Lagrangian Traffic Models

- Underlying Assumptions Common to all Models

Cars indexed by the integers and x_m, is the location of the front of the m^{th} car.

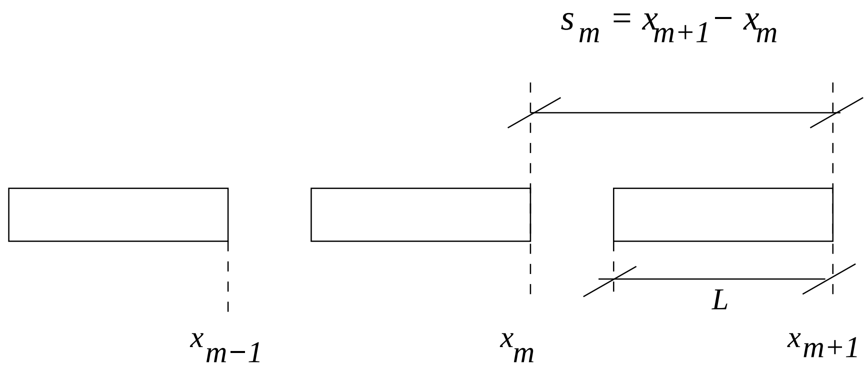


Figure 1

$0 \leq u_m$ speed of the m^{th} car

All cars of the same length L

$L \leq s_m = x_{m+1} - x_m$

Common Equations

$$\dot{x}_m = u_m \Rightarrow \dot{s}_m = u_{m+1} - u_m \tag{1}$$

The distinguishing thing about these models is how the cars accelerate.

BANDO - et. al. Model

$$\dot{u}_m = (V(s_m) - u_m)/\epsilon \tag{B – DISC.}$$

In simulations I use ϵ in the range (5 sec, 10 sec)

Aw-Rascle-Greenberg

$$\dot{u}_m = (V(s_m) - u_m)/\epsilon + c(s_m)(u_{m+1} - u_m) \tag{ARG – DISC.}$$

