Quasi-Optimal convergence rate of an Adaptive Discontinuous Galerkin method on Elliptic problems

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Outline

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Equidistribution of the error (1D case: $\Omega = [0, 1]$)

Smooth case $u \in W^{1,\infty}(0, 1)$. Given an uniform partition $x_i = i/N$, $i = 0, ..., N$. Let

$$U_N(x) := u(x_{n-1}), \qquad x_{n-1} \leq x < x_n.$$

It follows for $x \in [x_{n-1}, x_n)$ that

$$|u(x) - U_N(x)| = |u(x) - u(x_{n-1})| = \left| \int_{x_{n-1}}^{x_n} u'(t) \right| \leq h||u'||_{L^\infty(x_{n-1},x_n)}.$$

So that

$$||u - U_N||_{L^\infty(0,1)} \leq \frac{1}{N}||u'||_{L^\infty(0,1)}.$$

Equidistribution of the error (1D case: $\Omega = [0,1]$)

Rough case $u \in W^{1,1}(0,1)$. Assume $||u'||_{L^1(0,1)} = 1$ and define

$$\phi(x) := \int_0^x |u'|$$

inducing the partition given by

$$\int_{x_n}^{x_{n+1}} |u'| = \phi(x_{n+1}) - \phi(x_n) = \frac{1}{N}.$$

Then, for $x \in [x_n, x_{n+1}]$,

$$|u(x) - U_N(x)| = |u(x) - u(x_{n-1})| = \left| \int_{x_{n-1}}^{x} u'(t) \right| \leq \int_{x_{n-1}}^{x_n} |u'(t)| = \frac{1}{N}$$

or

$$||u - U_N||_{L^\infty(0,1)} \leq \frac{1}{N} ||u'||_{L^1(0,1)}.$$

Adaptive Finite Element Method (AFEM)

Let $\mathcal{T}_0$ be an initial triangulation of Ω a bounded set of $\mathbb{R}^d$, $k = 0$.

SOLVE : Compute the solution $U_k \in \mathbb{V}_k := \mathbb{V}(\mathcal{T}_k)$ of the discrete problem.

ESTIMATE : Compute an estimator for the error.
in terms of the discrete solution U_k and given data.

MARK : Use the estimator to mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ for refinement.

REFINE : Refine the marked subset $\mathcal{M}_k$ to obtain $\mathcal{T}_{k+1}$, increment k and go to step SOLVE.

$$\lim_{k \to \infty} U_k = U \ ?$$

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$$\lim_{k\to\infty} U_k = U \ ?$$

Continuous piecewise linear finite element for the Laplacian

SOLVE:

Seek $U_k \in \mathbb{V}_k \cap H_0^1(\Omega)$ solving

$$\int_\Omega \nabla U_k \nabla V = \int_\Omega fV, \qquad \forall V \in \mathbb{V}_k \cap H_0^1(\Omega).$$

ESTIMATE:

Compute

$$\eta_k(U_k, T)^2 := \|h_T \; f\|^2_{L^2(T)} + \left\| h_\sigma^{1/2} \, [\![\partial_\nu U_k]\!] \right\|^2_{L^2(\partial T)}.$$

MARK:

For $\theta \in (0,1)$, select a subset $\mathcal{M}_k$ of $\mathcal{T}_k$ so that

$$\sum_{T\in\mathcal{M}_k} \eta_k^2(U_k, T) \geq \theta \sum_{T\in\mathcal{T}_k} \eta_k^2(U_k, T).$$

REFINE:

Bissect each element of $\mathcal{M}_k$ and others to obtain a conforming mesh.

History of AFEM, focus on interior node property and oscillation

- Babuska, Vogelius, 1986 (1D problem).
- Dörfler, 1996.
- Morin, Nochetto, Siebert, Siam J.Numer.Anal., 2000: MARK by oscillation

$$\operatorname{osc}(f,\mathcal{T}) := \left\| h(f-\overline{f}) \right\|_{L^2(\mathcal{T})}, \quad \overline{f} := \fint_T f \quad (\text{for} \quad \mathbb{P}^1),$$

  interior node needed and require a certain refinement depth but implies the discrete lower bound

$$\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\Omega)} \preceq \sum_{T\in\mathcal{T}} \left(\eta_k^2(U_k,T) + \operatorname{osc}^2(f,\mathcal{T})\right).$$

$$\|\nabla(u-U_k)\|_{L^2(\Omega)} \leq C\alpha^k, \qquad 0<\alpha<1.$$

- Stevenson, Found. Comput. Math, 2006: MARK by oscillation only if the oscillation is small with respect to the estimator.
- Kreuzer, Cascon, Nochetto, Siebert, submitted, 2007: MARK by oscillation not needed and there exists $\delta > 0$ such that

$$\|\nabla(u-U_{k+1})\|_{L^2(\Omega)} + \delta\eta_{k+1} \leq \alpha\left(\|\nabla(u-U_k)\|_{L^2(\Omega)} + \delta\eta_k\right).$$

  No interior node needed.

