Exploiting Structure: Asymptotically-Reduced and Low-Order Models of Convective and Shear Turbulence

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Modeling/Analysis Spectrum for Spatiotemporally Complex (“Turbulent”) Flows

Ad hoc physically motivated models. . .

. . . Formal systematic analysis (“1st principles”?). . .

. . . . . . Rigorous analysis of Navier–Stokes equations

Themes:

- Geophysical applications (ocean surface boundary layer).
- Deterministic models.
- Mechanistic viewpoint: Turbulence structure (anisotropy, localization). . .

“The mind imbued with Newtonian mechanics seeks simple deterministic explanations of phenomena.” (S. Pope, *Turbulent Flows*, p. 323)

3 Classes of Systematic Models... and 3 Questions

1. Systematic multiscale models of upper ocean turbulence.
   Question: Reliance on scale separation?

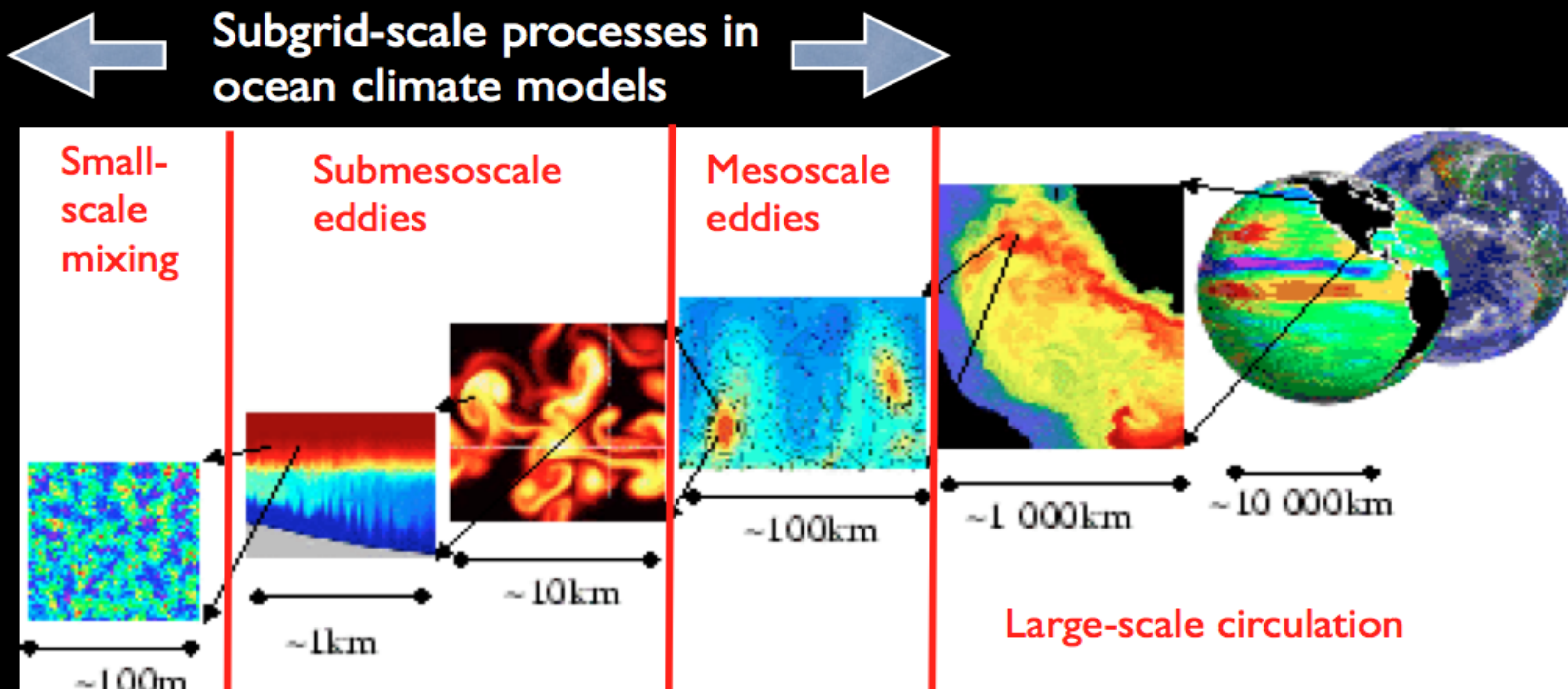
2. Asymptotically-exact reduced PDEs for constrained turbulent flows.
   Question: Application to wall-bounded shear flow turbulence?

3. Low-order ODE models of thermal convection from upper bound theory.
   Question: What if upper bounds are poor (or don't exist!)?

I. Systematic Multiscale Models of Geophysical Flows

Ocean Turbulence (R. Ferrari)

- mesoscale turbulence (10 km – 100 km)
- submesoscale turbulence (100 m – 10 km)
- small-scale turbulent mixing (10 cm – 100 m)



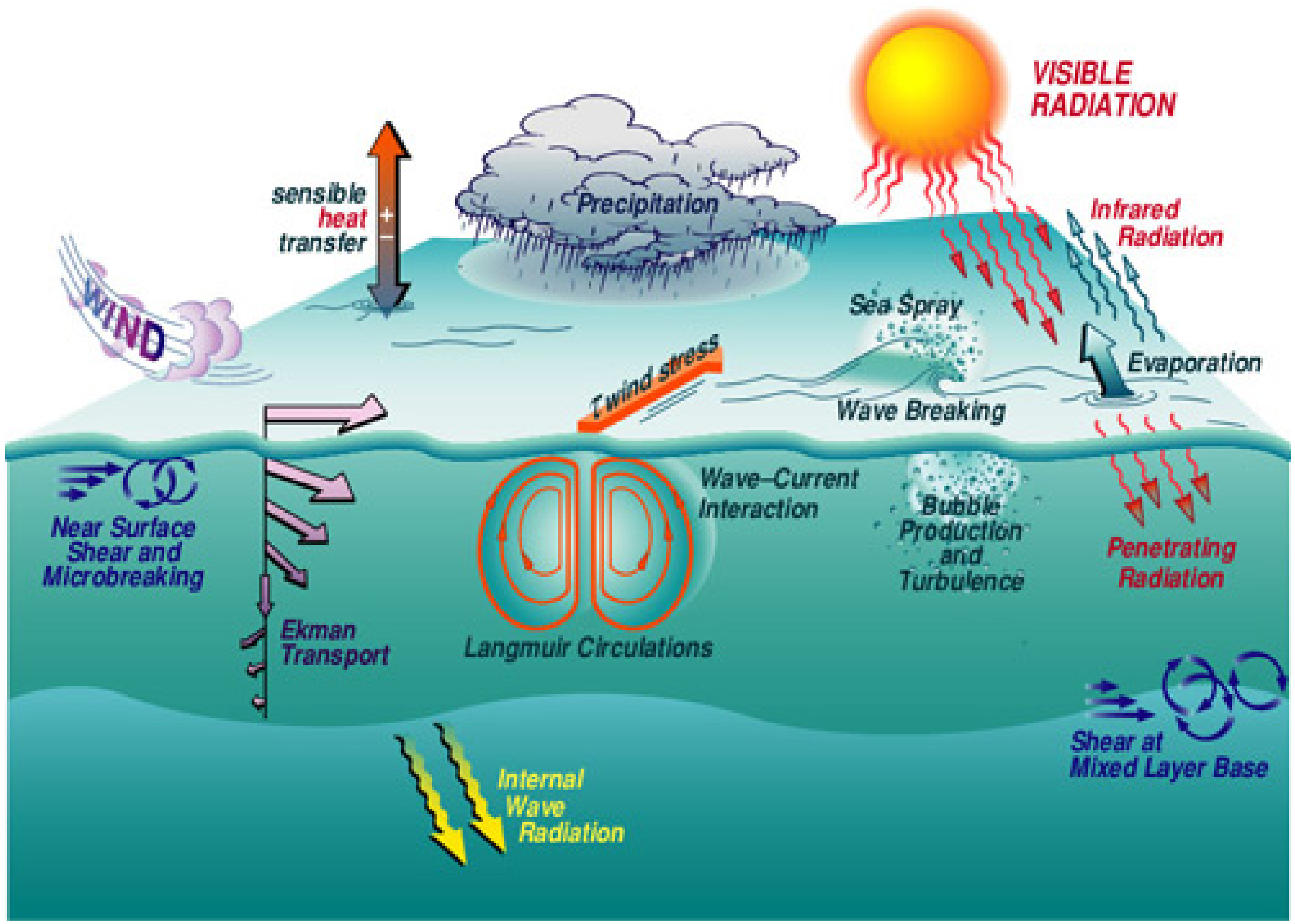
Approaches

(i) Physically motivated, *ad hoc* parameterizations (or complete omission!) of SGS processes in OGCMs, etc.

