The Multiagent Rendez-Vous Problem under Limited Communication Length

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Outline

Introduction

Problems of **coordinated control** deal with the behaviour of networks of individuals ('the agents'), who can exchange partial information among them, and who aim at reaching or preserving a particular configuration.

Examples of such situations are:

- *Rendez-vous*
- *Formation*
- *Flocking*

The mathematical modeling of such situations is a challenging task, with a wide range of applications.

The rendez-vous problem

Assume we have a large number N of agents in the space $\mathbb{R}^q$ (typically with $q = 1, 2, 3$).

The position of agent j at time t will be denoted by $x_j(t)$ and we will assume that they will all undergo the same dynamics governed by a linear input/output law which can be in either discrete or continuous time.

The discrete-in-time model

The evolution of all agents takes place over the lattice $0, \tau, 2\tau, \ldots$ where $\tau > 0$ is a fixed time step.
For the sake of notational simplicity, we assume $\tau = 1$, so that each $x_j(t)$ depends upon $t \in \mathbb{N}$.

We assume a first-order evolution law of every agent:

$$x_j(t+1) = Ax_j(t) + Bu_j(t)$$

The vector $u_j(t) \in \mathbb{R}^k$ plays the role of a control input function that each agent can autonomously choose on the basis of the information available at time t: in general, it will be a function of the position $x_j(t)$ as well of the information transmitted by its neighbors.

A and B are constant matrices of dimension $q \times q$ and $q \times k$, respectively.

If we use the notation

$$x(t) = (x_1(t), \dots x_N(t))^*\,, \qquad u(t) = (u_1(t), \dots u_N(t))^*\,,$$

we can rewrite the whole model as

$$x(t+1) = \bar{A}x(t) + \bar{B}u(t)\,,$$

for suitable matrices $\bar{A}$, $\bar{B}$.

Equivalently,

$$x(t+1) = x(t) + \hat{A}x(t) + \bar{B}u(t)\,,$$

with $\hat{A} = \bar{A} - I$.

A **feedback control law** for such a system consists of a sequence of matrices $K(t)$ to which there correspond controls of type

$$u(t) = K(t)x(t)\,.$$

Inserting in the above model we obtain the autonomous system

$$x(t+1) = (\bar{A} + \bar{B}K(t))x(t)\,.$$

We say that the feedback control law $K(t)$ satisfies the **rendez-vous problem** if for every initial condition $z(0)$, there exists $\alpha \in \mathbb{R}^q$ such that

$$\lim_{t \to +\infty} x_j(t) = \alpha \quad \forall j = 1, \ldots, N\,.$$

Moreover, we say that it satisfies the **barycentral rendez-vous problem** if $\alpha = N^{-1} \sum_j x_j(0)$.

The continuous-in-time problem

The scheme

$$x(t+1) = x(t) + (\hat{A} + \bar{B}K(t))x(t)$$

can be considered as the *Explicit Euler* time discretization of the continuous dynamical system (control problem with feedback)

$$x'(t) = (\hat{A} + \bar{B}K(t))x(t)\,, \qquad t \in \mathbb{R}_+\,.$$

The right-hand side is a *velocity*, which depends on the position via the feedback law.

The communication model

We assume constraints on the matrices $K(t)$, deriving from communication limitations.

Precisely, we assume that at every time instant t we have an undirected communication graph $\mathcal{G}_t = (V, \mathcal{E}_t)$ where $V = \{1, 2, \ldots, N\}$ and where $\mathcal{E}_t$ is a family of unordered pairs of vertices in V.

Given $j \in V$ we consider the neighborhood of j at time t formally defined by

$$\mathcal{N}_t(j) = \{k \in V \setminus \{j\} \mid \{j, k\} \in \mathcal{E}_t\}\,;$$

we also define

$$\nu_j(t) = |\mathcal{N}_t(j)|\,.$$

The communication model (cont'd)

On the other hand, each matrix $K(t)$ has a natural block structure induced by the blocks $u_i(t)$ and $x_j(t)$. We denote by $K_{ij}(t)$ the $k \times q$ block corresponding to $u_i(t)$ and $x_j(t)$.

The feedback control law $K(t)$ is said to be **adapted to the graph** $\mathcal{G}_t$ if, for every time t and every pair of indices i and j, we have that

$$K_{ij}(t) \neq 0 \;\Rightarrow\; i \in \mathcal{N}_t(j)\,.$$

The geometric graph

Assumption: The communication graph $\mathcal{G}_t$ is linked to the positions of the agents, as a consequence of a limitation in the communication length among agents.