Outline

Continuous Problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open polygonal domain. We consider the following linear elliptic symmetric PDE

$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

In weak formulation: Find $u \in \mathbb{V} := H_0^1(\Omega)$, so that

$$a\left(u, v\right) := \int_\Omega \mathbf{A}\nabla u \cdot \nabla v = \int_\Omega f\, v \qquad \text{for all } v \in \mathbb{V}. \qquad \text{(PDE)}$$

Assumptions on Data:

Let $\mathcal{T}_0$ be an conforming initial triangulation of Ω.

- $\mathbf{A} : \Omega \mapsto \mathbb{R}^{d\times d}$ is piecewise Lipschitz over $\mathcal{T}_0$ and is symmetric positive definite with eigenvalues $0 < a_* \leq a^* < \infty$ a.e. in Ω.
- $f \in L^2(\Omega)$.

Energy-norm is defined by

$$|\!|\!| v - w |\!|\!|_\Omega := a\left(v - w, v - w\right)^{1/2}.$$

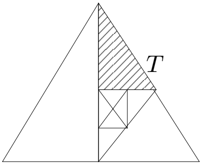
Existence and uniqueness is established by the Riesz Representation Theorem.

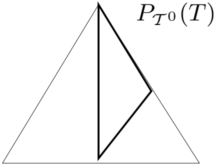
Finite Element Discretization

Let $\mathcal{T}_0$ be an initial, conforming triangulation of Ω.

Subdivisions

- We define $\mathbb{T}$ as the set of all conforming or noncomforming refinements of $\mathcal{T}_0$, that can be generated from $\mathcal{T}_0$ using iterative or recursive bisection.
- Given $\mathcal{T}$, let $\mathcal{T}^0 \leq \mathcal{T}$ be the underlying conforming subdivision.
- The level $l(T)$ of an element $T \in \mathcal{T}$ is the number of recursive bisection needed to obtain T starting from an element of the initial mesh.
- The level of nonconformity of $\mathcal{T}$ is $L := \max_{T\in\mathcal{T}}(\text{level}(T) - \text{level}(P_{\mathcal{T}^0}(T)))$.





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FEM

- For $\mathcal{T} \in \mathbb{T}$ we define $\mathbb{V}(\mathcal{T}) := \prod_{T\in\mathcal{T}} \mathbb{P}^n(T)$, $n \geq 1$, as a discontinuous piecewise polynomial finite element space over $\mathcal{T}$.
- For $\mathcal{T} \in \mathbb{T}$ we denote by $\mathcal{S}$ the set of faces (sides).
- We define $(.,.)_\omega := \sum_{T\in\omega} (.,.)_T$ for any $\omega \subset \Omega$, and $\langle .,.\rangle_\sigma := \sum_{s\in\sigma} \langle .,.\rangle_s$ for any $\sigma \subset \mathcal{S}$.
- $\{V\}_\sigma = \frac{1}{2}(V^+ + V^-)$, $[\![V]\!]_\sigma = V^+\nu_T^+ - V^-\nu_T^-$, with natrual extension to $W \in \mathbb{V}(\mathcal{T})^d$.

The Interior Penalty Method (IPM)

The discrete problem reads: Find $U \in \mathbb{V}(\mathcal{T})$, so that

$$a_{\mathcal{T}}(U,V) = (f,V), \qquad \forall V \in \mathbb{V}(\mathcal{T}), \qquad (\mathsf{PDE}_k)$$

where

$$\begin{aligned} a_{\mathcal{T}}^{\text{Arnold}}(U,V) &:= (\mathbf{A}\nabla U, \nabla V)_{\mathcal{T}} - \langle \{\mathbf{A}\nabla U\}, [\![V]\!]\rangle_{\mathcal{S}} \\ &\quad - \langle \{\mathbf{A}\nabla V\}, [\![U]\!]\rangle_{\mathcal{S}} + \gamma \left\langle h^{-1}[\![U]\!], [\![V]\!]\right\rangle_{\mathcal{S}} \\ &\equiv (\mathbf{A}\nabla U, \nabla V)_{\mathcal{T}} - (L_{\mathcal{T}}(U), A\nabla V)_{\mathcal{T}} \\ &\quad - (L_{\mathcal{T}}(V), A\nabla U)_{\mathcal{T}} + \gamma \left\langle h^{-1}[\![U]\!], [\![V]\!]\right\rangle_{\mathcal{S}} := a_{\mathcal{T}}(U,V), \end{aligned}$$

where $L_{\mathcal{T}} : \mathbb{V}(\mathcal{T}) + H_0^1(\Omega) \to \mathbb{V}(\mathcal{T})^d$, $(L_{\mathcal{T}}(V), AW)_{\mathcal{T}} = \langle [\![V]\!], \{\mathbf{A}W\}\rangle_{\mathcal{S}}$, $\forall W \in \mathbb{V}(\mathcal{T})^d$.