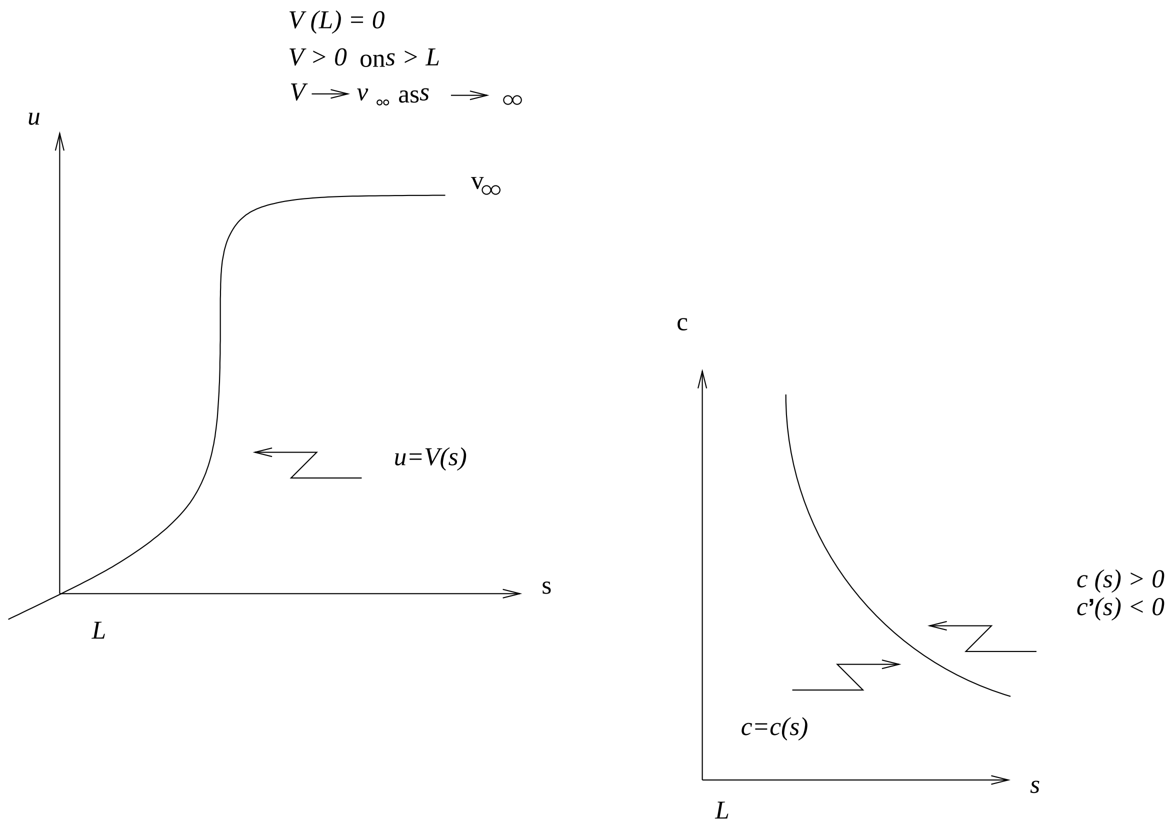


Figure 2

$V(\cdot)$ is referred to as the optimal or desired velocity.

The presence of the term $c(s_m)(u_{m+1} - u_m)$ in the ARG-DISC. Model reflects the fact that the driver of the m^{th} car knows his/her speed and can estimate the speed of the car ahead. Given these data, the m^{th} driver will (accelerate-decelerate) according as $((u_{m+1} - u_m > 0) - (u_{m+1} - u_m < 0))$

On the face of things both models seem quite reasonable and you pay your money and make your choice. If this were the case, I'd sit down now – there are qualitative differences. I introduce $P(s) = \int_L^s c(\eta)d\eta$ and note

$$P(L) = 0, \quad P'(s) = c(s) > 0 \text{ and } P''(s) = c'(s) < 0, \ s \geq L.$$

Throughout I'll assume

$$P(s) \geq V(s), \quad s \geq L$$

and

$$\lim_{s \to \infty} P(s) = P_\infty > v_\infty$$

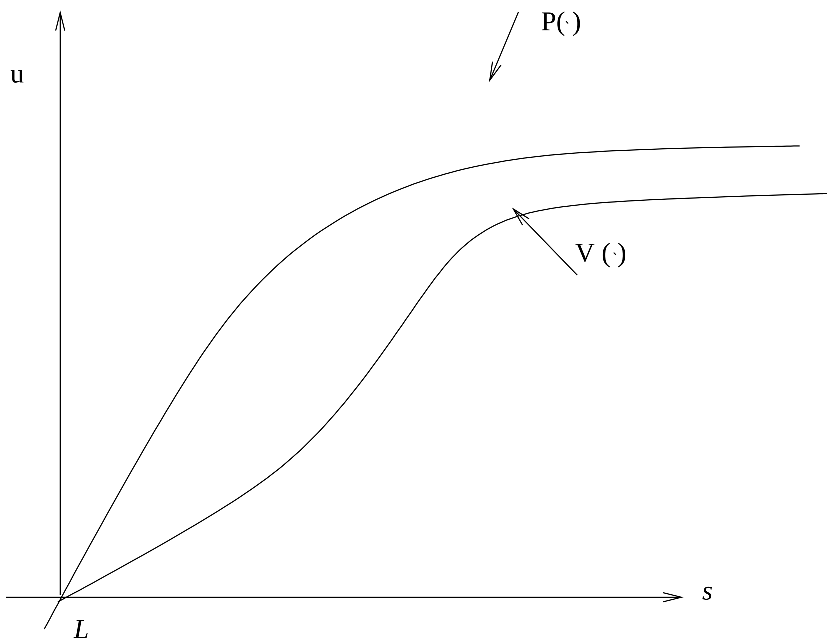


Figure 3

Continuum Versions of Discrete Systems

Continuum car index		λ	
$x(\lambda, t)$	...	location of	λ^{th} car
$u(\lambda, t) \geq 0$	...	speed of	λ^{th} car
replace differences	$f_{m+1} - f_m$	*by*	$f_{,\lambda}$

Bando-Continuum

$$
\begin{aligned}
x_{,t} &= u \Rightarrow s_{,t} = u_{,\lambda} \text{ where } s = x_{,\lambda} \\
u_{,t} &= (V(s) - u)/\epsilon
\end{aligned}
$$

System is hyperbolic with both wave speeds 0.

