The Amalgamation Property for Semilinear Commutative Idempotent Residuated Lattices

José Gil-Férez[1], Peter Jipsen[2], George Metcalfe[1], and Constantine Tsinakis[3]

[1] University of Bern, Switzerland
[2] Chapman University, CA, U.S.A.
[3] Vanderbilt University, TN, U.S.A.
jose.gil-ferez@math.unibe.ch

The main goal of this work is to present a proof of the Amalgamation Property for the variety SCIpRL of semilinear commutative idempotent residuated lattices, that is, the variety generated by the totally ordered commutative residuated lattices satisfying the equation $x \approx x^2$, also called commutative idempotent chains.

In [3], Raftery studied the variety SCIpRL and showed that it is locally finite. The proof of this result was based on a description of all commutative idempotent chains. In his representation, the negative cone of a commutative idempotent chain is divided into a family of segments indexed by the positive elements, although for some positive elements, the associated segment could be empty.

In order to prove the Amalgamation Property for SCIpRL, we give an alternative description that refines Raftery's representation, in that it captures better the inner structure of commutative idempotent chains. Both negative and positive cones of such chains are divided into families of nonempty upper-bounded segments. These upper bounds form a retract, which turns out to be an odd Sugihara monoid.

Recall that, given a residuated lattice $\mathbf{A}$ and an element $a \in A$, the map $\gamma_a : A \to A$ defined by $\gamma_a(x) = (a/x)\backslash a$ is a closure operator on $\mathbf{A}$ which in addition satisfies $y \cdot \gamma_a(x) \leq \gamma_a(y \cdot x)$. Therefore, in the commutative case, γ_a is a nucleus of $\mathbf{A}$. (While discussing commutative residuated lattices, we will use only the residual $\backslash$.) In particular, for $a = 1$, the identity of $\mathbf{A}$, the nucleus retract $\mathbf{A}_{\gamma_1} = \langle A_{\gamma_1}, \wedge, \vee_{\gamma_1}, \cdot_{\gamma_1}, \backslash, 1\rangle$ is a commutative residuated lattice, where $A_{\gamma_1} = \{x \in A : \gamma_1(x) = x\}$, $x \vee_{\gamma_1} y = \gamma_1(x \vee y)$, and $x \cdot_{\gamma_1} y = \gamma_1(x \cdot y)$.

Proposition 1. *Given a commutative idempotent chain $\mathbf{A}$, we have the following:*

(i) $\mathbf{A}_{\gamma_1}$ *is a totally ordered odd Sugihara monoid.*
(ii) For every $c \in A_{\gamma_1}$, the set $A_c = \{x \in a : \gamma_1(x) = c\}$ is a segment of $\mathbf{A}$ whose top is c.
(iii) For every $c, c' \in A_{\gamma_1}$, $x \in A_c$, and $y \in A_{c'}$,
   - *(a) if $c = c' \leq 1$, then $x \cdot y = x \wedge y$;*
   - *(b) if $1 < c = c'$, then $x \cdot y = x \vee y$;*
   - *(c) if $c \neq c'$, then $x \cdot y = x \iff c \cdot c' = c$.*

(iv) For every $x, y \in A$, with $x \in A_c$ for some $c \in A_{\gamma_1}$,

$$x\backslash y = \begin{cases} \ln c \vee x \vee y & \text{if } x \leq y, \\ \ln c \wedge y & \text{if } y < x. \end{cases}$$

For the other direction, given a totally ordered odd Sugihara monoid $\mathbf{S}$ and a family of (disjoint) chains (totally ordered sets) whose top elements are the members of $\mathbf{S}$, we can define a total order over the union of these chains and also operations $\cdot$ and $\backslash$ inspired by the previous proposition. We obtain in that way a commutative idempotent chain.

Proposition 2. *Given a totally ordered odd Sugihara monoid* $\mathbf{S}$ *and a family of (disjoint) chains* $\{\langle X_c, \leq_c\rangle : c \in S\}$ *such that for every* $c \in S$ *the greatest element of* X_c *is* c*, there is a commutative idempotent chain* $\mathbf{A}$ *satisfying that, retaining the notation of the previous proposition,* $\mathbf{S} = \mathbf{A}_{\gamma_1}$*, and for every* $c \in S$*,* $A_c = X_c$*.*

The proof of the Amalgamation Property for SCIpRL proceeds then in three stages. In first place, we prove that the class of totally ordered odd Sugihara monoids has the Amalgamation Property by constructing the corresponding amalgams. (The Amalgamation Property for this class was already proven by Marchioni and Metcalfe (see [1]) using the model-theoretic technique of quantifier elimination.)

Next, using the fact that the class of totally ordered sets also satisfies the Amalgamation Property and the representation given by Propositions 1 and 2, we prove that the class of commutative idempotent chains also has the Amalgamation Property, by constructing explicitly the amalgams.

Finally, the fact that the whole variety SCIpRL enjoys the Amalgamation Property is then a direct consequence of a result by Metcalfe, Montagna, and Tsinakis (see [2]) establishing that a variety of semilinear commutative residuated lattices has the Amalgamation Property whenever the class of its totally ordered members has the Amalgamation Property.

Theorem 3.

(i) The class of odd Sugihara chains has the Amalgamation Property.
(ii) The class of commutative idempotent chains has the Amalgamation Property.
(iii) The variety SCIpRL *has the Amalgamation Property.*

References

[1] E. Marchioni, G. Metcalfe. Craig interpolation for semilinear substructural logics, Mathematical Logic Quarterly 58(6), 2012, 468–481

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