References:

J. Nitsche (1971), J. Douglas and T. Dupont (1975), G.A. Baker, (1977), D.N. Arnold, (1982), F.Bassi and S.Rebay, (1997), P. Houston and D. Schötzau and T. Whiler (2007).

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Some properties

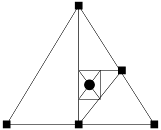
Consistency

$$a_{\mathcal{T}}(u-U,V)=0, \qquad \forall V \in \mathbb{V}^0(\mathcal{T}) := \mathbb{V}(\mathcal{T}) \cap H^1_0(\Omega).$$

Decomposition of the finite element space

$$\mathbb{V}(\mathcal{T}) := \mathbb{V}^0(\mathcal{T}) + \mathbb{V}^{\perp}(\mathcal{T}), \mathbb{V}^0(\mathcal{T}) := \mathbb{V}(\mathcal{T}) \cap H^1_0(\Omega).$$

Difficulty: $\mathbb{V}^0(\mathcal{T}^0) \subset \mathbb{V}^0(\mathcal{T})$



Lifting operator: Only need to assume that $u \in H^1_0(\Omega)$!

Outline

Adaptive Finite Element Method (AFEM)

Let $\mathcal{T}_0$ be an initial triangulation of Ω, $k = 0$.

SOLVE : Compute the solution $U_k \in \mathbb{V}_k := \mathbb{V}(\mathcal{T}_k)$ of the discrete equation (PDE$_k$).

ESTIMATE : Compute an estimator for the error $a_{\mathcal{T}_k}(u - U_k, u - U_k)$ in terms of the discrete solution U_k and given data.

MARK : Use the estimator to mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ for refinement.

REFINE : Refine the marked subset $\mathcal{M}_k$, with at least one bisection but without the interior node property, to obtain $\mathcal{T}_{k+1}$, increment k and go to step SOLVE.

Main Result:

$$\lim_{k\to\infty} a_{\mathcal{T}_k}(u - U_k, u - U_k) = 0, \qquad \gamma \geq \tilde{\gamma}.$$

The Module SOLVE

Define the energy norm (related to $\mathbb{E}(\mathcal{T}_k) := \mathbb{V}_k + H_0^1(\Omega)$)

$$\| V \|_k := \left\| \mathbf{A}^{1/2} \nabla V \right\|_{L^2(\mathcal{T}_k)} + \gamma^{1/2} \left\| h^{-1/2} [\![V]\!] \right\|_{L^2(\mathcal{S}_k)}$$

Continuity:

$$a_{\mathcal{T}_k}(U, V) \preceq \| U \|_k \| V \|_k, \qquad \forall\, U, V \in \mathbb{E}(\mathcal{T}_k).$$

Coercivity:

$$\| V \|_k^2 \preceq a_{\mathcal{T}_k}(V, V), \qquad \forall\, V \in \mathbb{E}(\mathcal{T}_k),$$

for all $\gamma \geq \gamma_0 > 0$.
Hence, Riesz Representation Theorem also implies existence and uniqueness of the discrete solution $U_k \in \mathbb{V}_k$ of (PDE$_k$).

The Module ESTIMATE

Standard residual based error estimator $\eta_k(U_k, T), U_k \in \mathbb{V}_k, T \in \mathcal{T}_k$

$$\eta_k^2(U_k, T) := \|h(f + \text{div}(\mathbf{A}\nabla U_k))\|_{L^2(T)}^2 + \left\| h^{1/2} [\![\mathbf{A}\nabla U_k]\!] \right\|_{L^2(\partial T \cap \Omega)}^2 .$$

For any subset $\mathcal{M} \subset \mathcal{T}_k$ we use the conventions

$$\eta_k^2(U_k, \mathcal{M}) := \sum_{T \in \mathcal{M}} \eta_k^2(U_k, T) \qquad \text{and} \qquad \eta_k := \eta_k(U_k, \mathcal{T}_k).$$

The estimator is monotone:

$$\eta_{k+1}(U_k, \mathcal{T}_{k+1}) \leq \eta_k(U_k, \mathcal{T}_k).$$

The "error" is bounded by

$$a_{\mathcal{T}_k}(u - U_k, u - U_k) \preceq \eta_k^2 + \gamma^2 \left\| h^{-1/2} [\![U_k]\!] \right\|_{L^2(\mathcal{S}_k)}^2 . \qquad \text{(UPPER)}$$

The Module MARK - Dörfler's Marking

For $\theta \in (0,1)$, we mark a minimal subset $\mathcal{M}_k$ of $\mathcal{T}_k$, so that

$$\eta_k^2(\mathcal{M}_k) \geq \theta\, \eta_k^2.$$

The Module REFINE

"Sketch of the algorithm" Given $L \geq 0$ be the maximum level of nonconformity. For each marked element $T \in \mathcal{M}_k$:

- if the level of nonconformity of T is $< L$, then bissect T (newest vertex) .
- else bissect $P_{\mathcal{T}^0}(T)$ and make the new subdivision conform (Stevenson). Bissect T is not already bissected.

