Anomalous scaling and large deviations in diffusion and Lagrangian transport

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SUMMARY

*** Diffusion Processes**

*** Standard Diffusion**

*** Anomalous Diffusion**:
a) *weak anomalous diffusion*
b) *strong anomalous diffusion*
d) *relative dispersion in turbulence*

*** Consequences of anomalous diffusion**
in reaction/diffusion

THANKS to

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A diffusion (transport) process is:

* *From a Lagrangian point of view:*
A deterministic or stochastic rule for the time evolution $\mathbf{x}(0) \to \mathbf{x}(t) = \mathcal{S}^t \mathbf{x}(0)$, e.g.

$$\mathbf{A} \quad x(t+1) = x(t) + w(t) \ , \ w(t) = \textit{random variable}$$

$$\mathbf{B} \quad x(t+1) = f(x(t)) \quad , \quad f(x(t)) = \textit{chaotic map}$$

$$\mathbf{C} \quad \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) + \sqrt{2D_0}\eta \quad , \quad \eta(t) = \textit{white noise}$$

* *From an Eulerian point of view:*
A rule for the time evolution of the PdF $\rho(\mathbf{x}, t)$, e.g. in the case **C**, for the incompressible flow $\nabla \cdot \mathbf{u} = 0$, one has the advection diffusion equation (Fokker-Planck eq.)

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho = D_0 \Delta \rho \ .$$

The typical scenario: Standard Diffusion

At large scale and asymptotically in time, usually one has the so called **standard diffusion** i.e. a Fick's law holds (just for simplcity we considere the case $< \mathbf{x} >= 0$)

$$\frac{\partial \Theta}{\partial t} = \sum_{i,j} \mathcal{D}_{i,j} \frac{\partial^2 \Theta}{\partial x_i \partial x_j}$$

and a Gaussian behavior.

Θ is the spatial coarse graining of ρ, and $\mathcal{D}_{i,j}$ is the effective (eddy) diffusion tensor, depending (often in a non trivial way) from D_0 and the field $\mathbf{u}$:

$$\Theta(\mathbf{x}, t) \sim exp - \frac{1}{4t} \sum_{i,j} x_i [\mathcal{D}^{-1}]_{i,j} x_j$$

$$< x_i(t) x_j(t) > \simeq 2 \mathcal{D}_{i,j} t \, .$$

QUESTIONS

* Is the standard diffusion generic?

* How violate the standard diffusion?

A) For incompressible velocity field $\nabla \cdot \mathbf{u} = 0$, if $D_0 > 0$ one has standard diffusion if the infrared contribution is not "too large" (Majda-Avellaneda), i.e.

$$\int \frac{|\mathbf{V}(\mathbf{k})|^2}{k^2} d\mathbf{k} < \infty \tag{1}$$

where $\mathbf{V}(\mathbf{k})$ is the Fourier transform of $\mathbf{u}(\mathbf{x})$.

B) Standard diffusion is present if the lagrangian correlations decay fast enough (Taylor), i.e.

$$\int_0^\infty < v_L(t) v_L(0) > dt < \infty \tag{2}$$

where $v_L(t)$ is the lagrangian velocity $v_L(t) = dx(t)/dt$.

Anomalous diffusion is, somehow, a pathology: it is necessary the violate the hypothesis for the validity of central limit theorem.

EXAMPLES OF ANOMALOUS DIFFUSION:

Ex 1: Trasversal diffusion in a random shear (Matheron and de Marsily): $\mathbf{u}(\mathbf{x}) = (U(y), 0)$, where $U(y)$ is a spatial random walk; it is possible to show that

$$< x(t)^2 > \sim t^{3/2} \ , \ \rho(x,t) \sim \frac{1}{t^{3/4}} exp - C\frac{x^4}{t^3} \ .$$

Ex 2: Levy walk

$$x(t+1) = x(t) + v(t)$$

where $v(t)$ is a random variable which can assume two values $\pm u_0$ for a duration T given by a random variable whose PdF is $\psi(T) \sim T^{-(\alpha+1)}$.