(ii) Presuming scale separation, systematic (but formal) multiple space and time scale asymptotic analysis (Klein, Majda, Julien. . .):

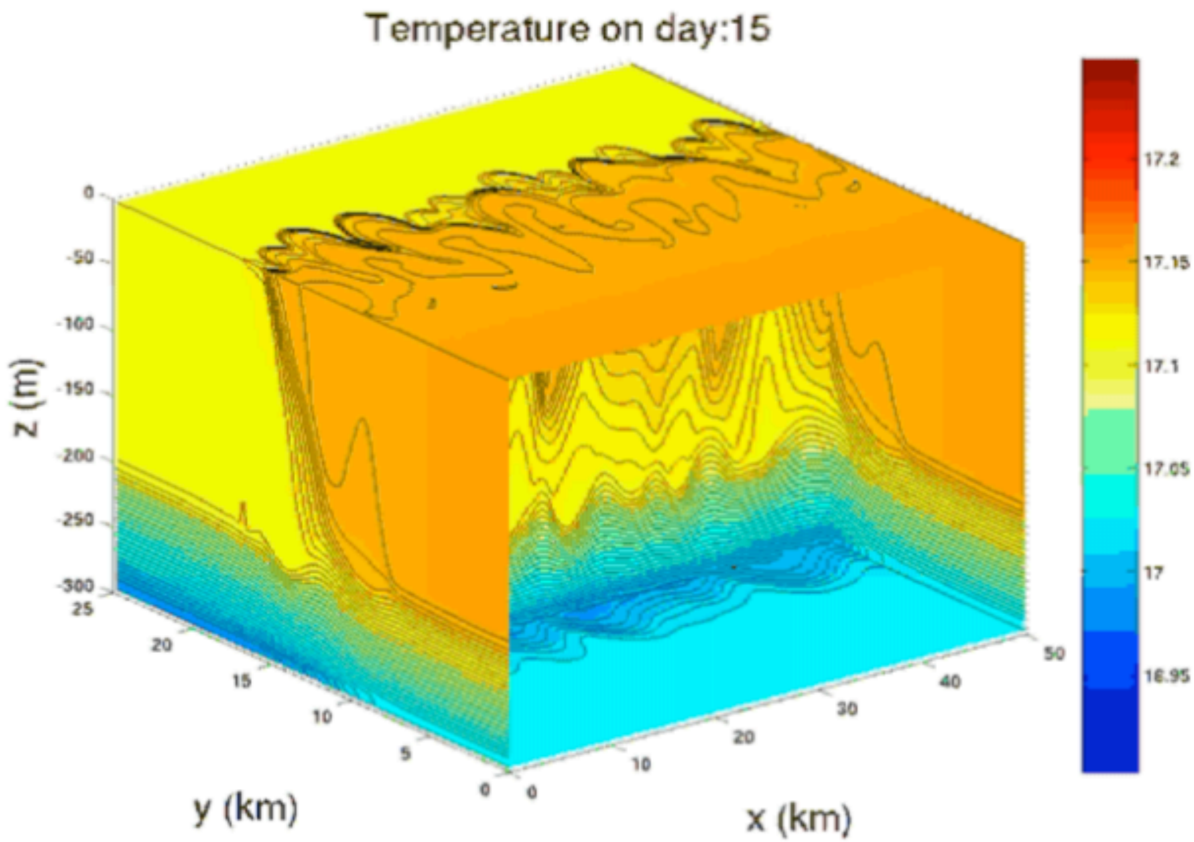
- To mitigate closure problems;
- To provide valuable guidelines regarding the reduced or averaged info that must be communicated from small scales to large;
- To provide the basis for multiscale computational strategies (e.g. the Heterogeneous Multiscale Method of E & Engquist).
- Successes: Madden–Julien oscillation in atm. dynamics (Majda).
- Illustrative example: upper ocean turbulence.

Turbulence in Ocean Surface BL (R. Weller)



- $O(10\text{–}100)$-m non-hydrostatic, weakly stratified, “fully” 3D turbulence.

Submesoscale Phenomena (B. Fox-Kemper)



- $O(10)$-km rotationally influenced (if not constrained), stratified, hydrostatic internal waves (IWs), eddies, fronts, and their associated instabilities.

Master PDEs: Stratified Rotating Craik–Leibovich (CL) Eqns

$$
\begin{aligned}
\partial_T u + (\mathbf{v}_\perp \cdot \nabla_\perp)\, u - \frac{1}{Ro} v &= \frac{1}{Re}\left(\partial_Y^2 + \frac{1}{\delta^2}\partial_z^2\right) u \\
\partial_T v + (\mathbf{v}_\perp \cdot \nabla_\perp)\, v + \frac{1}{Ro} u &= -\partial_Y p + U_s \partial_Y u + \frac{1}{Re}\left(\partial_Y^2 + \frac{1}{\delta^2}\partial_z^2\right) v \\
\partial_T w + (\mathbf{v}_\perp \cdot \nabla_\perp)\, w &= -\frac{1}{\delta^2}\partial_z p + \frac{1}{\delta^2} U_s \partial_z u + \frac{\Gamma}{\delta^2} b + \frac{1}{Re}\left(\partial_Y^2 + \frac{1}{\delta^2}\partial_z^2\right) w \\
\partial_T b + (\mathbf{v}_\perp \cdot \nabla_\perp)\, b &= \frac{1}{Pe}\left(\partial_Y^2 + \frac{1}{\delta^2}\partial_z^2\right) b \\
\nabla_\perp \cdot \mathbf{v}_\perp &= 0
\end{aligned}
$$

Free–Surface BC

$$
\frac{1}{Re\delta}\partial_z u = \left(\frac{u_*}{U}\right)^2
$$

Scaling & Parameters

Scale	Submesocale	Value	BL	Value
Horizontal Length	L	1–10 km	$l = h$	50-100 m
Horizontal Velocity	U	0.1 m/s	U	0.05–0.1 m/s
Wind Stress	u_*	0.01 m/s	u_*	0.01 m/s
BL Depth	h	50-100 m	h	50-100 m
Stokes Drift Velocity	u_{s0}	0.1 m/s	u_{s0}	0.1 m/s
Buoyancy Anomaly	$B = g\|\Delta\rho\|/\rho_0$	0.001 m/s^2	B	0.001 m/s^2
Vertical Velocity	Uh/L	$<$0.01 m/s	U	0.05–0.1 m/s
Advection Time	L/U	5–10 hr	h/U	0.5 hr
Dynamic Pressure	$\rho_0 U^2$	10 Pa	$\rho_0 U^2$	10 Pa

Parameters: $\delta = h/L$ $\quad Ro = U/fL$ $\quad \Gamma = Bh/U^2$ $\quad (Re, Pe) = UL/(\nu, \kappa)$

Asymptotic Analysis I: Distinguished Limit

1. Perform asymptotic analysis by exploiting smallness of $\delta \approx 0.01 - 0.1$.

2. Consider distinguished limit in which $\delta \to 0$ with:

$$Ro = O(1); \Gamma = O(1)$$
$$(Re, Pe) = \delta^{-2}(\hat{R}, \hat{P}), \text{ where } (\hat{R}, \hat{P}) = O(1)$$
$$(u_*/U)^2 = \delta\tau, \text{ where } \tau = O(1)$$

3. Introduce fast space and time scales: $y = Y/\delta$ and $t = T/\delta$.

4. Multiscale differentiation: $\partial_Y \to \partial_Y + \delta^{-1}\partial_y; \quad \partial_T \to \partial_T + \delta^{-1}\partial_t$.