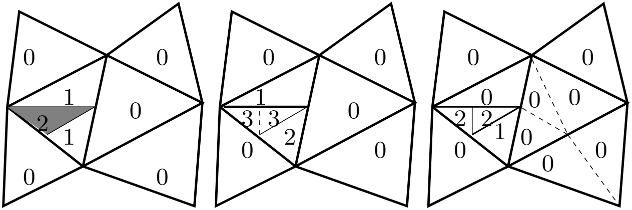


Figure: The algorithm Refine for $L = 2$.

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Properties

- Level of nonconformity is preserved and shape-regularity of $\mathcal{T}_k$ solely depends on the shape-regularity of $\mathcal{T}_0$ and L.No interior node property.
- The sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k, \mathcal{T}_{k+1}, \ldots$ of generated triangulations is nested which implies nested spaces $\mathbb{V}_0 \subset \mathbb{V}_1 \subset \ldots \mathbb{V}_k \subset \mathbb{V}_{k+1} \subset \ldots$
- Mesh-size of refined elements is strictly decreased: For the two children T_1, T_2 of any bisected element $T \in \mathcal{T}_k$ we have $|T_i| = \frac{1}{2}|T|$, $i = 1, 2$, i.e., the mesh-size reduction

$$h_{T_i} \leq 2^{-1/d} h_T.$$

Outline

Proposition (Linear Convergence)

There exist constants $\delta > 0$, $\tilde{\gamma} \geq \gamma_0$, $0 < \alpha < 1$, *depending solely on the shape regularity of* $\mathcal{T}_0$, L, *and the marking parameter* $0 < \theta < 1$, *such that for* $\gamma > \tilde{\gamma}$,

$$a_{\mathcal{T}_{k+1}}(u - U_{k+1}, u - U_{k+1}) + \delta\, \eta_{k+1}^2 \leq \alpha^2 \left(a_{\mathcal{T}_k}(u - U_k, u - U_k) + \delta\, \eta_k^2 \right).$$

Properties:

- Only one bisection needed. No interior node property.
- Valid for any $f \in L^2(\Omega)$.
- No assumptions on how the procedure REFINE behaves.
- No smallness assumption on the data or the meshsize.

History of AFEMs (discontinuous)

- Karakashian and Pascal, SIAM J.Numer. Anal., 2007

$$a_{\mathcal{T}_{k+1}}\left(u - U_{k+1}, u - U_{k+1}\right) < \alpha a_{\mathcal{T}_k}\left(u - U_k, u - U_k\right).$$

  - $f \in \mathbb{V}_0$, γ big enough.
  - Refinement:

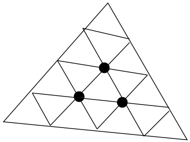

Figure: Refinenement of the triangles for $\mathbb{P}^1$.

- Hoppe, Kanschat and Warburton (preprint)
  - Only one bissection: Crouzeix-Raviart decomposition.
  - Works in 2D and on conforming subdivisions.

What we do not have?

Mesh independent Galerkin orthogonality

$$a_{\mathcal{T}_{k+1}}\left(u - U_{k+1}, V^0\right) = 0 \quad \forall V^0 \in \mathbb{V}^0_{k+1}, \qquad \left(V = U^0_{k+1} - U^0_k\right).$$

Therefore, we have

$$\begin{aligned} a_{\mathcal{T}_{k+1}}(u - U_{k+1}, u - U_{k+1}) - \frac{\gamma C}{\epsilon}\left\| h^{-1/2} [\![U_{k+1}]\!] \right\|^2_{L^2(\mathcal{S}_{k+1})} \\ = (1+\epsilon) a_{\mathcal{T}_{k+1}}(u - U_k, u - U_k) - a_{\mathcal{T}_{k+1}}(U_{k+1} - U_k, U_{k+1} - U_k) \\ + \frac{\gamma C}{\epsilon}\left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)}, \end{aligned}$$

Clean Upper bound

$$a_{\mathcal{T}_k}(u - U_k, u - U_k) \preceq \eta_k^2 + \gamma^2 \left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)}.$$

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Remedy

Control of the Jumps: There exists $\tilde{\gamma} \geq \gamma_0$ such that for $\gamma \geq \tilde{\gamma}$

$$\gamma^2 \left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)} \preceq \eta_k^2(U_k, \mathcal{T}_k).$$

Idea of the proof, inspired from [KP2007]: Let $\chi \in \mathbb{V}_k \cap H_0^1$,

$$\begin{aligned}
\gamma \left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)} &\preceq a_{\mathcal{T}_k}(U_k - \chi, U_k - \chi) \\
&= (f, U_k - \chi)_{\mathcal{T}_k} - a_{\mathcal{T}_k}(\chi, U_k - \chi) \\
&= (f, U_k - \chi)_{\mathcal{T}_k} - (\nabla \chi, \nabla (U_k - \chi))_{\mathcal{T}_k} \\
&\quad + (L_{\mathcal{T}_k}(\chi), A\nabla(U_k - \chi))_{\mathcal{T}_k} \\
&\quad + (L_{\mathcal{T}_k}(U_k - \chi), A\nabla U_k)_{\mathcal{T}_k} \\
&\quad - \gamma \left\langle h^{-1} [\![\chi]\!], [\![U_k - \chi]\!] \right\rangle_{\mathcal{S}_k}.
\end{aligned}$$