ARG-Continuum

$$
\begin{aligned}
x_{,t} &= u \Rightarrow s_{,t} = u_{,\lambda} \text{ where } s = x_{,\lambda} \\
u_{,t} &= c(s)u_{,\lambda} \quad + (V(s) - u)/\epsilon
\end{aligned}
$$

System is hyperbolic with wave speeds $-c < 0$ and 0.

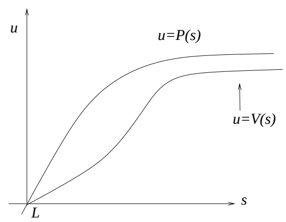
All of these Models have the common equilibrium solutions

$$s \equiv \overline{s} \geq L \quad \text{and } u \equiv V(\overline{s}).$$

Contrast

B-DISC.	ARG-DISC.
B-Cont.	ARG-Cont.

ARG-DISC.



$$
\begin{aligned}
\dot{s}_m &= (u_{m+1} - u_m) \\
\dot{u}_m &- P'(s_m)(u_{m+1} - u_m) = (V(s_m) - u_m)/\epsilon \Leftrightarrow \\
\dot{\alpha}_m &+ \alpha_m/\epsilon = (P(s_m) - V(s_m))/\epsilon \geq 0 \quad , \quad P \geq V \text{ on } s \geq L \\
\alpha &= P(s) - u
\end{aligned}
$$

Numerics FO Euler $t_n = n(dt), \mu = \frac{(dt)}{\epsilon}$.

$$
\begin{aligned}
s_m^{n+1} &= s_m^n + (dt)(u_{m+1}^n - u_m^n) \\
\alpha_m^{n+1} &= (1-\mu)\alpha_m^n + \mu(P(s_m^n) - V(s_m^n)),\ \alpha = P(s) - u \\
u_m^{n+1} &= (1-\mu)u_m^n + P(s_m^{n+1}) - (1-\mu)P(s_m^n) - \mu P(s_m^n) + \mu V(s_m^n) \Rightarrow \\
u_m^{n+1} &= (1-\mu)u_m^n + (dt)\mathbb{C}_m^{n,n+1}(u_{m+1}^n - u_m^n) + \mu V(s_m^n) \\
0 &\leq \mathbb{C}_m^{n,n+1} = \tfrac{(P(s_m^{n+1}) - P(s_m^n))}{(s_m^{n+1} - s_m^n)} \leq P'(L)
\end{aligned}
$$

Recall $P(L) = 0,\ P(s) \geq V(s), P'(s) \geq 0,\ P''(s) \leq 0,$ and $P(\infty) < \infty$.

ARG-Continuum - Integrated with Downwind diferencing which reflects the fact that the wave speeds are 0 and $-P'(s) < 0$.

$$
\begin{aligned}
s_{,t} &= u_{,\lambda} \text{ and } u_{,t} - P'(s)u_{,\lambda} = (V(s) - u)/\epsilon,\ \alpha = P(s) - u \\
t_n &= n(dt) \quad , \quad \lambda_j = j(d\lambda) \\
\beta &= (dt)/(d\lambda) \text{ and } \mu = (dt)/\epsilon
\end{aligned}
$$

$$
\begin{aligned}
s_j^{n+1} &= s_j^n + \beta(u_{j+1}^n - u_j^n) \\
\alpha_j^{n+1} &= (1-\mu)\alpha_j^n + \mu(P(s_j^n) - V(s_j^n)) \Leftrightarrow \\
u_j^{n+1} &= (1-\mu)u_j^n + \beta\mathbb{C}_j^{n,n+1}(u_{j+1}^n - u_j^n) + \mu V(s_j^n) \\
0 &\leq \mathbb{C}_j^{n,n+1} = \frac{(P(s_j^{n+1}) - P(s_j^n))}{(s_j^{n+1} - s_j^n)} \leq P'(L)
\end{aligned}
$$

Theorem 1. Suppose $0 \leq (d\lambda) \leq 1,\ 0 \leq \mu,\ 0 < \beta,$ and $\mu + \beta P'(L) < 1$.