Asymptotic Analysis II: Multiscale Asymptotic Expansions

5. Decompose fields into fast-(y,t) mean plus fluctuation:

$$
\begin{aligned}
u(Y,z,T) &\sim \overline{u}(Y,z,T) + u'(y,Y,z,t,T) \\
v(Y,z,T) &\sim \overline{v}(Y,z,T) + v'(y,Y,z,t,T) \\
w(Y,z,T) &\sim \overline{w}(Y,z,T) + \frac{1}{\delta} w'(y,Y,z,t,T) \\
p(Y,z,T) &\sim \overline{p}(Y,z,T) + p'(y,Y,z,t,T) \\
b(Y,z,T) &\sim \overline{b}(Y,z,T) + b'(y,Y,z,t,T)
\end{aligned}
$$

$$
\overline{f}(Y,z,T) = \frac{lim}{\substack{\mathcal{L}\to\infty \\ \mathcal{T}\to\infty}} \frac{1}{4\mathcal{T}\mathcal{L}} \int_{-\mathcal{T}}^{\mathcal{T}} \int_{-\mathcal{L}}^{\mathcal{L}} f(y,Y,z,t,T)\, dy dt
$$

Mean Equations: Submesoscale Dynamics

$$\bar{D}_T\bar{u}+\partial_Y\left(\overline{v'u'}\right)+\delta^{-1}\partial_z\left(\overline{w'u'}\right)-Ro^{-1}\bar{v} = \hat{R}^{-1}\partial_z^2\bar{u}$$

$$\bar{D}_T\bar{v}+\partial_Y\left(\overline{v'v'}\right)+\delta^{-1}\partial_z\left(\overline{w'v'}\right)+Ro^{-1}\bar{u} = -\partial_Y\bar{p}+U_s\partial_Y\bar{u}+\hat{R}^{-1}\partial_z^2\bar{v}$$

$$0 = -\partial_z\bar{p}+\bar{b}+U_s\partial_z\bar{u}-\partial_z\left(\overline{w'w'}\right)$$

$$\bar{D}_T\bar{b}+\partial_Y\left(\overline{v'b'}\right)+\delta^{-1}\partial_z\left(\overline{w'b'}\right) = \hat{P}^{-1}\partial_z^2\bar{b}$$

$$\partial_Y\bar{v}+\partial_z\bar{w} = 0$$

Free-Surface BC: $$\hat{R}^{-1}\partial_z\bar{u} = \bar{\tau}$$

where $\bar{D}_T \equiv \partial_T + \bar{v}\partial_Y + \bar{w}\partial_z$.

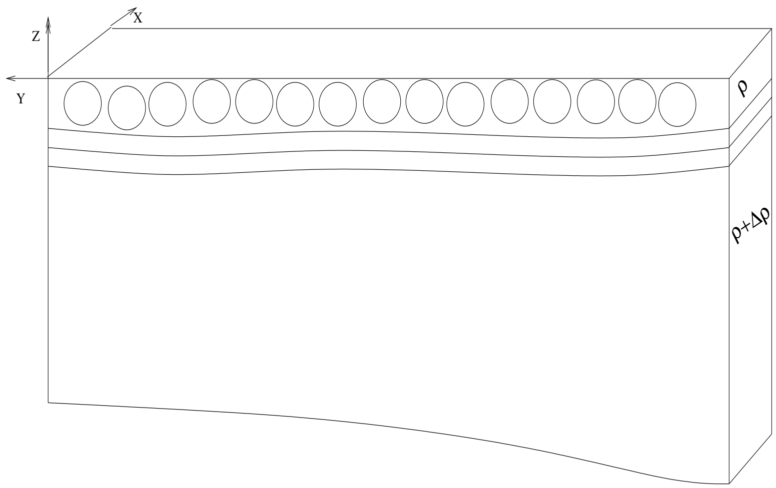
Fluctuation Equations: Langmuir Turbulence

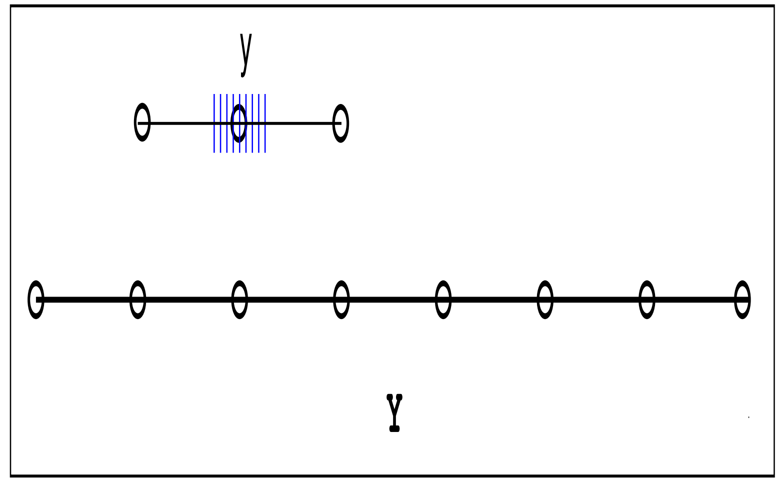
$$
\begin{aligned}
D_t' u' + w' \partial_z \overline{u} &= \delta \widehat{R}^{-1} \left(\partial_y^2 + \partial_z^2 \right) u' \\
D_t' v' + w' \partial_z \overline{v} &= -\partial_y p' + U_s \partial_y u' + \delta \widehat{R}^{-1} \left(\partial_y^2 + \partial_z^2 \right) v' \\
D_t' w' - \partial_z \left(\overline{w' w'} \right) &= -\partial_z p' + U_s \partial_z u' + b' + \delta \widehat{R}^{-1} \left(\partial_y^2 + \partial_z^2 \right) w' \\
D_t' b' + w' \partial_z \overline{b} &= \delta \widehat{P}^{-1} \left(\partial_y^2 + \partial_z^2 \right) b' \\
\partial_y v' + \partial_z w' &= 0
\end{aligned}
$$

Free-Surface BC: $\widehat{R}^{-1} \partial_z u' = 0$

where $D_t' \equiv \partial_t + (\overline{v} + v')\partial_y + w'\partial_z$.

Comments





- Explicit identification of dominant multiscale coupling terms.
- Pattern-forming geophysical BLs: LC, buoyancy-driven convection, Ekman rolls, even uni-directional shear flows?
- Opportunity to borrow ideas, techniques from pattern formation theory: Phase-diffusion/mean-drift equations via nonlinear WKBJ analysis.
- Informs design of multiscale numerical algorithms (e.g. HMM).

Discussion Topic # 1: Scale Separation?

- How small does δ have to be in practice? Dynamic assessment?

- How wide does **micro-scale** domain have to be for stable averages/fluxes?

- "There may very well be clean scale separations even if they are not visible in *spectral* decompositions" (R. Klein, Annu. Rev. Fluid. Mech. 2010).

- Extension of homogenization and related multiscale asymptotic techniques to problems lacking (spectral) scale separation (T. Hou)...?

Discussion Topic # 1: Scale Separation? (Cont'd)

Adaptation of T. Hou Strategy

Consider passive scalar advection for $Pe \gg 1$.

$$\begin{aligned} \partial_T C + (\mathbf{V} \cdot \nabla) C &= \frac{1}{Pe} \nabla^2 C \\ C(X, Y, 0) &= C_0(X, Y) \\ C(X, Y), & \quad L\text{–periodic in } 0 \leq X \leq L \text{ and } 0 \leq Y \leq L. \end{aligned}$$

1. Re-partition C_0, C, and $\mathbf{V}$ into formal 2-scale representation; e.g.:

$$C_0(\mathbf{X}) = \bar{C}_0(\mathbf{X}) + C_0'(\mathbf{X}, \mathbf{X}/\varepsilon), \text{ where } \varepsilon = \Delta X/L = \Delta Y/L.$$

2. Introduce phase variable and perform Lagrangian (not Eulerian) averaging:

$$\begin{aligned} \partial_T \mathbf{\Theta} + (\mathbf{V} \cdot \nabla) \mathbf{\Theta} &= 0 \\ \mathbf{\Theta}(\mathbf{X}, 0) &= \mathbf{X} \end{aligned}$$

Discussion Topic # 1: Scale Separation? (Cont'd)

$$
\begin{aligned}
C_0^{'} &= \sum_{|\mathbf{k}|>1/2\varepsilon} \widehat{C}_0(\mathbf{k}) \exp\left(i\frac{2\pi}{L}\mathbf{k}\cdot\mathbf{X}\right) \\
&= \sum_{\frac{1}{\varepsilon}\mathbf{k}^{(s)}+\mathbf{k}^{(l)}} \widehat{C}_0\left(\frac{1}{\varepsilon}\mathbf{k}^{(s)}+\mathbf{k}^{(l)}\right) \exp\left(i\frac{2\pi}{L}\mathbf{k}^{(l)}\cdot\mathbf{X}\right) \exp\left(i\frac{2\pi}{L}\mathbf{k}^{(s)}\cdot\frac{\mathbf{X}}{\varepsilon}\right) \\
&= \sum_{\mathbf{k}^{(s)}\neq 0} \widehat{C}_{0s}\left(\mathbf{k}^{(s)},\mathbf{X}\right) \exp\left(i\frac{2\pi}{L}\mathbf{k}^{(s)}\cdot\frac{\mathbf{X}}{\varepsilon}\right) \\
&= C_0^{'}\left(\mathbf{X},\frac{\mathbf{X}}{\varepsilon}\right)
\end{aligned}
$$

where: $\widehat{C}_{0s}\left(\mathbf{k}^{(s)},\mathbf{X}\right) = \sum_{\mathbf{k}^{(l)}} \widehat{C}_0\left(\frac{1}{\varepsilon}\mathbf{k}^{(s)}+\mathbf{k}^{(l)}\right) \exp\left(i\frac{2\pi}{L}\mathbf{k}^{(l)}\cdot\mathbf{X}\right)$

$$\Rightarrow \qquad C^{'} = C^{'}\left(\mathbf{X},\frac{\mathbf{\Theta}}{\varepsilon},T,\frac{T}{\varepsilon}\right)$$

II. Asymptotically-Reduced PDEs for Constrained Turbulent Flows

Reduced Models of Constrained Flows: Old Idea. . .