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&= (f, U_k - \chi)_{\mathcal{T}_k} - (\nabla \chi, \nabla(U_k - \chi))_{\mathcal{T}_k} \\
&\quad + (L_{\mathcal{T}_k}(\chi), A\nabla(U_k - \chi))_{\mathcal{T}_k} \\
&\quad + (L_{\mathcal{T}_k}(U_k - \chi), A\nabla U_k)_{\mathcal{T}_k} \\
&\quad - \gamma \left\langle h^{-1} [\![\chi]\!], [\![U_k - \chi]\!] \right\rangle_{\mathcal{S}_k}.
\end{aligned}$$

Convergence of $a_{\mathcal{T}_k}(u-U_k,u-U_k)+\delta\eta_k^2$: Main Ingredients

Upper Bound

$$a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)\preceq \eta_k^2+\gamma^2\left\|h^{-1/2}\left[\!\left[U_k\right]\!\right]\right\|_{L^2(\mathcal{S}_k)}^2.$$

Coercivity (discrete)

$$\|\nabla(U_{k+1}-U_k)\|_{L^2(\mathcal{T}_{k+1})}^2\preceq a_{\mathcal{T}_{k+1}}\left(U_{k+1}-U_k,U_{k+1}-U_k\right).$$

Quasi Galerkin Orthogonality

$$\begin{aligned}
&a_{\mathcal{T}_{k+1}}\left(u-U_{k+1},u-U_{k+1}\right)-\frac{\gamma C}{\epsilon}\left\|h^{-1/2}\left[\!\left[U_{k+1}\right]\!\right]\right\|_{L^2(\mathcal{S}_{k+1})}^2\\
&\le(1+\epsilon)a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)-a_{\mathcal{T}_{k+1}}\left(U_{k+1}-U_k,U_{k+1}-U_k\right)\\
&\qquad+\frac{\gamma C}{\epsilon}\left\|h^{-1/2}\left[\!\left[U_k\right]\!\right]\right\|_{L^2(\mathcal{S}_k)}^2.
\end{aligned}$$

Estimator Reduction For all $\xi>0$

$$\delta\eta_{k+1}^2\le(1+\xi)\delta\left\{\eta_k^2-(1-2^{-1/d})\,\eta_k^2(\mathcal{M}_k)\right\}+\delta(1+\xi^{-1})\,C\,\|\nabla(U_{k+1}-U_k)\|_{L^2(\mathcal{T}_{k+1})}^2$$

No discrete lower bound needed!

Convergence of $a_{\mathcal{T}_k}(u-U_k,u-U_k)+\delta\eta_k^2$: Main Ingredients

Upper Bound

$$a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)\preceq\eta_k^2.$$

Coercivity (discrete)

$$\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1})}\preceq a_{\mathcal{T}_{k+1}}\left(U_{k+1}-U_k,U_{k+1}-U_k\right).$$

Quasi Galerkin Orthogonality

$$\begin{aligned}a_{\mathcal{T}_{k+1}}\left(u-U_{k+1},u-U_{k+1}\right)&-\frac{\gamma C}{\epsilon}\left\|h^{-1/2}\left[\!\left[U_{k+1}\right]\!\right]\right\|^2_{L^2(\mathcal{S}_{k+1})}\\ \leq(1+\epsilon)a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)&-a_{\mathcal{T}_{k+1}}\left(U_{k+1}-U_k,U_{k+1}-U_k\right)\\ &+\frac{\gamma C}{\epsilon}\left\|h^{-1/2}\left[\!\left[U_k\right]\!\right]\right\|^2_{L^2(\mathcal{S}_k)}.\end{aligned}$$

Estimator Reduction For all $\xi>0$

$$\delta\eta^2_{k+1}\leq(1+\xi)\delta\left\{\eta_k^2-\left(1-2^{-1/d}\right)\eta_k^2(\mathcal{M}_k)\right\}+\delta(1+\xi^{-1})\,C\,\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1})}$$

No discrete lower bound needed!