Precisley, we assume that there is a constant $R > 0$ such that the communication graph available at time t is the so-called **geometric graph** $\mathcal{G}_t(R) = (V, \mathcal{E}_t(R))$ for which

$$\{i,j\} \in \mathcal{E}_t(R) \;\Leftrightarrow\; \|x_i(t) - x_j(t)\| \leq R\,.$$

The simplest realization of such a dynamics is

$$x_j(t+1) = \frac{1}{\nu_j(t)} \sum_{k \in N_j(t)} x_k(t) \,.$$

This means that at the new time step each agent places itself in the barycenter of the position of all agents which it sees in a neighborhood of radius R around it.

Equivalently, one has

$$x_j(t+1) = x_j(t) + u_j(t)$$

with

$$u_j(t) = \left(\frac{1}{\nu_j(t)} \sum_{k \in N_j(t)} x_k(t) \right) - x_j(t) \quad = \quad \frac{\displaystyle\sum_{k \in N_j(t)} (x_k(t) - x_j(t))}{\displaystyle\sum_{k \in N_j(t)} 1} \,,$$

i.e., the velocity is the difference between the barycenter indicated above and the current position of the agent.

A vaste literature on this and similar problems is available, e.g.,

Lorenz (2005), Blondel, Hendricks and Tsitsiklis (2007), ...

with different points of view and contributions from mathematicians, statisticians, physicists, engineers, social scientists, ...

Almost invariably, these investigations are based on the **Lagrangean** point of view, which is inherent to the formulation presented so far.

The continuous-in-space model

When the number of agents N is very large, one can identify the set of agents at time t with a **mass distribution** μ_t in $\mathbb{R}^q$.

Since agents are neither created nor destroyed, the total mass of μ_t is preserved, hence (up to a normalization) it is not restrictive to assume that μ_t is, at every time $t \geq 0$, a probability measure in $\mathbb{R}^q$.

In principle, μ_t can be any Borel probability measure in $\mathbb{R}^q$,
such as a (normalized) Lebesgue measure in $[a, b]$, or a fully atomic measure $\mu_t = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j(t)}$.

A **velocity field** $V_t(x) = V_t(\mu_t)(x)$ is attached to any point $x \in \mathbb{R}^q$ at time t.

Lagrangean approach $\longrightarrow$ **Eulerian approach**

A mass transfer problem

In a discrete-in-time setting, the dynamical system takes the form

$$\mu_{t+1} = T(\mu_t)\mu_t = p_t \# \mu_t, \quad t = 0, 1, 2, \dots ,$$

where

$$T(\mu_t) = p_t \# \cdot$$

is the **push-forward** of a measure by the mapping

$$p_t : \mathrm{supp}\, \mu_t \subseteq \mathbb{R}^q \to \mathbb{R}^q, \qquad p_t(x) = x + V_t(\mu_t)(x) ,$$

which is formally defined as

$$p_t \# \mu_t(E) = \mu_t(p_t^{-1}(E)) \qquad \text{for every Borel set } E.$$

A mass transfer problem (cont'd)

Equivalently, we have

$$\int_{\mathbb{R}^q} \varphi(x)\, d\mu_{t+1} = \int_{\mathbb{R}^q} \varphi(x + V_t(x))\, d\mu_t$$

for every (bounded and Borel) function φ.

By choosing φ as the characteristic function of a set E one may substantiate the intuitive idea that a point x in the support of μ_t moves at time $t+1$ to the point $x + Vt(x)$ in the support of μ_{t+1}.

This formulation is an instance of **mass transport problem**, related to the classical **Monge-Kantorovich** problem.

The velocity field is defined, for any $x \in \mathrm{supp}(\mu_t)$, as

$$V_t(x) = V_t(\mu_t)(x) = \frac{\displaystyle\int_{\mathbb{R}^q} \xi_R(y-x)\, y\, d\mu_t(y)}{\displaystyle\int_{\mathbb{R}^q} \xi_R(y-x)\, d\mu_t(y)} - x = \frac{\displaystyle\int_{\mathbb{R}^q} \xi_R(y-x)\,(y-x)\, d\mu_t(y)}{\displaystyle\int_{\mathbb{R}^q} \xi_R(y-x)\, d\mu_t(y)},$$

where $\xi_R : \mathbb{R}^q \to R$ is supported and positive in the ball $B(0,R)$ of radius R around the origin (e.g., the characteristic function of the ball).

Non-local dependence on μ_t.

A simplified model uses

$$V_t(x) = \int_{\mathbb{R}^q} \xi_R(y-x)\,(y-x)\, d\mu_t(y)$$

instead.

The continuous-in time model

The continuous-in-time counterpart is the **conservation law** (continuity equation)

$$\frac{\partial}{\partial t}\mu_t + \operatorname{div} V_t \mu_t = 0 \,,$$

to be meant in the sense of measures, i.e.,

$$\frac{d}{dt}\int_{\mathbb{R}^q} \eta(x)\, d\mu_t(x) = \int_{\mathbb{R}^q} \nabla\eta(x)\cdot V_t(x)\, d\mu_t(x)$$

for any test function $\eta \in \mathcal{D}(R^q)$.