For $\alpha > 2$, one has the usual standard diffusion, on the contrary if $\alpha \leq 2$ one has a superdiffusion:

$$< x^2(t) > \sim t^{2\nu}$$

where

$$\nu = 1 \ , \ if \ \alpha < 1 \ , \ \ \nu = \frac{(3-\alpha)}{2} \ , \ if \ 1 < \alpha < 2 \ .$$

Ex 3: Lagrangian chaos in $2d$

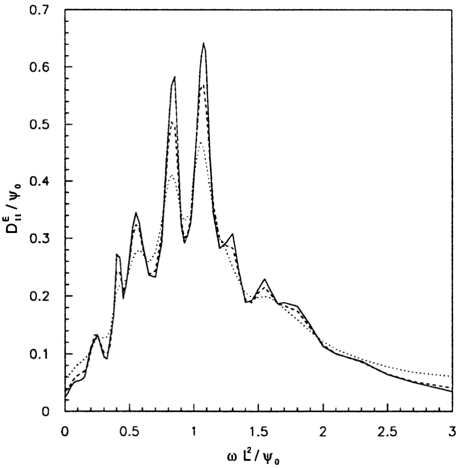
$$\frac{dx}{dt} = \frac{\partial \psi(x,y,t)}{\partial y} + \sqrt{2D_0}\,\eta_1 \ ,$$

$$\frac{dy}{dt} = -\frac{\partial \psi(x,y,t)}{\partial x} + \sqrt{2D_0}\,\eta_2 \ ,$$

where

$$\psi(x,y,t) = \psi_0 \sin\left(\frac{2\pi x}{L} + B\sin\omega t\right)\sin\left(\frac{2\pi y}{L}\right)$$

the term $B\cos\omega t$ represents the lateral oscillation of the rolls.
For $B \neq 0$ one has chaos (mechanism of the homoclinic intersection).
The effective diffusion coefficient depends from D_0 and ω in a non trivial way.



Lagrangian chaos in $2d$: $\mathcal{D}_{11}$ vs ω (rescaled), $D_0/\psi_0 = 3 \times 10^{-3}$ (dotted curve); $D_0/\psi_0 = 10^{-3}$ (broken curve); $D_0/\psi_0 = 4 \times 10^{-4}$ (full curve).

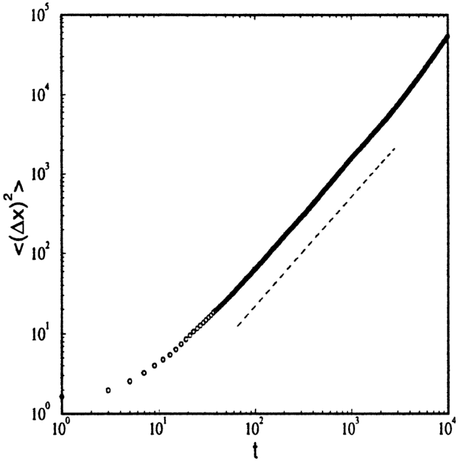
Two different ways to have anomalous diffusion

I) In the random shear flow the anomalous diffusion is due to the violation of (1) i.e. the infrared contributions are dominant.

II) In the Levy walk, the "violation" of the central theorem is due to the non integrable long tail of the velocity-velocity correlation function which determines, for $\alpha < 2$ (superdiffusion):

$$< v_L(t) v_L(0) > \sim t^{-\beta} \ , \ \ with \ \ \beta < 1 \, .$$

The same mechanism is present, for $D_0 = 0$ and special values of ω, in the Lagrangian chaos in the oscillating rolls.



Lagrangian chaos in $2d$: $< x^2(t) >$ vs t, with $D_0 = 0$ and $\omega = 1.1$, the dashed line indicates $t^{1.3}$.

The result in the previous system is non an isolated case.
Such kind of mechanism is rather common in low dimensional symplectic chaotic systems, e.g. in the standard map

$$\theta_{t+1} = \theta_t + J_t \ \ , \ \ J_{t+1} = J_t + \sin(\theta_{t+1})$$

for some peculiar values of K.

The long tail in the correlation function is due to the presence of (weakly unstable) ballistic trajectories.

SOME NATURAL QUESTIONS

* Does the value of the scaling exponent ν allow to determine the shape of $\rho(x,t)$?

* Is the scaling exponent ν (for $< x^2(t) >$) the unique relevant quantity?

In the standard diffusion one has $\nu = 1/2$ and a gaussian feature:

$$\Theta(x,t) \sim \frac{1}{t^{1/2}}\, exp - C\left(\frac{x}{t^{1/2}}\right) \; ,$$

$$< |x(t)|^q > \sim t^{q/2}$$

Naively, in the case of anomalous diffusion, one could guess the simplest generalization:

$$\Theta(x,t) \sim \frac{1}{t^{\nu}} F_{\nu}\left(\frac{x}{t^{\nu}}\right) , \qquad (3)$$

$$< |x(t)|^q > \sim t^{q\nu}$$

where $F_{\nu}(\,)$ is a suitable function, in the Gaussain case $F_{1/2}(z) = exp - Cz^2$.

The above scenario is called **weak anomalous diffusion**:
the exponent ν is sufficient to describe the scaling features, and in addition the PdF has a scaling shape.