Convergence of $a_{\mathcal{T}_k}(u-U_k,u-U_k)+\delta\eta_k^2$: Main Ingredients

Upper Bound

$$a_{\mathcal{T}_k}(u-U_k,u-U_k)\preceq \eta_k^2.$$

Coercivity (discrete)

$$\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1})}\preceq a_{\mathcal{T}_{k+1}}(U_{k+1}-U_k,U_{k+1}-U_k).$$

Quasi Galerkin Orthogonality

$$\begin{aligned}a_{\mathcal{T}_{k+1}}(u-U_{k+1},u-U_{k+1})-\frac{C}{\gamma\epsilon}\eta_{k+1}^2\\ \leq (1+\epsilon)a_{\mathcal{T}_k}(u-U_k,u-U_k)-a_{\mathcal{T}_{k+1}}(U_{k+1}-U_k,U_{k+1}-U_k)\\ +\frac{C}{\epsilon\gamma}\eta_k^2.\end{aligned}$$

Estimator Reduction For all $\xi>0$

$$\delta\eta_{k+1}^2\leq(1+\xi)\delta\left\{\eta_k^2-(1-2^{-1/d})\,\eta_k^2(\mathcal{M}_k)\right\}+\delta(1+\xi^{-1})\,C\,\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1}}$$

No discrete lower bound needed!

Convergence of $a_{\mathcal{T}_k}(u-U_k,u-U_k)+\delta\eta_k^2$: Main Ingredients

Upper Bound

$$a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)\preceq\eta_k^2.$$

Coercivity (discrete)

$$\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1})}\preceq a_{\mathcal{T}_{k+1}}\left(U_{k+1}-U_k,U_{k+1}-U_k\right).$$

Quasi Galerkin Orthogonality

$$\begin{aligned}a_{\mathcal{T}_{k+1}}\left(u-U_{k+1},u-U_{k+1}\right)-\frac{C}{\gamma\epsilon}\eta^2_{k+1}\\ \leq(1+\epsilon)a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)-C\left\|\nabla(U_{k+1}-U_k)\right\|_{L^2(\mathcal{T}_{k+1})}\\ +\frac{C}{\epsilon\gamma}\eta_k^2.\end{aligned}$$

Estimator Reduction For all $\xi>0$

$$\delta\eta^2_{k+1}\leq(1+\xi)\delta\left\{\eta_k^2-(1-2^{-1/d})\,\eta_k^2(\mathcal{M}_k)\right\}+\delta(1+\xi^{-1})\,C\,\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1}}$$

No discrete lower bound needed!

Convergence of $a_{\mathcal{T}_k}(u-U_k,u-U_k)+\delta\eta_k^2$: Main Ingredients

Upper Bound

$$a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)\preceq\eta_k^2.$$

Coercivity (discrete)

$$\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1})}\preceq a_{\mathcal{T}_{k+1}}\left(U_{k+1}-U_k,U_{k+1}-U_k\right).$$

Quasi Galerkin Orthogonality

$$\begin{aligned}&a_{\mathcal{T}_{k+1}}\left(u-U_{k+1},u-U_{k+1}\right)-\frac{C}{\gamma\epsilon}\eta_{k+1}^2\\ \leq{}&(1+\epsilon)a_{\mathcal{T}_k}\left(u-U_k,u-U_k\right)-C\left\|\nabla(U_{k+1}-U_k)\right\|_{L^2(\mathcal{T}_{k+1})}\\ &+\frac{C}{\epsilon\gamma}\eta_k^2.\end{aligned}$$

Estimator Reduction For all $\xi>0$

$$\delta\eta_{k+1}^2\leq(1+\xi)\delta\left\{\eta_k^2-(1-2^{-1/d})\,\theta\eta_k^2\right\}+\delta(1+\xi^{-1})\,C\,\|\nabla(U_{k+1}-U_k)\|^2_{L^2(\mathcal{T}_{k+1})}.$$

No discrete lower bound needed!

Proposition (Linear Convergence)

There exist constants $\delta > 0$, $\tilde{\gamma} \geq \gamma_0$, $0 < \alpha < 1$, *depending solely on the shape regularity of* $\mathcal{T}_0$ *and the marking parameter* $0 < \theta < 1$, *such that for* $\gamma > \tilde{\gamma}$,

$$a_{\mathcal{T}_{k+1}}(u - U_{k+1}, u - U_{k+1}) + \delta\, \eta_{k+1}^2 \leq \alpha^2 \left(a_{\mathcal{T}_k}(u - U_k, u - U_k) + \delta\, \eta_k^2 \right).$$

Properties:

- Only one bisection needed. No interior node property.
- Valid for any $f \in L^2(\Omega)$.
- No assumptions on how the procedure REFINE behaves.

Outline

Prior Optimality Results (for continuous FEM)

- Binev, Dahmen, De Vore, 2004: Optimal cardinality of AFEM with coarsening.
- Stevenson, 2005: Optimal cardinality of AFEM for piecewise constant coefficient matrix. Additional refinement for oscillation is needed.
- Kreuzer, Cascon, Nochetto, and Siebert, 2008.