Then, if

$$L \leq s_j^n,\ 0 \leq \alpha_j^n,\ \text{and } 0 \leq u_j^n$$

the same inequalities hold at t_{n+1}, i.e.

$$L \leq s_j^{n+1},\ 0 \leq \alpha_j^{n+1},\ \text{and } 0 \leq u_j^{n+1} \leq P(s_j^{n+1})$$

Remark

What is really required for the last theorem to be true is that

$$P(L) = 0, \quad P'(s) > 0, \quad \text{and} \quad P(s) \geq V(s) \quad \text{on} \quad s \geq L.$$

One would replace $P'(L)$ with $\max\limits_{L \leq s} P' < \infty$.

These remarks imply we could take $P(s) \equiv V(s)$. In this latter case $\alpha(\lambda, t) = e^{-t/\epsilon}\alpha(\lambda, 0)$,

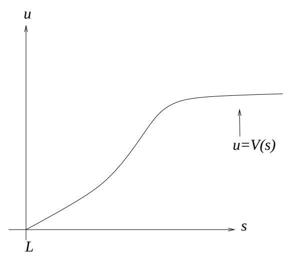
$$u(\lambda, t) = V(s) \quad - e^{-t/\epsilon}\alpha(\lambda, 0)$$

and the continuum equation for s becomes

$$s_{,t} - V(s)_{,\lambda} = -e^{-t/\epsilon}(\alpha_{,\lambda}(\lambda, 0)) \to 0 \text{ as } t \to \infty.$$

This converges to the LWR eqn for large t.

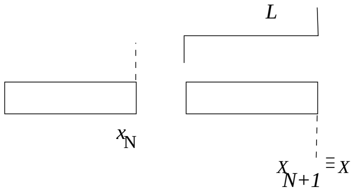
For BANDO-Continuum the analogus integration scheme is

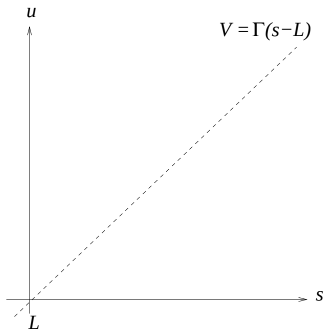


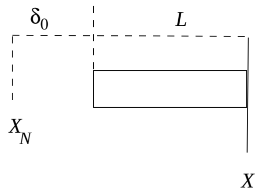
$$\begin{aligned} s_j^{n+1} &= s_j^n + \beta(u_{j+1}^n - u_j^n) \\ u_j^{n+1} &= (1-\mu)u_j^n + \mu V(s_j^n) \\ 0 &\leq (d\lambda) \leq 1, \quad 0 < \mu = (dt)/\epsilon\ ,\ \beta = (dt/d\lambda) \end{aligned}$$

and there are no estimates comparable to those of Theorem 1.

BANDO EXAMPLE







$$
\begin{aligned}
\dot{X}_N &= u_N \\
\dot{u}_N &= (V(X - X_N) - u_N)/\epsilon \\
\dot{u}_N &= (\Gamma(X - X_N - L) - u_N)/\epsilon \\
\ddot{X}_N &= (\Gamma(X - X_N - L) - \dot{X}_N)/\epsilon \\
X_N(0) &= X - (L + \delta_0), \quad \delta_0 > 0 \\
\dot{X}_N(0) &= U > 0 \\
\delta &= (X - L - X_N) \\
-\dot{\delta} &= \dot{X}_N \quad , \quad -\ddot{\delta} = \ddot{X}_N \text{ and}
\end{aligned}
$$

$$\ddot{\delta} \; + \; \dot{\delta}/\epsilon + \Gamma\delta/\epsilon = 0$$

$$
\begin{aligned}
\delta(0) &= \delta_0 \text{ and } \dot{\delta}(0) = -U < 0 \\
\delta &= e^{Kt} \quad : \quad K^2 + \frac{K}{\epsilon} + \frac{\Gamma}{\epsilon} = 0
\end{aligned}
$$