- “Large-scale” geophysical and astrophysical flows constrained by rotation, stratification, and/or magnetic fields.
- Strong constraint imposed by dominant force on flow drives near two-dimensionalization, reduced mode coupling in certain directions.
- Departures from 2D dynamics remain fundamentally important, and flows remain strongly nonlinear.
- Reduced models obtained by linking emergence of strongly anisotropic flow structures to imposed constraint and exploiting asymptotic disparity in length and time scales.
- Example: Quasi-Geostrophic equations in oceanic/atmospheric flows.

Reduced Models of Constrained Flows: Modern Developments

Extensions to constrained but non-hydrostatic flows:

- Julien, Knobloch have greatly extended these techniques to:

  (i) rapidly rotating thermal convection;

  (ii) thermal convection in a strong magnetic field;

  (iii) the magnetorotational instability (in accretion disks).

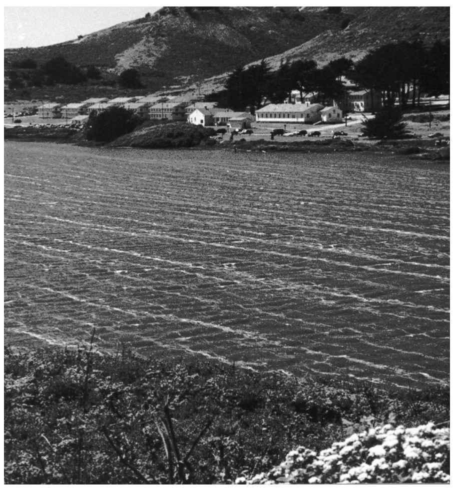
- These flows are both constrained **and** centrifugally unstable.

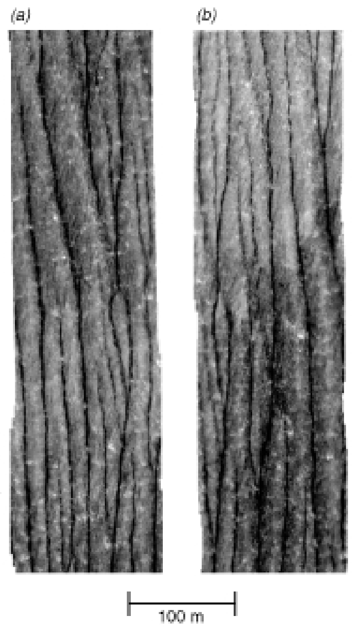
Are extensions to shear flows possible?

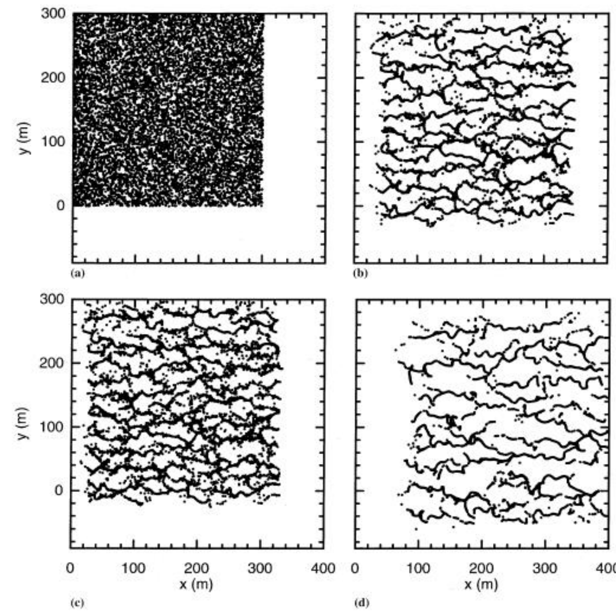
- Anisotropic Langmuir turbulence. . . (Tandon & Leibovich, *JPO* 1995, Chini *et al. GAFD* 2009)

- Plane Couette flow (PCF) turbulence? (Waleffe, Nagata, Busse. . .)

Anisotropic LC Dynamics







–Szeri (1996) –Marmorino *et al.* (2005) –McWilliams *et al.* (1997)

Isotropically Scaled CL Equations

- Consider full 3D, isotropically-scaled CL equations, where two parameters $R_* \equiv u_* H/\nu_e$, $La_t = \sqrt{u_*/u_{s_0}}$ replace single parameter $La \equiv La_t R_*^{-3/2}$:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \frac{1}{La_t^2}(\mathbf{U}_s \times \boldsymbol{\omega}) + \frac{1}{R_*}\nabla^2 \mathbf{u}$$

- Two turbulence regimes:

Shear flow turbulence regime: $La_t \gg 1$ with $R_* \gg 1$.

Langmuir turbulence regime: $La_t = O(0.1)$ with $La \ll 1$.

- Motivates consideration of formal limit $La_t \to 0$ with R_* or La fixed.

Implications of Leading-Order Balance

- Upon rescaling the pressure, the leading-order balance is:

$$\nabla\mathcal{P} = \mathbf{U}_s \times \mathbf{\Omega} \quad \Rightarrow \quad \mathbf{U}_s \cdot \nabla\mathbf{\Omega} = \mathbf{\Omega} \cdot \nabla\mathbf{U}_s$$

- From this balance, the following deductions can be made:

$$\begin{aligned} \partial_x \mathcal{P} &= 0 \\ U_s \partial_x \Omega_x &= \Omega_z U_s^{'}(z) \\ U_s \partial_x \Omega_y &= 0 \\ U_s \partial_x \Omega_z &= 0 \end{aligned}$$

- 2D dynamics: $\Omega_x \neq 0$, $\partial_x \Omega_x = 0$, u–fluctuations $\ll$ (v, w)–fluctuations.

Anisotropic Velocity Scalings

- Employ anisotropic velocity scales to capture nonlinear, spatially anisotropic reduced dynamics:

$$L_x = H, \quad (L_y, L_z) = H, \quad \mathcal{T} = H/\mathcal{V}$$
$$\mathcal{U} = u_* R_*, \quad (\mathcal{V}, \mathcal{W}) = \sqrt{\mathcal{U} u_{s_0}}, \quad \mathcal{P} = \rho \mathcal{V}^2$$

- In essence, perturbing off of strictly 2D [$\partial(\cdot)/\partial x = 0$] problem.

- Identify $\mathcal{U}/\mathcal{W} = La_t (La_t/La)^{1/3} \equiv \varepsilon \ll 1$

(cf. Tejada-Martinez & Grosch JFM 2007, Teixeira & Belcher JFM 2002).

Rescaled CL Equations in Strong CL Vortex-Force Limit

$$\begin{aligned}
\partial_t u + \varepsilon\, u \partial_x u + (\mathbf{v}_\perp \cdot \nabla_\perp)\, u &= -\varepsilon^{-1} \partial_x P + La \left[\partial_x^2 + \nabla_\perp^2\right] u \\
\partial_t \mathbf{v}_\perp + \varepsilon\, u \partial_x \mathbf{v}_\perp + (\mathbf{v}_\perp \cdot \nabla_\perp)\, \mathbf{v}_\perp &= -\nabla_\perp P + La \left[\partial_x^2 + \nabla_\perp^2\right] \mathbf{v}_\perp \\
&\quad + U_s \left(\nabla_\perp u - \varepsilon^{-1} \partial_x \mathbf{v}_\perp\right) \\
\varepsilon\, \partial_x u + \nabla_\perp \cdot \mathbf{v}_\perp &= 0
\end{aligned}$$

- Wind stress BC: $\partial_z u = 1$ along $z = 0, -1$.
- x-invariance at leading-order: $\partial_x P = \partial_x v = \partial_x w = 0$ and $\nabla_\perp \cdot \mathbf{v}_\perp = 0$.