Class of approximation

Total error

$$\mathcal{E}(V,\mathcal{T})^2 := \|\nabla(u-V)\|^2_{L^2(\mathcal{T})} + \gamma \left\| h^{-1/2} [\![V]\!] \right\|^2_{L^2(\mathcal{S})} + \operatorname{osc}^2_{\mathcal{T}}(V).$$

where

$$\operatorname{osc}^2_{\mathcal{T}}(V) := \|h(I-P_{n-1})(f-\mathsf{div}(AV))\|^2_{L^2(\mathcal{T})} + \left\| h^{1/2}(I-P_{2n-1}) [\![A\nabla V]\!] \right\|^2_{L^2(\mathcal{S}\cap\Omega)}$$

Rem: the global lower bound $\eta_{\mathcal{T}_k}(U_k,\mathcal{T}_k)^2 \preceq |||u-U_k|||^2_{\mathcal{T}_k} + \operatorname{osc}^2_{\mathcal{T}_k}(U_k,\mathcal{T}_k)$ implies

$$\mathcal{E}(U_k,\mathcal{T}_k)^2 \approx \|\nabla(u-U_k)\|^2_{L^2(\mathcal{T}_k)} + \gamma \left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)} + \eta^2_{\mathcal{T}_k}(U_k,\mathcal{T}_k)^2.$$

Class of approximation

$$\mathcal{E}(V,\mathcal{T})^2 := \|\nabla(u-V)\|^2_{L^2(\mathcal{T})} + \gamma \left\| h^{-1/2} [\![V]\!] \right\|^2_{L^2(\mathcal{S})} + \mathrm{osc}^2_{\mathcal{T}}(V).$$

"Non linear" class of approximation Define

$$\mathbb{T}_N := \Big\{ \mathcal{T},\ \mathcal{T} \text{ refinement of } \mathcal{T}_0, \text{ level of nonconformity bounded by } L, \quad \dim(\mathbb{V}(\mathcal{T})) \le N \Big\}$$

$$\mathbb{A}_s := \left\{ (u,f,A) : \sup_{N>0} \left(N^s \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{V\in\mathbb{V}(\mathcal{T})} \mathcal{E}(V,\mathcal{T}) \right) < \infty \right\},\ 0<s<1$$

i.e. for all $N>0$, there exists $(\mathcal{T}^*, V^*)$ such that $\#\mathcal{T}^* = N$ and

$$\|\nabla(u-V^*)\|^2_{L^2(\mathcal{T}^*)} + \gamma \left\| h^{-1/2} [\![V^*]\!] \right\|^2_{L^2(\mathcal{S}^*)} + \mathrm{osc}^2_{\mathcal{T}^*}(V^*) \preceq (\#\mathcal{T}^*)^{-2s}.$$

Lemma

$$\mathbb{A}_s \equiv \mathbb{A}_s^0.$$

Main ingredients:

- Binev et al. (2004) and Stevenson (2007), for conforming meshes:

$$\#\mathcal{T}_k - \mathcal{T}_0 \preceq \sum_{i=1}^{k} \#\mathcal{M}_i.$$

- There exists $\pi_{\mathcal{T}} : \mathbb{V}(\mathcal{T}) \to \mathbb{V}^0(\mathcal{T})$ such that for any $V \in \mathbb{V}(\mathcal{T})$ there holds

$$\|\nabla(V - \pi_{\mathcal{T}} V)\|_{L^2(\mathcal{T})} \preceq \left\| h^{-1/2} [\![V]\!] \right\|_{L^2(\mathcal{S})}.$$

Proposition (Quasi-optimality)

There exists $\theta^* \in (0,1)$, $\gamma^* > 1$ *such that for* $\theta \in (0,\theta^*)$ *and* $\gamma > \gamma^*$, *if* $(u, f, A) \in \mathbb{A}_s$, *then* $\{U_k; \mathcal{T}_k\}_{k\geq 0}$ *satisfies*

$$\|\nabla(u - U_k)\|^2_{L^2(\mathcal{T}_k)} + \gamma \left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)} + \mathrm{osc}^2_{\mathcal{T}_k}(U_k) \preceq (\#\mathcal{T}_k)^{-2s}.$$

One of the Main Ideas

FEM solution

$$(U_k, \mathcal{T}_k) \to \mathcal{E}(U_k, \mathcal{T}_k) := \|\nabla(u - V)\|^2_{L^2(\mathcal{T})} + \gamma \left\| h^{-1/2} [\![V]\!] \right\|^2_{L^2(\mathcal{S})} + \operatorname{osc}^2_{\mathcal{T}}(V)$$

Optimal solution $(u, f, A) \in \mathbb{A}_s$

$$(U_\epsilon \ , \mathcal{T}_\epsilon \ ) : \mathcal{E}(U_\epsilon \ , \mathcal{T}_\epsilon \ ) \preceq \epsilon := \mathcal{E}(U_k, \mathcal{T}_k); \qquad \epsilon \preceq (\#\mathcal{T}_\epsilon \ )^{-s}.$$

