# A perturbation approach to turbulence modeling

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## beginning of turbulence theory

* **Reynolds** – the velocity field in a turbulent flow should be decomposed into a *mean* and a *fluctuation* $\mathbf{u} = \mathbf{U} + \mathbf{u}'$ where $\mathbf{U} = \langle \mathbf{u} \rangle$ is some appropriate average (ensemble preferably).

* **goal** – predict the flow *statistics*; most simply, the mean flow $\mathbf{U}$.

* **closure problem** – the equation derived from Navier-Stokes for $\mathbf{U}$ depends on $\langle \mathbf{uu} \rangle$ through the quadratic nonlinearity (convective term) – a new unknown. We start from

$$\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \quad \nabla \cdot \mathbf{u} = 0$$

and obtain

$$\dot{\mathbf{U}} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P - \nabla \left\langle \mathbf{u}' \mathbf{u}' \right\rangle + \nu \nabla^2 \mathbf{U}$$

$$\dot{\mathbf{u}}' + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} + \mathbf{u}' \cdot \nabla \mathbf{u}' - \left\langle \mathbf{u}' \cdot \nabla \mathbf{u}' \right\rangle = -\nabla p' + \nu \nabla^2 \mathbf{u}'$$

*underdetermined* equations: true, but useless for prediction

Problem for Reynolds - he started with a good equation and derived two bad ones (so, unhappy reviewers).

Turbulence is described by unclosed *Reynolds-averaged Navier-Stokes* (RANS) equations. 'RANS models' seek to close these equations.

**A first step**: Boussinesq (1877) had already proposed

$$\langle u_i' u_j' \rangle = \nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

where $\nu_t$ is an **eddy viscosity** (flow property, $\nu_t / \nu \gg 1$)

How to find $\nu_t$ ?

## Prandtl's mixing length model

In the *boundary-layer approximation* ($\partial/\partial x \ll \partial/\partial y$):

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\left\langle u'v'\right\rangle + \nu\frac{\partial^2 U}{\partial^2 y}$$

Close this mean flow equation, ignoring fluctuation equation.

$$\left\langle u'v'\right\rangle = \nu_t\frac{\partial U}{\partial y} \qquad \nu_t = \mathsf{C}L^2\left|\frac{\partial U}{\partial y}\right|$$

where $L$ is a 'characteristic length:'

(distance to wall, width of shear layer, ...)

## limitations of the mixing length model

* 'Natural' choice for $L$ only exists for thin shear flows.

* The constant C is flow-dependent.

* We ignored the fluctuation equations entirely; consequently

* the turbulent viscosity was determined by mean flow properties, not by turbulence properties.

(Such modeling works because of self-similarity.)

## Harlow, Launder, Spaulding: the two-equation model

Recall the eddy viscosity hypothesis $\left\langle u_i' u_j' \right\rangle = \nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$.

**research program**:

* Express $\nu_t$ in terms of two *turbulence properties* equivalent to a length- and a time-scale.

* Derive *transport equations* for these properties from the fluctuation equation.

We consider only these equations, not the validity of the eddy viscosity as a representation of turbulence.

Turbulent fluctuations are the subject of

## G I Taylor's Statistical Theory of Turbulence

Turbulence is a random field, each realization of which is governed by the Navier-Stokes equations.

Suppose there is no mean flow. Attention is focused entirely on the fluctuations. Their simplest statistical property is the two-point correlation $\mathrm{U}(\mathbf{x},\mathbf{x}') = \langle \mathbf{u}(\mathbf{x})\mathbf{u}(\mathbf{x}')\rangle$.

kinematic simplifications:

1. homogeneity $\mathrm{U}(\mathbf{x},\mathbf{x}') = \mathrm{U}(\mathbf{x}-\mathbf{x}') = \mathrm{U}(\mathbf{r})$
2. Fourier transformation $\mathrm{U}(\underline{\kappa}) = \int d\mathbf{r}\ \mathrm{U}(\mathbf{r})\exp(i\underline{\kappa}\cdot\mathbf{r})$
3. isotropy: $\mathrm{U}(\underline{\kappa}) = U(\kappa)\mathrm{P}(\underline{\kappa})$ P takes care of the solenoidal condition

## analytical formulation

Fluctuations are characterized by the *energy spectrum*
$E(\kappa) = 2\pi\kappa^2 U(\kappa)$ total energy $= \int_0^\infty d\kappa\ E(\kappa)$

NSE imply an (unclosed) spectral evolution equation for homogeneous isotropic turbulence

$$\dot{E}(\kappa,t) = P(\kappa,t) - T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t)$$

1. $P(\kappa,t)$ is a forcing ('production') term
2. $T(\kappa,t)$ is energy transfer between different wavenumbers; it is a third-order moment satisfying

$$\int_0^\infty d\kappa\ T(\kappa,t) = 0$$

In view of energy conservation property, $T(\kappa) = \dfrac{\partial}{\partial\kappa}\mathcal{F}(\kappa)$ where $\mathcal{F}(0) = \mathcal{F}(\infty) = 0$. ($\mathcal{F}$ is the scale-to-scale *energy flux*). Then

$$\dot{E}(\kappa) = P(\kappa) - \frac{\partial}{\partial\kappa}\mathcal{F}(\kappa) - 2\nu\kappa^2 E(\kappa)$$

Two important moments are

$$\int_0^\infty d\kappa\ E(\kappa) = k \qquad \int_0^\infty d\kappa\ 2\nu\kappa^2 E(\kappa) = \epsilon$$

total energy and dissipation.