Multiple Scale Expansion

1. Limit process: $\varepsilon \to 0$ with La fixed.
2. Introduce slow x scale: $X \equiv \varepsilon x$ so that $\partial_x \to \partial_x + \varepsilon \partial_X$.
3. Expand fields:

$$\begin{aligned} u(x,y,z,t) &= u_0(x,X,y,z,t) + \varepsilon u_1(x,X,y,z,t) + \ldots \\ \mathbf{v}_\perp(x,y,z,t) &= \mathbf{v}_{0\perp}(X,y,z,t) + \varepsilon \mathbf{v}_{1\perp}(x,X,y,z,t) + \ldots \\ P(x,y,z,t) &= P_0(X,y,z,t) + \varepsilon P_1(x,X,y,z,t) + \ldots \end{aligned}$$

4. Substitute into PDEs, collect terms of like order and **average** over fast x.
5. Obtain **closed** set of equations for $\bar{u}_0 \equiv U(X,y,z,t)$, $\mathbf{v}_{0\perp} \equiv \mathbf{V}_\perp(X,y,z,t)$ and $P_0 \equiv \Pi(X,y,z,t)$.

Reduced PDEs

- Define:

$$D_t^{\perp}(\cdot) \;\equiv\; \partial_t(\cdot)+(\mathbf{V}_\perp\cdot\nabla_\perp)(\cdot) \equiv \partial_t(\cdot)+J[(\cdot),\psi],$$

  where $J[(\cdot),\psi] = \partial_z\psi\partial_y(\cdot) - \partial_y\psi\partial_z(\cdot)$.

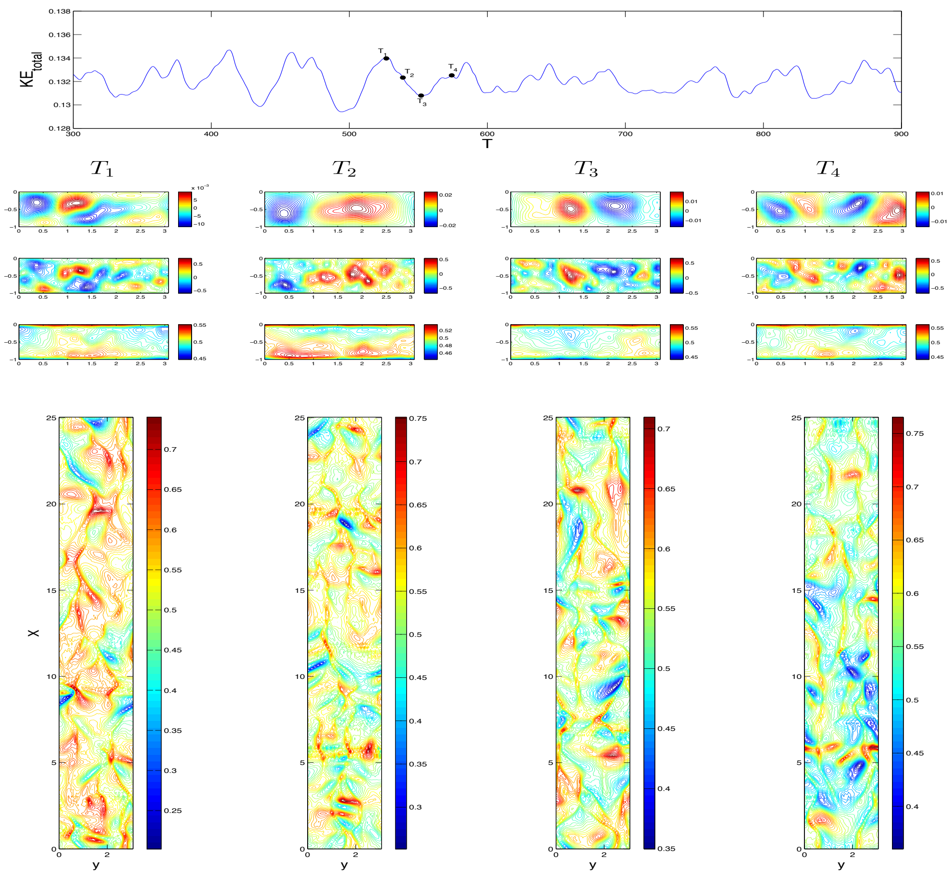
- Reduced dynamics governed by:

$$\begin{aligned}
D_t^{\perp}U &= -\partial_X \Pi + La\nabla_\perp^2 U \\
D_t^{\perp}\Omega + U_s(z)\partial_X\Omega &= U_s'(z)(\partial_X V - \partial_y U) + La\nabla_\perp^2\Omega \\
\nabla_\perp^2 \Pi &= 2J[\partial_y\psi,\partial_z\psi] + \nabla_\perp\cdot(U_s(z)\nabla_\perp U) + U_s'(z)\partial_X(\partial_y\psi) \\
\nabla_\perp^2\psi &= -\Omega, \quad \mathbf{V}_\perp \equiv \nabla_\perp\times\psi\hat{\imath}
\end{aligned}$$

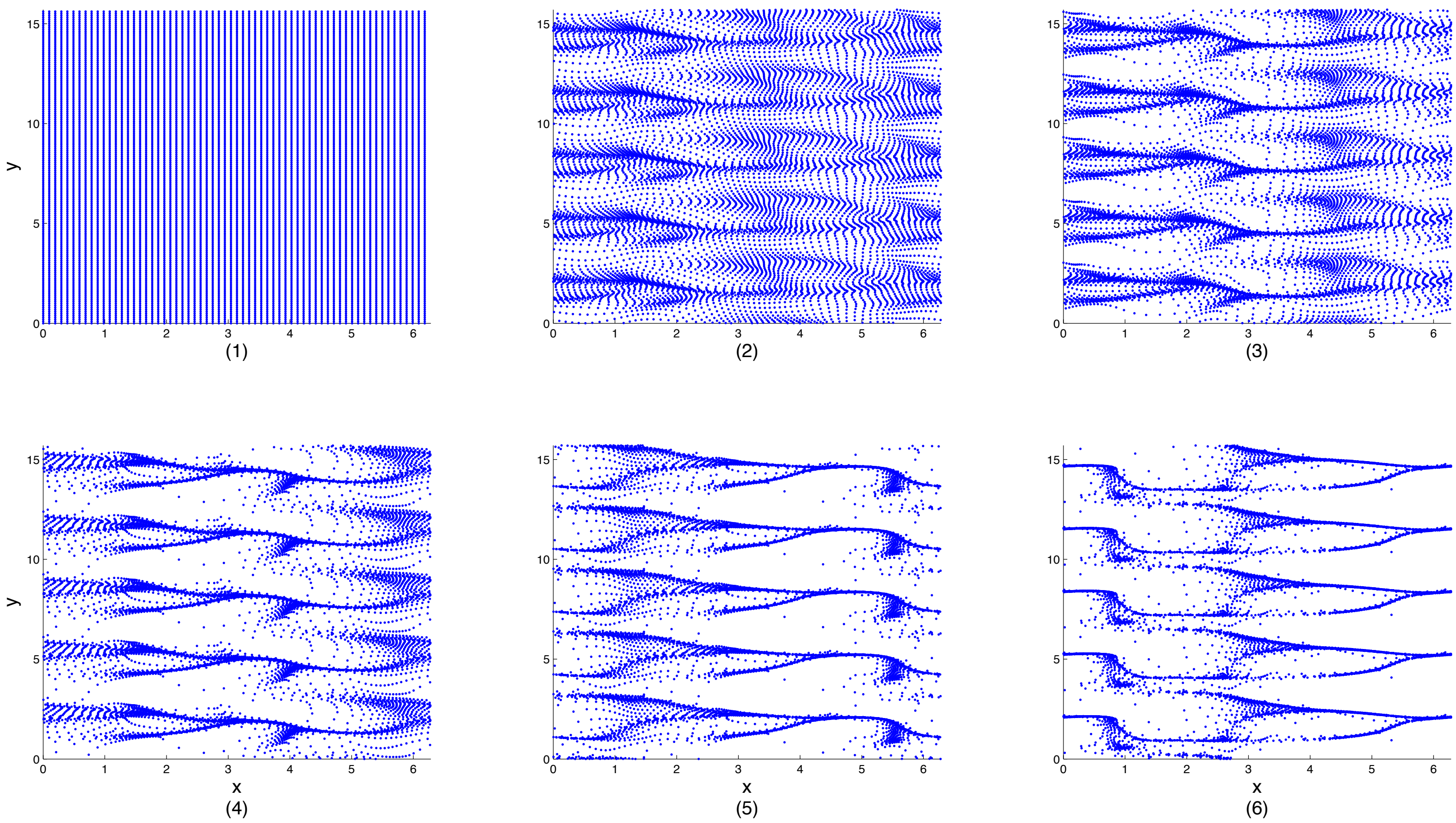
- Fast x averaged BCs along $z=0,-1$: $\partial_z U = 1, \;\; \Omega = 0, \;\; \psi = 0$.