Overlay

- $\mathcal{T}_* := \mathcal{T}_k \oplus \mathcal{T}_\epsilon \ , \#\mathcal{T}_* \leq \#\mathcal{T}_k + \#\mathcal{T}_\epsilon \ - \#\mathcal{T}_0$
- U^0_* the continuous FEM sol on $\mathcal{T}^0_*$
- $\mathcal{E}(U^0_*, \mathcal{T}_*) \preceq \mathcal{E}(U^0_\epsilon, \mathcal{T}^0_\epsilon) \preceq \epsilon = \mathcal{E}(U_k, \mathcal{T}_k)$ where for $\gamma > \gamma^*$, $\theta < \theta^*$, the constant hidden in "$\preceq$" is $< 1/2$.
- $||h^{-1/2} \left[\!\left[U_k - U^0_* \right]\!\right] ||_{\Sigma_k} = ||h^{-1/2} [\![U_k]\!] ||_{\Sigma_k}$, then

$$|||U^0_* - U_k|||^2_{\mathcal{T}_k} \preceq \eta^2_{\mathcal{T}_k}(U, \overline{\mathcal{R}}_{\mathcal{T}^0_k \to \mathcal{T}_*}) + \gamma^{-1} \eta^2_{\mathcal{T}_k}(U_k, \mathcal{T}_k).$$

Bringing back the optimality (Stevenson)

$$\#\mathcal{M}_k \preceq \#\mathcal{T}^0_\epsilon \preceq \epsilon^{-s}$$

One of the Main Ideas

FEM solution

$$(U_k, \mathcal{T}_k) \to \mathcal{E}(U_k, \mathcal{T}_k) := \|\nabla(u - V)\|^2_{L^2(\mathcal{T})} + \gamma \left\| h^{-1/2} [\![V]\!] \right\|^2_{L^2(\mathcal{S})} + \text{osc}^2_{\mathcal{T}}(V)$$

Optimal solution $(u, f, A) \in \mathbb{A}_s \equiv \mathbb{A}^0_s$

$$({U_\epsilon}^0, {\mathcal{T}_\epsilon}^0) : \mathcal{E}({U_\epsilon}^0, {\mathcal{T}_\epsilon}^0) \preceq \epsilon := \mathcal{E}(U_k, \mathcal{T}_k); \qquad \epsilon \preceq (\#{\mathcal{T}_\epsilon}^0)^{-s}.$$

Overlay

- $\mathcal{T}_* := \mathcal{T}_k \oplus {\mathcal{T}_\epsilon}^0$, $\#\mathcal{T}_* \leq \#\mathcal{T}_k + \#{\mathcal{T}_\epsilon}^0 - \#\mathcal{T}_0$
- U^0_* the continuous FEM sol on $\mathcal{T}^0_*$
- $\mathcal{E}(U^0_*, \mathcal{T}_*) \preceq \mathcal{E}(U^0_\epsilon, \mathcal{T}^0_\epsilon) \preceq \epsilon = \mathcal{E}(U_k, \mathcal{T}_k)$ where for $\gamma > \gamma^*$, $\theta < \theta^*$, the constant hidden in "$\preceq$" is $< 1/2$.
- $||h^{-1/2} \left[\!\left[U_k - U^0_* \right]\!\right] ||_{\Sigma_k} = ||h^{-1/2} [\![U_k]\!] ||_{\Sigma_k}$, then

$$\left|\left|\left| U^0_* - U_k \right|\right|\right|^2_{\mathcal{T}_k} \preceq \eta^2_{\mathcal{T}_k}(U, \overline{\mathcal{R}}_{\mathcal{T}^0_k \to \mathcal{T}_*}) + \gamma^{-1} \eta^2_{\mathcal{T}_k}(U_k, \mathcal{T}_k).$$

Bringing back the optimality (Stevenson)

$$\#\mathcal{M}_k \preceq \#\mathcal{T}^0_\epsilon \preceq \epsilon^{-s}$$

Conclusions

The result is “Half” satisfactory

- Quasi-optimal rate of convergence for discontinuous Galerkin FEM $(u, f, A) \in \mathbb{A}_s$:

$$\|\nabla(u - U_k)\|^2_{L^2(\mathcal{T}_k)} + \gamma \left\| h^{-1/2} [\![U_k]\!] \right\|^2_{L^2(\mathcal{S}_k)} + \operatorname{osc}^2_{\mathcal{T}_k}(U_k) \preceq (\#\mathcal{T}_k)^{-2s}.$$

  Quasi-Galerkin orthogonality, quasi-localized upper bound (and upper bound), global lower bound, monotone estimator (with respect to the meshsize), Dörfler marking, contraction property.

- $\mathbb{A}_s \equiv \mathbb{A}^0_s$.

The result is “Half” unsatisfactory

- Nonrobust upper and lower bounds with respect to the level of nonconformity. Is the full power/freedom of dG used?
- Nonrobust interpolant.
  There exists $\pi_{\mathcal{T}} : \mathbb{V}(\mathcal{T}) \to \mathbb{V}^0(\mathcal{T})$ such that for any $V \in \mathbb{V}(\mathcal{T})$ there holds

$$\|\nabla(V - \pi_{\mathcal{T}} V)\|_{L^2(\mathcal{T})} \preceq \left\| h^{-1/2} [\![V]\!] \right\|_{L^2(\mathcal{S})}.$$