Assume : $\quad 4\Gamma\epsilon - 1 \stackrel{def}{=} \omega^2 > 0$ and let $\omega = \sqrt{4\Gamma\epsilon - 1} > 0$

$$
\begin{aligned}
K_\pm &= -\frac{1}{2\epsilon} \pm \frac{i\omega}{2\epsilon} \\
\delta(t) &= \delta_0 e^{-t/2\epsilon} \cos\frac{\omega}{2\epsilon}t + \left(\frac{\delta_0}{\omega} - \frac{2\epsilon U}{\omega}\right) e^{-t/2\epsilon} \sin\frac{\omega}{2\epsilon}t
\end{aligned}
$$

Remark: If $\delta_0 - 2\epsilon U < 0$, then

$$\delta\left(t = \frac{\pi\epsilon}{\omega}\right) = \frac{1}{\omega}(\delta_0 - 2\epsilon U)e^{-\frac{\pi}{2\omega}} < 0$$

and this is interpreted as a crash having taken place before $t = \dfrac{\pi\epsilon}{\omega}$.

Linear Stability of the Uniformly Spaced Solutions and Dispersion Relations

$$s \equiv \overline{s} > L \quad \text{and} \quad u \equiv V(\overline{s})$$

We linearize the various evolutions about the above solution; specifically we let

$$s = \overline{s} + \delta r \quad \text{and} \quad u = V(\overline{s}) + \delta q, \quad 0 < \delta << 1$$

LB-DISC.

$$\dot{r}_n = q_{m+1} - q_m \quad \text{and} \quad \dot{q}_m = (V'(\overline{s})r_m - q_m)/\epsilon$$

LARG-DISC.

$$\dot{r}_n = q_{m+1} - q_m \quad \text{and} \ \dot{q}_m = (V'(\overline{s})r_m - q_m)/\epsilon + c(\overline{s})(q_{m+1} - q_m)$$

LB-CONT.

$$r_{,t} = q_{,\lambda} \quad \text{and} \quad q_{,t} = (V'(\overline{s})r - q)/\epsilon$$

LARG-CONT.

$$r_{,t} = q_{,\lambda} \quad \text{and} \quad q_{,t} = (V'(\overline{s})r - q)/\epsilon + c(\overline{s})q_{,\lambda}$$

We'll restrict our attention to the ring-road scenario where

- in the discrete cases: $(r_{m+M}, q_{m+M}) = (r_m, q_m)$
- in the continuum cases $(r, q)(\lambda + M, t) = (r, q)(\lambda, t)$

M is the number of cars on the ring road.

For definiteness I'll restrict the analysis to the ARG models but will record the results for the Bando models which are obtained in exactly the same way.

LARG-DISCRETE

$$\begin{aligned}
(r_m, q_m) &= e^{i\left(\frac{2\pi k}{M}\right)m}(R_k, Q_k) \quad , \quad 0 \le k \le M-1 \\
\omega_k &= \frac{2\pi k}{M} \\
\dot{R}_k &= \left(e^{i\omega_k} - 1\right) Q_k \quad \text{and} \quad \dot{Q}_k = (V'(\overline{s})R_k - Q_k)/\epsilon + c(\overline{s})(e^{i\omega_k} - 1)Q_k
\end{aligned}$$

We look for solutions of the form

$$(R_k, Q_k) = e^{\kappa_k t}(\tilde{R}_k, \tilde{Q}_k).$$

Insertion of this ansatz into the above equations yields the following equation for κ_k:

$$\kappa_k = \epsilon c(\overline{s})(e^{i\omega_k} - 1)\kappa_k + V'(\overline{s})(e^{i\omega_k} - 1) - \epsilon \kappa_k^2$$