- Advection by U and stretching of Ω are subdominant processes.

Pseudospectral Numerical Simulations of Reduced PDEs



Surface Tracer Evolution: Windrows and Y-Junctions



Comments

- Derived reduced PDEs for anisotropic turbulent Langmuir circulation in strong vortex-force limit.

- Reduced PDEs capture dominant linear and secondary instabilities.

- Reduced PDEs offer several analytical and computational advantages:

  1. Filter fine x-scale variability $\Rightarrow$ larger Δx and possibly Δt.

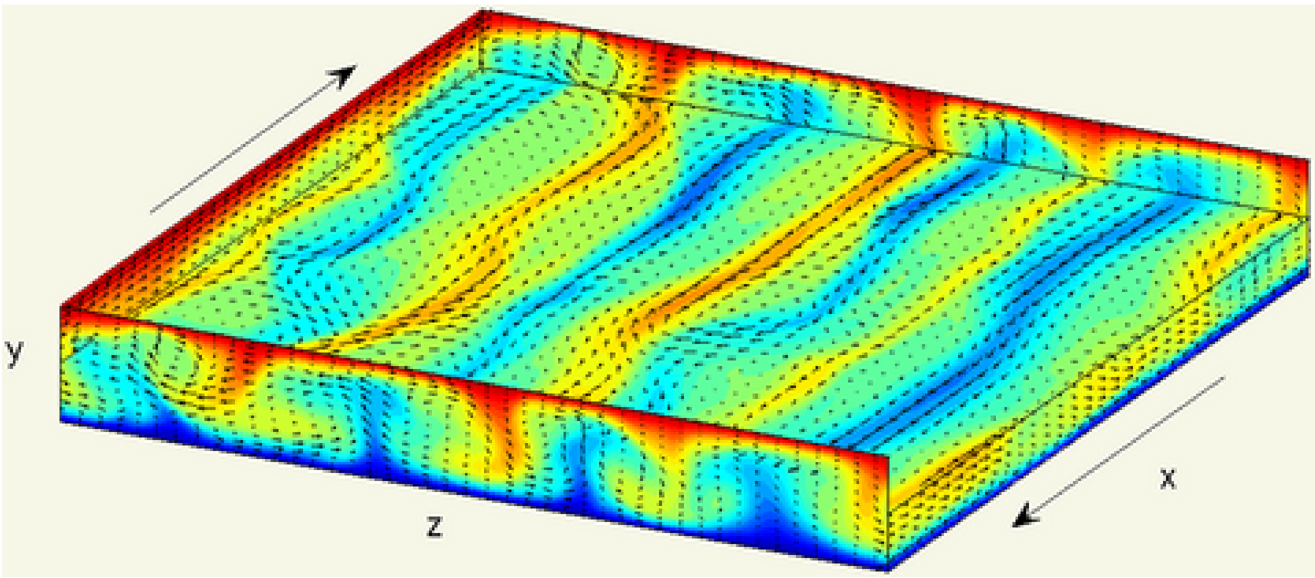
  2. Vortex stretching is sub-dominant process $\Rightarrow$ facilitates analysis (e.g. homogenization theory [T. Hou]).

  3. *Limiting procedure suppresses "self-sustaining process" proposed by Waleffe for wall-bounded shear-flow turbulence.*

  4. Methodology being applied to PCF turbulence...

Discussion Topic # 2: Extension to Wall-Bounded Shear Flow Turbulence?

Low Reynolds Number Plane Couette Flow (PCF) Turbulence



$$\mathcal{U} = U_w, \quad (\mathcal{V}, \mathcal{W}) = \frac{1}{Re} U_w$$

– J. Gibson

Reduction of NSE for Lower-Branch PCF "Turbulence"

Averaged Equations

$$
\begin{aligned}
\partial_T \bar{u}_0 + (y+\bar{u}_0)\,\partial_X \bar{u}_0 + (\bar{\mathbf{v}}_{1\perp}\cdot\nabla_\perp)\,\bar{u}_0 + \bar{v}_1 &= \frac{1}{R}\nabla_\perp^2 \bar{u}_0 \\
\partial_T \bar{\mathbf{v}}_{1\perp} + \partial_X\left[(y+\bar{u}_0)\,\bar{\mathbf{v}}_{1\perp}\right] + \nabla_\perp\cdot\left[\bar{\mathbf{v}}_{1\perp}\bar{\mathbf{v}}_{1\perp} + \overline{\mathbf{v}'_{1\perp}\mathbf{v}'_{1\perp}}\right] &= -\nabla_\perp \bar{p}_2 + \frac{1}{R}\nabla_\perp^2 \bar{\mathbf{v}}_{1\perp} \\
\partial_X \bar{u}_0 + \nabla_\perp\cdot\bar{\mathbf{v}}_{1\perp} &= 0
\end{aligned}
$$

where $R \equiv \epsilon Re = O(1)$ is the reduced Reynolds number.

Fluctuation Equations

$$
\begin{aligned}
\partial_t u'_1 + (y+\bar{u}_0)\,\partial_x u'_1 + \left(\mathbf{v}'_{1\perp}\cdot\nabla_\perp\right)\bar{u}_0 + v'_1 &= -\partial_x p'_1 \\
\partial_t \mathbf{v}'_{1\perp} + (y+\bar{u}_0)\,\partial_x \mathbf{v}'_{1\perp} &= -\nabla_\perp p'_1 \\
\partial_x u'_1 + \nabla_\perp\cdot\mathbf{v}'_{1\perp} &= 0
\end{aligned}
$$

- See recent paper by Hall & Sherwin (*JFM* 2010).
- SGS modeling of "outer–inner" BL coupling, spatial dynamics studies?

III. Low-Order (ODE) Models from Upper Bound Theory

Motivation

- Low-dimensional (ODE) descriptions of nonlinear fluid phenomena required in increasing number of applications (flow control, parameter estimation, fine-scale dynamics in multiscale algorithms).

- Popular existing approaches fall into two broad categories:

  1. Fully predictive models generated by (e.g.) Galerkin projection (GP) onto orthogonal basis functions (e.g. spectral expansions), but these modes generally not well adapted to reduced description of highly nonlinear systems.

  2. Proper Orthogonal Decomposition (POD) based approaches are often better suited for the low order description of highly nonlinear dynamics, but the POD modes are empirical.

Key Heuristic of New Approach

- **Assertion:** Eigenfunctions from energy stability theory provide a suitable *a priori* basis for reduced description of highly nonlinear dynamics. . . ***if*** energy stability is enforced about a suitable nonlinear base state (e.g. temperature profile) rather than the "conduction" solution.

- Idea is that, on attractor, dynamical system adjusts so that fluctuations are marginally stable (Malkus 1954, Howard 1963).

- Poje & Lumley (1995,1997) closest in spirit to the methodology described, but they employed a semi-empirical mean profile and sought the fastest growing energy modes ($\approx$ coherent structures).

Case Study: (2D) Porous Medium Convection

PDEs

$$\begin{aligned} \partial_t T + \mathbf{u} \cdot \nabla T &= \nabla^2 T \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u} + \nabla P &= Ra T \hat{\mathbf{e}}_{\mathbf{z}} \end{aligned}$$

BCs $z = 0, 1:$ $T = 1, 0;\ w = 0$

x periodic

- The Nusselt number Nu is

$$Nu \equiv 1 + \langle wT \rangle = \left\langle |\nabla T|^2 \right\rangle = 1 + Ra^{-1} \left\langle |\mathbf{u}|^2 \right\rangle \geq 1,$$

where $\langle \cdot \rangle$ denotes a space–time average.

Algorithm

1. Following Doering & Constantin (1998), Otero *et al.* (2004), decompose:

$$T(x,z,t) \equiv \tau(z) + \theta(x,z,t), \quad \text{where} \quad \tau(0) = 1 \quad \text{and} \quad \tau(1) = 0.$$

2. Substituting yields dynamical evolution of θ:

$$\begin{aligned} \partial_t \theta + \mathbf{u} \cdot \nabla \theta &= \nabla^2 \theta - \tau' w + \tau'' \\ &= -\mathcal{S}(\theta) - \mathcal{A}(\theta) + \tau'', \end{aligned}$$

$$\text{where: } w = \mathcal{L}(\theta); \quad \mathcal{S}(\theta) = \frac{1}{2}\Big(\tau' \mathcal{L}(\theta) + \mathcal{L}(\tau' \theta)\Big) - \nabla^2 \theta.$$

3. Motivated by (rigorous) upper bound analysis, determine $\tau(z)$ by requiring $\Lambda(\mathcal{S}) \geq 0$ ("energy stability") and minimizing $\int_0^1 \tau'(z)^2 dz$ (heat flux, Nu).

