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in the positive Yamabe class with no conformal Killing vectors. Let τ be in $W^{1,p}$, while the fields σ^2, j^a and ρ be in L^p, with $p > 3$ and small enough norms so that there exist global barriers ϕ_- and ϕ_+ for the Hamiltonian constraint equation. Then, there exists a scalar field $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ and a vector field $w^a \in W^{2,p}$ solving the constraint equations.

Open problems

- Full parameterization of the solution space
- Manifolds with boundary
- Unbounded manifolds

AFEM for Hamiltonian constraint

Let us consider the Hamiltonian constraint on a closed connected flat manifold

$$(\nabla u, \nabla v) + \langle f(u, w), v \rangle = 0, \qquad \forall v \in H^1$$

- Residual error estimator [Verfürth '94]
- Red refinement [Stevenson '05]
- Quasi-orthogonality $\rightarrow$ Optimal convergence [Stevenson '07] or [Cascon, Kreuzer, Nochetto, Siebert '08]

Galerkin approximation

Let $\mathcal{X}$ be a Hilbert space, and let $u \in \mathcal{V} \subset \mathcal{X}$ be a solution to

$$a(u, v) + \langle f(u), v \rangle = 0, \qquad \forall v \in \mathcal{X},$$

where $a : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a symmetric, coercive, bounded bilinear form and the mapping $f : \mathcal{V} \to \mathcal{X}^*$ satisfies

$$\|f(v) - f(w)\|_{\mathcal{X}^*} \leqslant K\|v - w\|_{\mathcal{X}_-}, \qquad \forall v, w \in \mathcal{V},$$

with $\mathcal{X}_-$ being a Banach space such that $\mathcal{X} \hookrightarrow \mathcal{X}_-$ and $\| \cdot \|_{\mathcal{X}_-} \leqslant \| \cdot \|_{\mathcal{X}}$.

Galerkin approximation $u_h \in X_h \cap \mathcal{V}$ of u from $X_h \subset \mathcal{X}$:

$$a(u_h, v) + \langle f(u_h), v \rangle = 0, \qquad \forall v \in X_h \subset \mathcal{X}.$$

Quasi-Orthogonality

We have

$$\|u - u_h\|_{\mathcal{X}_-} \leqslant \alpha_h \|u - u_h\|_a, \tag{7}$$

where $\alpha_h \to 0$ as $h \to 0$.

Lemma

Let a and b satisfy the above assumptions. Then

$$\|u - u_h\|_a^2 \leqslant \Lambda_h \|u - u_H\|_a^2 - \|u_h - u_H\|_a^2,$$

where $\Lambda_h = (1 - \alpha_h m K)^{-1}$, *and m is the constant in* $\|\cdot\| \leqslant m \|\cdot\|_a$.

Discrete existence

If an invertible operator A satisfies the maximum principle, then one can obtain existence results for nonlinear equations of the form $Au = f(u)$, in the case that there exist sub- and super-solutions to that equation.
Here P is a mesh, and S_P is the piecewise linear finite element space.

Theorem

Consider the discrete Hamiltonian constraint with the sub- and super-solutions χ_- and χ_+. Assume that the discretized Laplacian satisfies the maximum principle.
Then, for all partitions with sufficiently small mesh-size h_P, there exists a solution $u_P \in [\chi_-, \chi_+] \cap S_P$ to the discrete Hamiltonian constraint.

Discrete maximum principle

Recall that the linear operator A satisfies the *maximum principle* if $A\chi \geqslant 0$ implies $\chi \geqslant 0$.

Lemma

Let the stiffness matrix $A_{\alpha\beta} = a(\phi_P^\beta, \phi_P^\alpha)$, $\alpha, \beta \in V_P$, be nonsingular, and satisfy the following conditions:

$$\sum_{\beta \in V_P} A_{\alpha\beta} \geqslant 0, \qquad \alpha \in V_P,$$

$$A_{\alpha\beta} \leqslant 0, \qquad \alpha, \beta \in V_P,\ \alpha \neq \beta.$$

Then, $[A_{\alpha\beta}]$ satisfies the maximum principle.

Discrete *a priori* L^∞ estimate

Lemma

Let the elements $A_{\alpha\beta} = a(\phi_P^\beta, \phi_P^\alpha)$ *satisfy*

$$\frac{\operatorname{dist}(\alpha,\beta)^2}{A_{\alpha\beta}} \to 0^- \qquad \textit{as the partition } P \textit{ is refined so that } \operatorname{dist}(\alpha,\beta) \to 0, \tag{8}$$

for all $\alpha, \beta \in V_P$ *with* $\alpha \neq \beta$*, and*

$$\textstyle\sum_{\beta \in V_P} A_{\alpha\beta} \geqslant 0, \qquad \textit{for all } \alpha \in V_P. \tag{9}$$

Let f *be non-decreasing and positive on* $[\chi_+, \infty)$*, and non-decreasing and negative on* $(0, \chi_-]$*, for some* $0 < \chi_- \leqslant \chi_+$*. Let* $\|u_P\|_{W^{1,p}} \lesssim 1$ *for some* $p > 3$*. Then for any* $\varepsilon > 0$ *there exists a partition* $\tilde{P}$ *such that when* P *is any refinement of* $\tilde{P}$ *it holds that* $u_P \in [\chi_- - \varepsilon, \chi_+ + \varepsilon]$*.*

Ongoing/Future

- Fast solution of the discretized system
- Geometry
- Boundary conditions
- Coupled system
- Higher order elements, flexible mesh
- Wavelets
- Evolution equation

Manuscripts, Collaborators, Acknowledgments

[HNT2] M. Holst, G. Nagy, and GT, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*.

[HNT1] M. Holst, , G. Nagy, and GT, *Rough solutions of the Einstein constraint equations on closed manifolds without near-CMC conditions*.

[HT2] M. Holst, and GT, *Convergent Adaptive Finite Element Approximation of Nonlinear Geometric PDE*. Preprint.

[HT1] M. Holst, and GT, *Convergent Adaptive Finite Element Approximation of the Einstein Constraints*. Preprint.

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