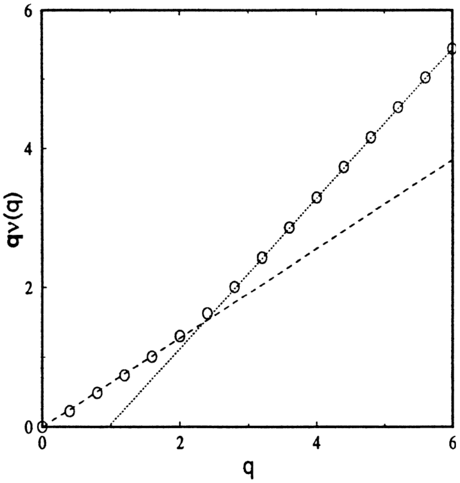
The existence of anomalous scaling in fully devolved turbulence (and other phenomena) suggests that a more complex scenario can appear, namely

$$< |x(t)|^q >\sim t^{q\nu(q)}$$

where $\nu(q)$ is not constant.
In such a case called **strong anomalous diffusion** *the PdF cannot have a scaling structure* as in (3).

Are there real examples of strong anomalous diffusion?

A first example: Lagrangian Chaos in Rayleigh-Benard convection; for $D_0 = 0$ for some values of ω, one has $\nu(q) \neq const$.



Lagrangian chaos in $2d$, $D_0 = 0$ and $\omega = 1.1$: $\nu(q)$ vs q, the dashed line corresponds to $0.65q$, the dotted line corresponds to $q - 1.04$.

The Lagrangian Chaos in Rayleigh-Benard convection is not an isolated case of strong anomalous diffusion (Artuso et al, Klages et al).

Other examples:
* Standard Map
* Levy walks

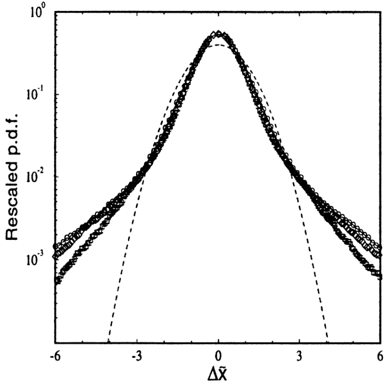
In particular it rather common the following shape:

$$q\nu(q) \simeq q\nu(0) \ , \ \mathit{for} \ \ q < q^* \ ,$$

$$q\nu(q) \simeq q - \mathit{const.} \ , \ \mathit{for} \ q > q^* \ .$$

In some stochastic processes it is possible to derive, the above shape:

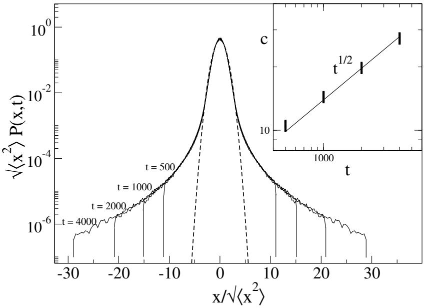
In presence of strong anomalous diffusion there is not a scaling structure of the PdF



Lagrangian chaos in $2d$, $D_0 = 0$ and $\omega = 1.1$: rescaled PdF: $p(x/t^{\nu(0)})$ vs $x/t^{\nu(0)}$ for three different times $(500, 1000, 2000)$.

Even in presence of standard diffusion, i.e. $< x^2(t) > \sim t$, the scenario can be not trivial

For instance in the Levy walk with $\alpha > 2$ one has $\nu(2) = 1/2$, but the PdF does not rescale and $\nu(q) \neq 1/2$ for large values of q



Levy walk, $\alpha = 2.2$, rescaled PdF: $p(x/t^{\nu(2)})$ vs $x/t^{\nu(2)}$ for different times.

RELATIVE DISPERSION IN TURBULENCE

The classical result of Richardson in the inertial range

$$< R^2(t) > \sim t^3$$

where $R(t) = |\mathbf{x}_1(t) - \mathbf{x}_2(t)|$. Now, a posteriori, the result is a simple consequence of the Kolmogoriv scaling $\delta v(\ell) \sim \ell^{1/3}$.

What about the effect of intermittency for the relative diffusion? Two possible scenarios:

* Weak anomalous diffusion:

$$< R^p(t) > \sim t^{\frac{3}{2}p} \; ;$$

* Strong anomalous diffusion:

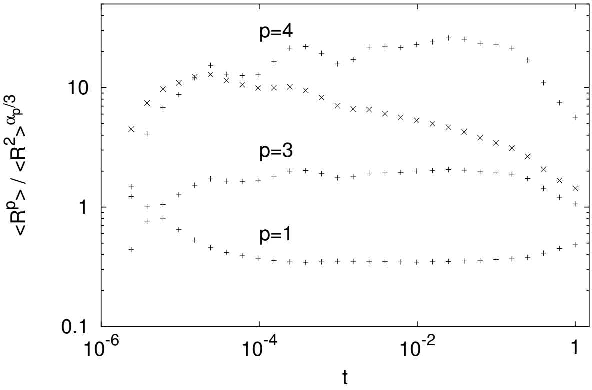
$$< R^p(t) >\sim t^{\alpha(p)}$$

with $\alpha(p) \neq \frac{3}{2}p$.