If we restrict our attention to the case $0 < \omega_k << 1$ we find that one root goes as $-\frac{1}{\epsilon}$ and the other has the asymptotic expansion

$$\kappa_k^{\mathrm{LARG-D}} \sim iV'(\overline{s})\omega_k + V'(\overline{s})\left(-\frac{1}{2} - \epsilon(c(\overline{s}) - V'(\overline{s}))\right)\omega_k^2 \quad + \text{hot}$$

$$\left\{\begin{array}{l}\text{stable}\\ \text{unstable}\end{array}\right\} \text{according as } c(\overline{s}) \left\{\begin{array}{c} > \\ \\ < \end{array}\right\} V'(\overline{s}) - \frac{1}{2\epsilon}$$

The analogous computation for LARG-Continuum yields the characteristic equation

$$\kappa_k = \epsilon c(\overline{s}) i\omega_k \kappa_k + V'(\overline{s}) i\omega_k - \epsilon \kappa_k^2$$

Again one root goes as $-1/\epsilon$ (as $\omega_k \to 0^+$) and the other as

$$\kappa_k^{LARG-C} \sim iV'(\overline{s})\omega_k - \epsilon V'(\overline{s})(c(\overline{s}) - V'(\overline{s}))\omega_k^2 + \text{hot}.$$

Here we obtain

$$\left\{\begin{array}{l}\text{stable}\\ \text{unstable}\end{array}\right\} \quad \text{according as } c(\overline{s}) \left\{\begin{array}{c} > \\ \\ < \end{array}\right\} V'(\overline{s}).$$

For Bando-Discrete we obtain

$$\kappa_k^{\text{LB-D}} \sim iV'(\overline{s})\omega_k + V'(\overline{s})\left(-\frac{1}{2} + \epsilon V'(\overline{s})\right)\omega_k^2 \quad + \text{hot}$$

and the solution is

$$\left\{ \begin{array}{l} \text{stable} \\ \text{unstable} \end{array} \right\} \text{according as } V'(\overline{s}) \left\{ \begin{array}{c} < \\ > \end{array} \right\} \frac{1}{2\epsilon}.$$

For Bando-Continuum

$$\kappa_k^{LB-C} \sim iV'(\overline{s})\omega_k + \epsilon(V'(\overline{s}))^2\omega_k^2 + \text{hot}$$

and all solutions are unstable.

In his change to the presenters, Michel asked if there were "fixes" (to the continuum models) which brought them more in line with the discrete models. Below, I'll show a "fix" to Bando-Continuum which brings us more in line with Bando-Discrete. A similar fix could be applied to ARG-Continuum which I'll leave to the workshop participants. The preceding calculations of the dispersion relations for Bando-Discrete and Bando-Continuum yielded

$$\kappa_k^{\text{LB-D}} \sim iV'(\overline{s})\omega_k + V'(\overline{s})\left(-\frac{1}{2} + \epsilon V'(\overline{s})\right)\omega_k^2 + \text{hot}$$

and

$$\kappa_k^{\text{LB-C}} \sim iV'(\overline{s})\omega_k + \epsilon(V'(\overline{s}))^2\omega_k^2 + \text{hot}$$

as

$$\omega_k = \frac{2\pi k}{M} \to 0.$$

We seek a modification to Bando-Continuum

$$s_{,t} = u_{,\lambda} \quad \text{and} \quad u_{,t} = (V(s) - u)/\epsilon$$

which will yield the same dispersion relation as Bando-Discrete through terms of order ω_k^2 as $\omega_k \to 0$. This is actually a rather simple task; the result is

Bando-Modified Continuum

$$s_{,t} = \left(u + \frac{1}{2}u_{,\lambda}\right)_{,\lambda} \quad \text{and} \quad u_{,t} = (V(s) - u)/\epsilon.$$