Note $k = \frac{1}{2}\langle \mathbf{u}' \cdot \mathbf{u}' \rangle$ and $\epsilon = \frac{1}{2}\nu\langle \nabla\mathbf{u}' : \nabla\mathbf{u}' \rangle$ are *single-point* moments.

To advance beyond kinematics, we need the Kolmogorov theory.

## highlights of the Kolmogorov theory

In steady-state forced isotropic turbulence,

$$\int_0^\infty d\kappa \; P(\kappa) = P = \epsilon$$

If forcing is concentrated at length scale $\ell$, then a universal spectrum $E(\kappa) = E(\kappa; \ell, \epsilon)$ exists.

For $\kappa \gg \ell^{-1}$, $E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$ (with a viscous cutoff at $\kappa_d \propto (\epsilon/\nu^3)^{1/4}$)

The Kolmogorov theory asserts that this spectrum is a 'universal equilibrium' (Batchelor): it is an attractor for the statistics of small scales of motion in *any* turbulent flow.

* Stress that it is an *attractor*, not necessarily a permanent feature.

* By characterizing the spectrum by two parameters $\ell$ and $\epsilon$, this theory supports the possibility of two-equation modeling.

## problem statement

Turbulent motion couples an infinite number of scales of motion.

Yet Kolmogorov theory suggests a description by a few parameters.

## modeling asks a theoretical question:

Can turbulence evolution be described by equations for these parameters alone?

Kolmogorov theory does not answer this question.

Let's try to find these equations by taking moments of

$$\dot{E}(\kappa) = P(\kappa) - \frac{\partial}{\partial \kappa}\mathcal{F}(\kappa) - 2\nu\kappa^2 E(\kappa)$$

The moments of order zero are

$$\begin{aligned}
\int_0^\infty d\kappa\ E(\kappa,t) &= k \text{ kinetic energy} \\
\int_0^\infty d\kappa\ P(\kappa,t) &= P \text{ production} \\
\int_0^\infty d\kappa\ T(\kappa,t) &= 0 \text{ (energy conservation)} \\
\int_0^\infty d\kappa\ 2\nu\kappa^2 E(\kappa,t) &= \epsilon \text{ dissipation}
\end{aligned}$$

They satisfy the energy balance $\dot{k} = P - \epsilon$ suggesting $k, \epsilon$ as natural variables. If so, we need an equation for dissipation rate $\epsilon$.

## towards a dissipation rate equation

Take another moment: the exact equation for $\epsilon$ is found from

$$\int_0^\infty d\kappa\ 2\nu\kappa^2\ \dot{E}(\kappa,t) = \int_0^\infty d\kappa\ 2\nu\kappa^2\ \left[P(\kappa,t) - T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t)\right]$$

Note

$$\begin{aligned}
\int_0^\infty d\kappa\ 2\nu\kappa^2 E(\kappa,t) &= \epsilon \text{ dissipation} \\
\int_0^\infty d\kappa\ 2\nu\kappa^2 P(\kappa,t) &\approx 0 \\
\int_0^\infty d\kappa\ 2\nu\kappa^2 T(\kappa,t) &= S \text{ vortex stretching} \\
\int_0^\infty d\kappa\ 4\nu^2\kappa^4 E(\kappa,t) &= G \text{ enstrophy destruction}
\end{aligned}$$

Therefore, $\dot{\epsilon} = S - G$ but $S, G$ are **new unknowns.**

## a basic principle about turbulence

At high Reynolds number, the large scales in a turbulent flow are independent of viscosity.

The $\kappa^{-5/3}$ range ends at the Kolmogorov scale: $\kappa_d^4 = \epsilon/\nu^3$. In the limit $\nu \longrightarrow 0$, $\kappa_d \longrightarrow \infty$ whilst $\epsilon$ is constant.

If $\nu \longrightarrow 0$ and $k_d \longrightarrow \infty$, turbulence creates more small scales, **but the large scales are unchanged**.

This property permits finite-dimensional descriptions of turbulence (models): the large scales and small scales know about each other only through the dissipation rate $\epsilon$.

**another formulation:**

Modeling requires all limits $\kappa_d \to \infty$ to be finite.

OR, At high $Re$, models are independent of $\kappa_d$ and $\nu$:
if $\kappa_d$ and $\nu$ are important, then we must consider the coupled dynamics of all scales, and modeling is impossible.

What about the dissipation rate equation?