We say that a family of probability measures μ_t, $t \geq 0$, is a solution, if for every test function $\eta(x) \in \mathcal{D}(\mathbb{R}^q)$, the function

$$t \mapsto \int_{\mathbb{R}^q} \eta(x)\, d\mu_t(x), \quad t \geq 0,$$

is continuous in $[0,\infty)$, differentiable in $(0,\infty)$ and satisfies the previous equation for every $t > 0$.

The case of absolutely continuous measures

Assume that the probability measures μ_t are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^q$, i.e., there exists a density function $\rho(t,x) \geq 0$, which for all t is compactly supported in x and satisfies $\int_{\mathbb{R}^q} \rho(t,x)\,dx = 1$, such that

$$d\mu_t = \rho(t,x)\,dx\,.$$

Then, the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \operatorname{div} F = 0\,,$$

where the flux F is the nonlocal function

$$F(\rho;t,x) = V(\rho;t,x)\rho(t,x)$$

depending on the velocity field

$$V(\rho;t,x) = \frac{\displaystyle\int_{\mathbb{R}^q} \xi_R(y-x)\,(y-x)\,\rho(t,y)\,dy}{\displaystyle\int_{\mathbb{R}^q} \xi_R(y-x)\,\rho(t,y)\,dy}\,.$$

Existence and Uniqueness

Assume that V is given by $V_t(x) = \int_{\mathbb{R}^q} \xi_R(y-x)\,(y-x)\,d\mu_t(y)$ and that the cut-off function ξ_R satisfies $\xi_R(-x) = \xi_R(x)$ for all x.

Theorem

Let μ_0 be any probability measure on $\mathbb{R}^q$ with compact support. Then the discrete-in-time dynamical system

$$\mu_{t+1} = T(\mu_t)\mu_t \ , \quad t = 0, 1, 2, \dots \ ,$$

generates a sequence of probability measures which converge, as $t \to \infty$, to a limit probability measure μ_∞. This is a purely atomic measure, whose atoms are a distance at least R apart from one another.

A similar result holds for the continuous-in-time dynamical system, with the additional result of the uniqueness of the solution.

Schetch of proof:

- Prove the bound

$$\sum_{t=0}^{\infty} \int_{\mathbb{R}^q} \int_{\mathbb{R}^q} \xi_R(y-x) \|y-x\|^2 \, d\mu_t(y) d\mu_t(x) < +\infty$$

- Prove that the second-order moments

$$\int_{\mathbb{R}^q} \|x\|^2 \, d\mu_t(x)$$

  form a non-increasing sequence as $t \to \infty$.

- Deduce from previous steps that there exists a probability measure μ_∞ such that

$$\lim_{t\to\infty} \int_{\mathbb{R}^q} \varphi(x) \, d\mu_t(x) = \int_{\mathbb{R}^q} \varphi(x) \, d\mu_\infty(x)$$

  for all bounded and continuous function φ.

- Prove by contradiction that the non-atomic part of μ_∞ is zero.

Numerical simulations

Assuming $\rho_0(x)$ a compactly supported piecewise smooth function, the initial-value problem

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(V\rho) = 0 \,, \qquad x \in \mathbb{R}^q, \quad t > 0 \,,$$
$$\rho(x,0) = \rho_0(x) \,, \qquad x \in \mathbb{R}^q$$

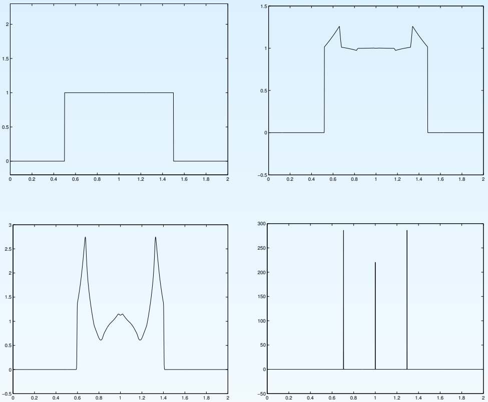
is discretized by a standard finite-volume scheme of upwind type on a uniform Cartesian grid, with Courant number $=1$.

Invariably, the cut-off function ξ_R has been chosen as the characteristic function of the ball $B(0,R)$.

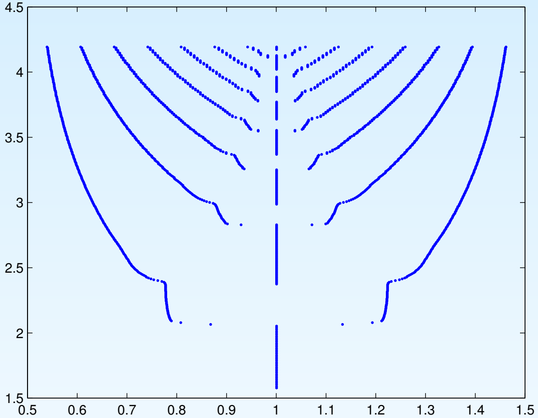
A significant robustness of the qualitative features of the discretized dynamics with respect to the discretization parameter(s) has been observed.