Nusselt–Rayleigh Scaling: Upper Bound Theory vs. DNS

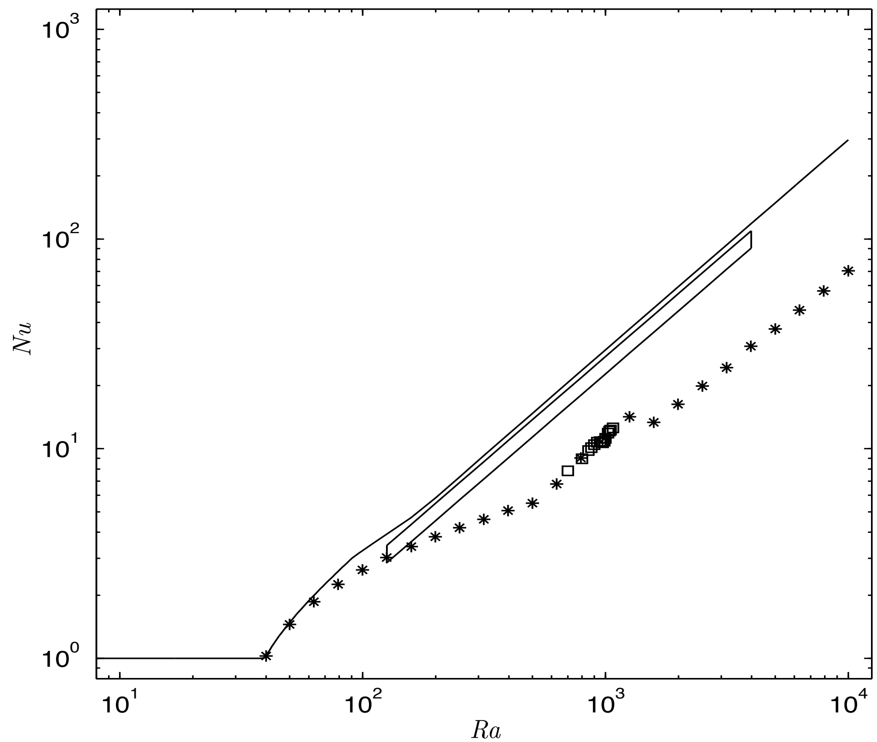


Figure 4.6: **Bounds and data for porous medium convection.** The solid line shows the lower envelope of the numerical bound and the rigorous bound shown in figure 4.4. The square boxes show the data from [Graham and Steen, 1994]. The data indicated by asterisks is from a direct numerical simulation, for the two-dimensional problem, carried out by Hans Johnston. The box indicates the range of Elder's 1967 experimental data where $Nu = .025Ra \pm 10\%$ for $100 < Ra < 5000$, see [Elder, 1967].

Computation of Basis Functions

- *A priori* basis functions obtained by solving **constrained, non-local** eigenvalue problem using Chebyshev spectral collocation method:

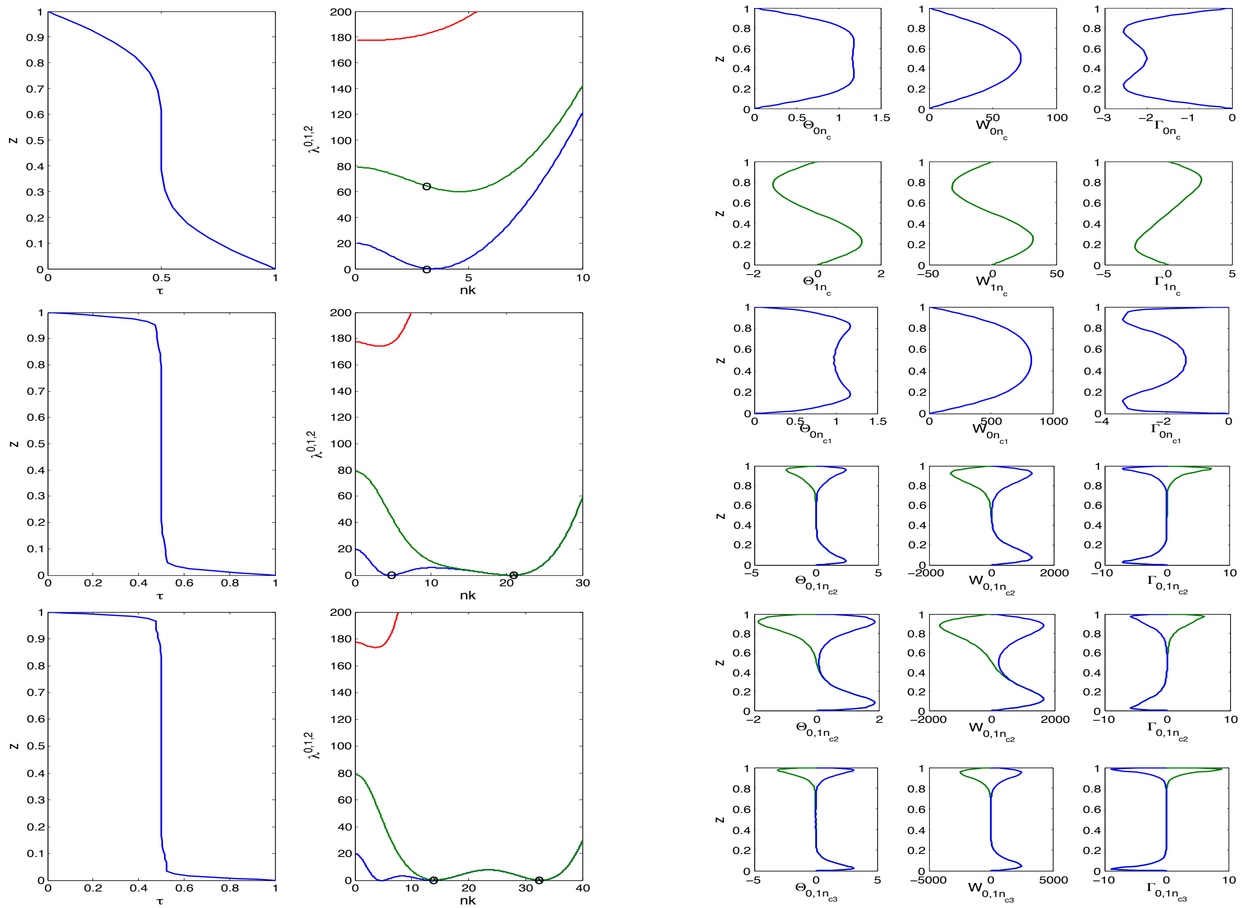
$$\theta(x,z) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \Theta_{mn}(z) e^{inkx}, \quad etc.$$

- 1. Guess initial background profile $\tau(z)$. 2. Solve e-value problem. 3. Repeat, subject to constraint $\lambda^0 \geq 0$, adjusting $\tau(z)$ to **minimize** upper bound functional

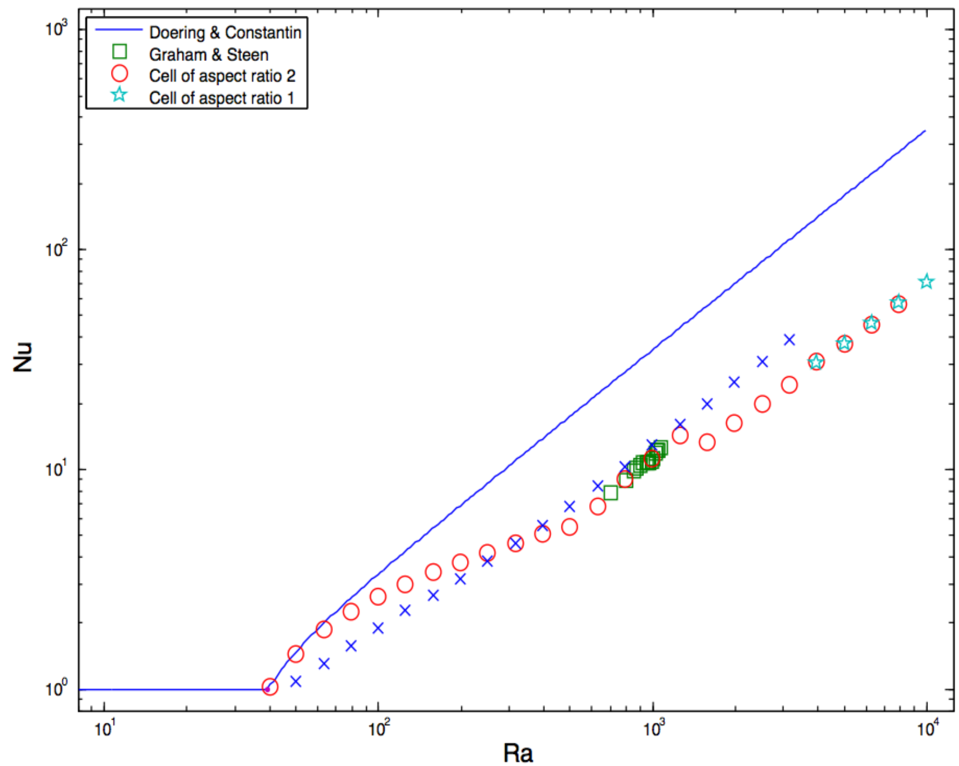
$$\mathcal{J}[\tau(z); Ra, k] \equiv \int_0^1 \tau'(z)^2 dz$$