This system is equivalent to the following second-order hyperbolic problem for s:

$$s_{,tt} - \frac{(V'(s)s_{,\lambda})_{,\lambda}}{2\epsilon} + \frac{1}{\epsilon}\underbrace{(s_{,t} - V'(s)s_{,\lambda})}_{\text{LWR}-\text{op}}$$

The speeds of propogation of this equation are $\pm\sqrt{\dfrac{V'(s)}{2\epsilon}}$ whereas the speed of propagation of the LWR equation

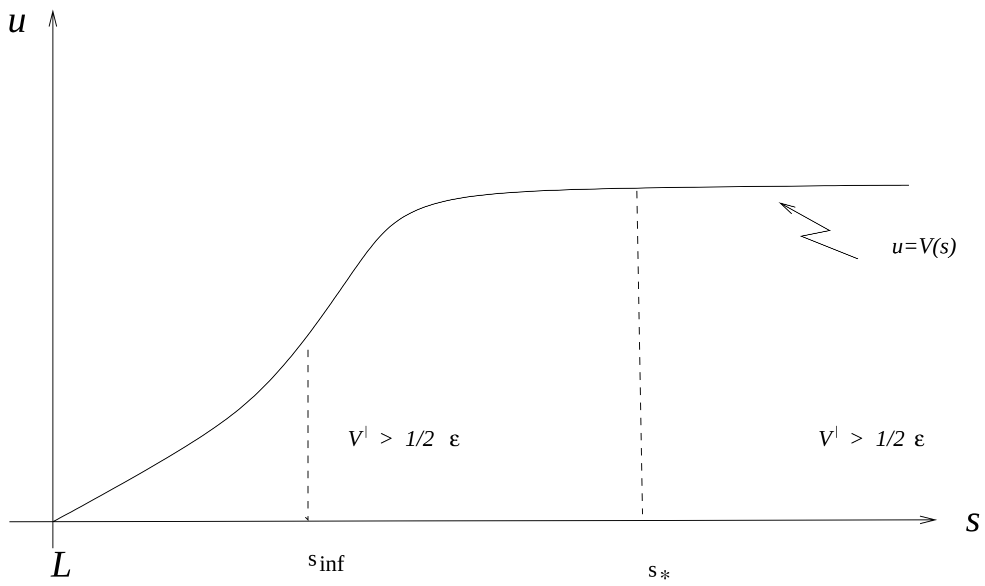
$$s_{,t} - V'(s)s_{,\lambda} = 0$$

is $-V'(s)$ and these speeds satisfy the “Whitham” stability or subcharacteristic condition provided

$$\begin{aligned}
-\sqrt{\frac{V'(s)}{2\epsilon}} \quad &< \quad -V'(s) < 0 \quad \Leftrightarrow \\
0 < V'(s) \quad &< \quad \sqrt{\frac{V'(s)}{2\epsilon}} \qquad\quad \Leftrightarrow \\
0 < V'(s) \quad &< \quad \frac{1}{2\epsilon}.
\end{aligned}$$

We note that if this condition holds for $s \equiv \overline{s} > L$, then the uniform solution $s(\lambda, t) \equiv \overline{s}$ and $u(\lambda, t) = V(\overline{s})$ is linearly stable for the Bando-Modified Continuum equation.