Numerical simulations (cont'd)

The evolution of a piecewise constant density towards steady state:

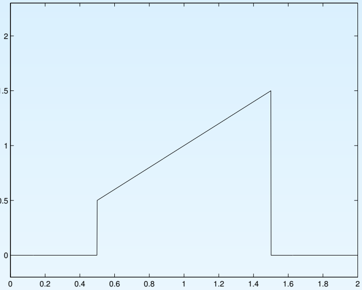


Constant initial density

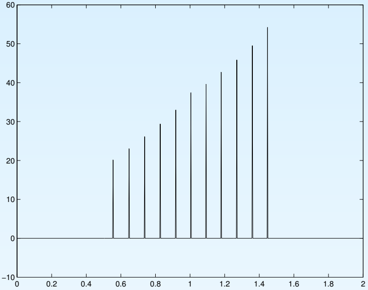


Position of deltas (horizontal axis) vs $|\log R|$ (vertical axis)

Linear density evolution



initial



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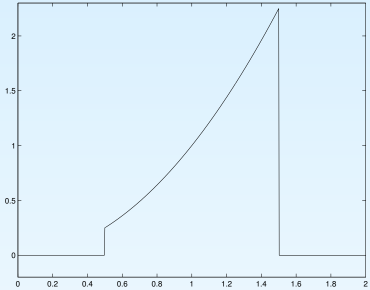
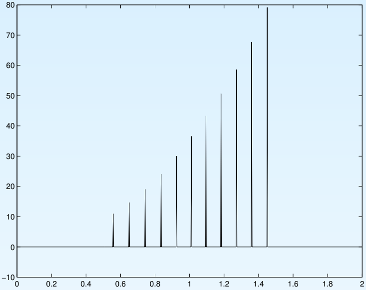
Parabolic density evolution



initial



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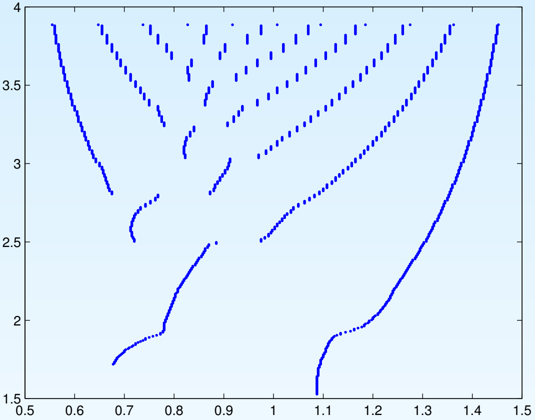
Linear initial density



Position of deltas (horizontal axis) vs $|\log R|$ (vertical axis)

Invariance and stability

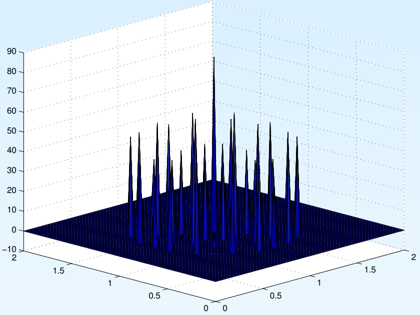
A probability measure μ is said to have a **radial symmetry** with respect to $x_0 \in \mathbb{R}^q$ if for any rotation U centered at x_0 one has $U\#\mu = \mu$.

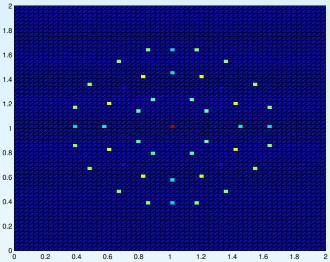
Theorem

In dimension $q > 1$, let ξ_R be a radial function. If the initial measure μ_0 has radial symmetry with respect to some x_0, then $\mu_\infty = \delta_{x_0}$.

Thus, in 2D, if μ_0 is the characteristic function of the unit circle, then $\mu_\infty = \delta_0$.
However, ...

Invariance and stability (cont'd)





Rotational invariance is unstable.

Further developments

The present investigation has shown that the rendez-vous problem can be succesfully solved only if the communication length is sufficiently large.
To enhance the possibility of success, one could think of

- having different species of agents, with different communication power (e.g., few 'master agents' with long-range communication, many 'slave agents' with short-range communication)

Potentially interesting additional features of the model may include

- the presence of noise (diffusion effects)
- the roughness of the 'ground' (friction effects)
- the topology of the 'ground' (boundary effects)
- ...