* For a given domain size (L) and Rayleigh number Ra, optimization code must sample **all** admissible modes to ensure spectral constraint satisfied for all relevant horizontal wavenumbers.

Background Profile, Spectra, Basis Functions

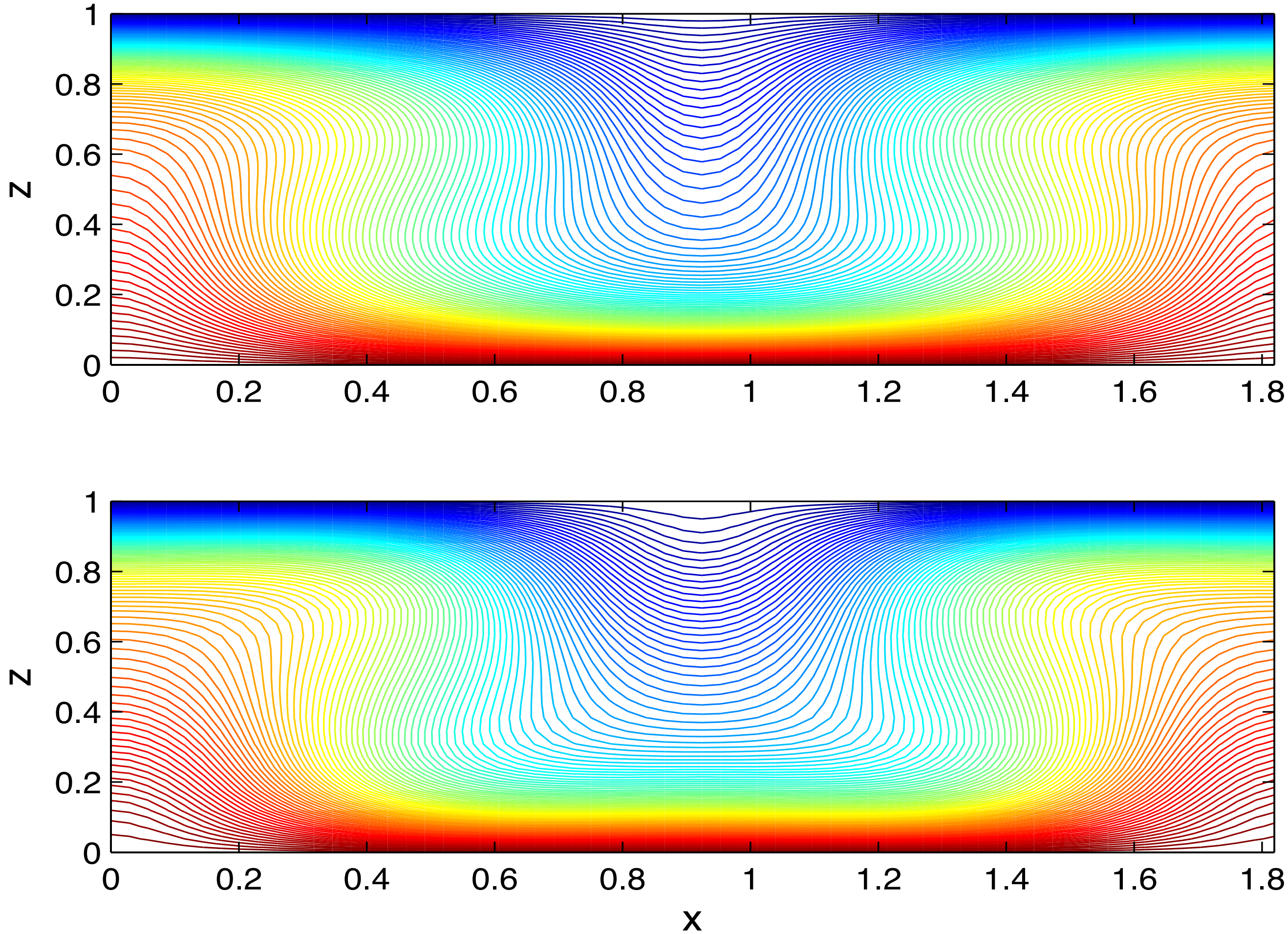


Nusselt–Rayleigh Predictions



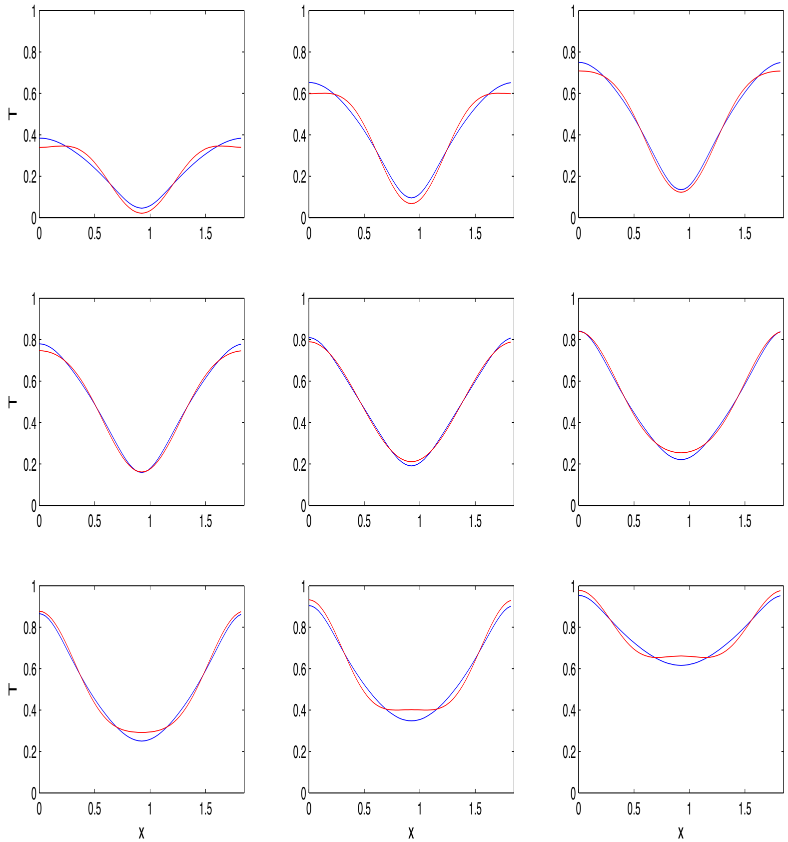
Steady Single Roll-Pair Solution at Ra=100

T(x,z) Contours: DNS (upper) vs. 6-Mode Model (lower)



Performance of 6-Mode Model at Ra=100 (Single Roll-Pair)

—— DNS $T(x, z_*)$ —- 6-Mode Model $T(x, z_*)$



Discussion Item #3: Challenges of New Low-Order Modeling Approach

- Ultimately, aim to construct low-order models of “turbulent” porous medium convection (e.g. at $Ra = O(5000)$) – with goal of reproducing turbulent statistics. May be able to exploit (i) mode slaving and (ii) minimal flow units.

- **BUT**. . . *“Only in a very limited sense can coherent structures ‘explain’ the behavior of near-wall turbulent flows. [...] The goal of developing a quantitative theory of near-wall turbulence based on dynamical interaction of a small number of structures has not been attained, and is likely unattainable.”* (S. Pope, *Turbulent Flows*, p.323)

- **What if bounds are poor or don’t exist. . . ?** Possibility of introducing physically (or asymptotically) motivated constraints? (R. Kerswell)

“Turbulence” at $Ra = 7924$, $k = \pi$