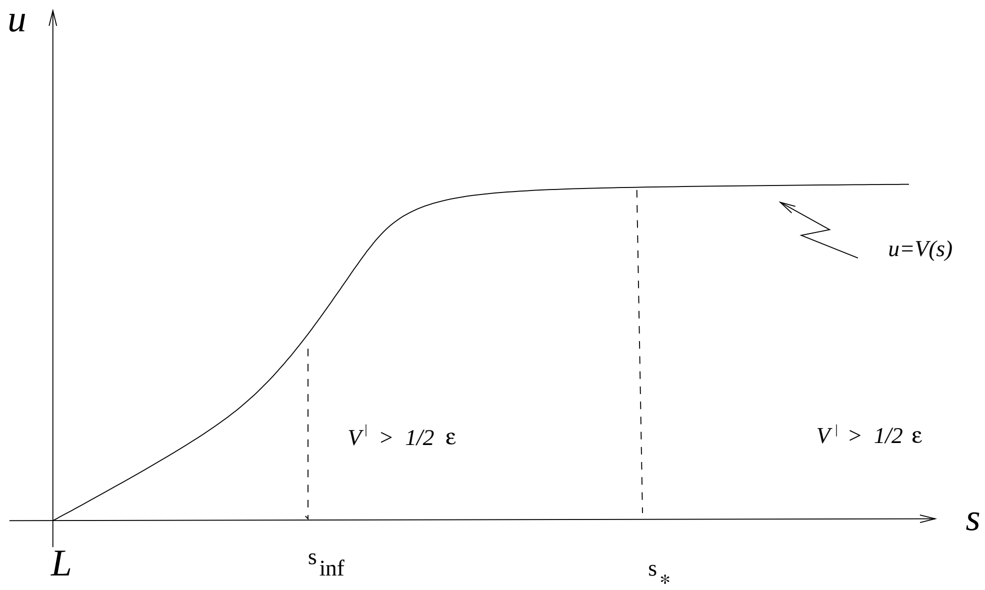
For optimal velocities, $V(\cdot)$, of the type I've been showing

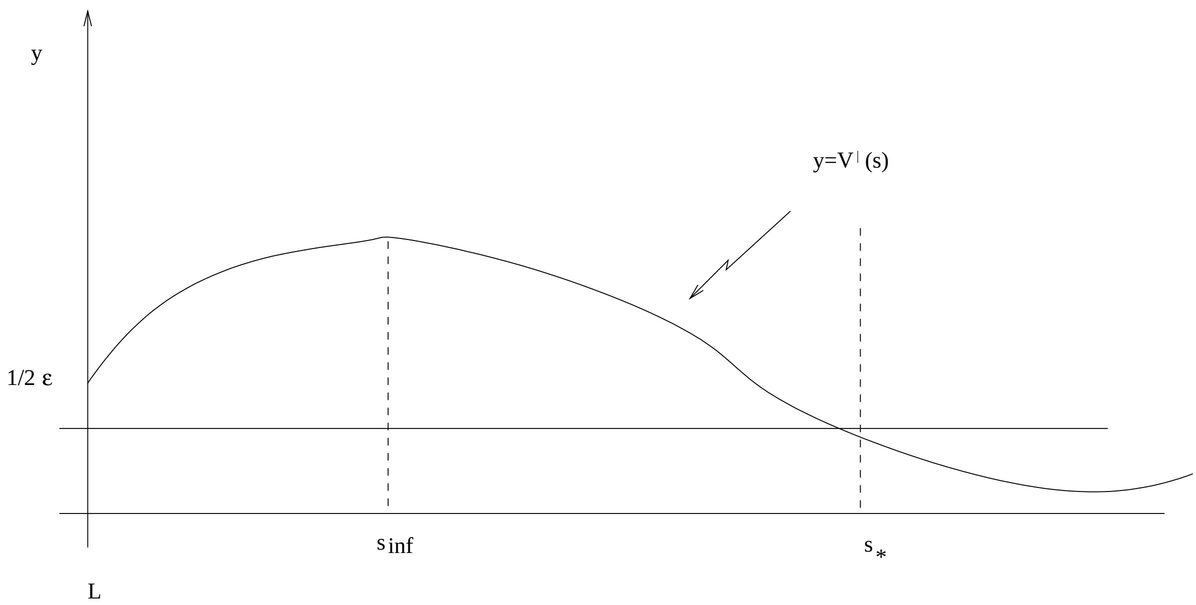


the more typical situation is that

$$V' > 1/2\epsilon \quad \text{for} \quad L \le s < s_* \quad \ldots \text{ unstable}$$

$$V' < 1/2\epsilon \quad \text{for} \quad s_* < s < \infty \quad \ldots \text{ stable.}$$


u
u=V(s)
V' > 1/2 ε
V' > 1/2 ε
s
L
s inf
s *


y
y=V'(s)
1/2 ε
s inf
s *
L