From the multifractal model one has a prediction(Boffetta et al) in terms of $D(h)$:

$$\alpha(p) = \inf_h \left[\frac{p+3-D(h)}{1-h} \right] . \tag{4}$$

It is remarkable that, even in presence of intermittency, the Richardson scaling $\alpha(2) = 3$ is exact; the (4) has been checked in synthetic turbulence, where the velocity field is random process with the proper statio-temporal statistical features (Boffetta et al) and in direct numerical simulation of the NS equations (Boffetta and Sokolov).



$< R^p(t) > / < R^2(t) >^{\alpha(p)/3}$ vs t, the $\alpha(p)$ are obtained (eq. 4) from $D(h)$; the plateaus is the mark of a nontrivial scaling, the x indicate for $p = 4$ the rescaling with the non intermittent prediction i.e. $\alpha(4) = 6$.

CONSEQUENCES OF ANOMALOUS DIFFUSION IN FRONT PROPAGATION

The simplest reaction-diffusion problem (FKPP): a system with standard diffusion and a reactive terms:

$$\frac{\partial \theta}{\partial t} = D_0 \frac{\partial^2}{\partial x^2}\theta + \frac{1}{\tau} f(\theta) \ , \qquad (5)$$

asymptotically one has a front propagation:

$$\theta(x,t) = F(x - v_f t)$$

where $F(1) = 1$, $F(0) = 0$ and, if $f^{''} < 0$, the front speed is $v_f = 2\sqrt{D_0 f'(0)/\tau}$.

$$\theta(x,t) \sim exp\Big[-\frac{(x - X_F(t))}{\zeta}\Big]$$

$$X_f(t) \simeq v_f t \ , \ \zeta = 8\sqrt{D_0 \tau / f'(0)}$$

What happen in presence of anomalous diffusion?

For instance we can replace the (5) with

$$\frac{\partial \theta}{\partial t} = \mathcal{L}\theta + \frac{1}{\tau} f(\theta)$$

where $\mathcal{L}$ is linear operator such that, in absence of the reaction term, the diffusion is anomalous.

For the relative diffusion according to Richardson one has

$$\mathcal{L}\theta = \frac{1}{r^{d-1}} \frac{\partial}{\partial r}\Big(K(r) r^{d-1} \frac{\partial}{\partial r}\theta\Big) \ , \ K(r) \propto r^{4/3} \ .$$

There is class of systems where, in spite of the presence of the anomalous diffusion, the front propagation is always standard i.e. $X_F(t) \simeq v_f t$ with a finite v_f, and $\zeta = const$.
For instance if $\nu \neq 1/2$ and the PfD has the shape (which holds for the random shear):

$$\rho(x,t) \sim \frac{1}{t^\nu} \, exp - C\Big(\frac{x}{t^\nu}\Big)^{\frac{1}{1-\nu}}$$

the front propagation is standard (Mancinelli et al).

On the other hand, there are cases where the front propagation can be non standard, i.e.

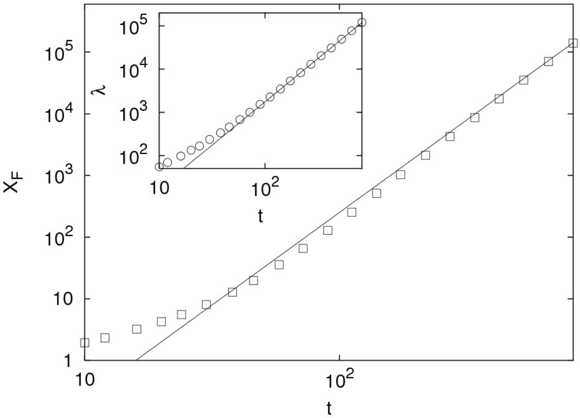
$$\theta(x,t) \sim exp\Big[- \frac{(x - X_F(t))}{\zeta(t)} \Big]$$

with

$$X_F(t) \sim t^{\gamma} \ , \ \zeta(t) \sim t^{\delta} \ with \ \delta = \gamma - 1 \ .$$

For instance in the Richardson diffusion one has

$$\gamma = 3 \ , \ \delta = 2 \ .$$



Front propagation in the Richardon diffusion (and reaction) equation: $X_F(t)$ vs t, in the insect $\zeta(t)$ vs t, the lines indicate the predictions $X_F(t) \sim t^3$, and $\zeta(t) \sim t^2$.

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