**classic Tennekes-Lumley analysis**

* well, $G = \int_0^{\kappa_d} \nu^2 \kappa^4 E(\kappa) d\kappa \sim \nu^2 \kappa_d^{10/3} \sim \kappa_d^{2/3}$ **diverges with** $\kappa_d$**.**

* and $S = \int_0^{\kappa_d} \nu \kappa^2 T(\kappa) d\kappa \sim \kappa_d^{2/3}$ **diverges with** $\kappa_d$**.**

* Modeling requires extraordinary cancellation of divergences:
$\dot{\epsilon} = S - G \sim \kappa_d^0 \neq 0 \quad (*)$

* $S = G$ in a steady state (Batchelor's skewness relation),

* but what can justify $(*)$ in an unsteady problem?

**No theory answers this question**
(except 'our theory,' to be explained.)

## – remark –

$\kappa_d^{2/3} \sim \nu^{-1/2} \sim Re^{1/2}$ therefore, in a Kolmogorov steady state $S \sim Re^{1/2}$.

– Doering has shown that enstrophy production is at most $O(Re^3)$.

– The bound can be realized by a suitable initial condition.

– Steady state: enstrophy production is $O\left(\frac{1}{\nu} Re^{1/2}\right) = O(Re^{3/2})$.

Steady state turbulence organizes itself so that enstrophy production is much less than what the Navier-Stokes equations permit.

## basic consequence of the Tennekes-Lumley analysis

A dissipation rate equation cannot be derived from the fluctuation equations because the **nonzero** $O(Re^0)$ difference $S - G$ is not a definite moment of the velocity field.

The dissipation rate equation is necessarily 'phenomenological.'

all well-known in modeling community (Wilcox, *Turbulence modeling for CFD*)

Michael Leschziner's talk summarized the standard RANS approach:
from fluctuation equations

$$\dot{\mathbf{u}}' + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} + \mathbf{u}' \cdot \nabla \mathbf{u}' - \left\langle \mathbf{u}' \cdot \nabla \mathbf{u}' \right\rangle = -\nabla p' + \nu \nabla^2 \mathbf{u}'$$

we can obtain (exact) equations for any correlation of fluctuations.

Such equations contain new unknown correlations (closure problem).

RANS models express unknown in terms of known correlations.

This program fails for the dissipation rate equation, because the basic unknown is the $O(1)$ difference of two $O(Re^{1/2})$ correlations.

## continuum mechanics approach

choose descriptors, say $k$ and $\epsilon$;
make closure assumption, say $\dot{\epsilon} = f(k, \epsilon, P)$;
postulate simple plausible form, say $\dot{\epsilon} = \frac{\epsilon}{k}[C_{\epsilon 1} P - C_{\epsilon 2} \epsilon]$;
find 'model constants' $C_{\epsilon 1}$ and $C_{\epsilon 2}$ from measurements in *self-similar* flows (isotropic decay, homogeneous shear...) in which finite dimensional modeling is *a priori* possible.

underlying heuristic:

Turbulence is a 'fluid' with same 'properties' in all flows.

many difficulties: what insures model validity away from calibration cases? model validity in self-similar flows follows from self-similarity, not from the excellence of the modeling assumptions; too many self-similar cases exist for calibration by two constants (*e.g.* different decay laws), leading to 'constants' becoming functions, 'blending functions' to combine different models,....

In any case,

this 'continuum mechanics' paradigm abandons the connection to the fluctuation equations, one of the arguments for superiority of two-equation modeling to mixing length models.

## 'our' viewpoint

A turbulence model should be the demonstrably approximate solution of some 'equations of motion,' valid under some explicitly known conditions.

As 'equations of motion,' we suggest two-point closures of Kraichnan's DIA (direct interaction approximation) family: they are based on general statistical hypotheses and make no assumptions about
– Reynolds number
– kinematics (homogeneity, isotropy)
– Kolmogorov scaling
– self-similarity

## an example

Kraichnan, *Direct-interaction approximation for shear and thermally driven turbulence* (PF 1964) obtains the eddy diffusivity

$$\kappa_{ij}(\mathbf{x},t) = \int_0^t ds \int d\mathbf{y}\ G(\mathbf{x},\mathbf{y};t,s)U_{ij}(\mathbf{x},\mathbf{y};t,s)$$

for passive scalar advection.

$G$ is the *scalar response function* of the DIA theory, and $U_{ij}$ is the two-point two-time velocity correlation. DIA provides evolution equations for both quantities.

This idea has been developed further by Yoshizawa.

two-scale direct interaction approximation (TSDIA)

–introduce slow space and time scales $X = \epsilon x$, $T = \epsilon t$

–assume lowest order field is (locally) homogeneous isotropic turbulence

–corrections contain unknown RANS correlations

–expand them in powers of $\epsilon$

–express results using direct interaction approximation (DIA) descriptors of isotropic turbulence (response and correlation functions)

–find linear and nonlinear viscosity models, transport coefficients in MHD.... Nonlinearity enters through DIA; the comparable expansion by Chini would probably give linear rapid distortion theory for fluctuation dynamics.

However, such expressions require solving the DIA equations. We require something much simpler, closer to a single-point model.

A second anticipated use [of the direct interaction equations] is to suggest improved qualitative descriptions which could supplement the existing mixing-length approaches when strong inhomogeneity and anisotropy exist. One way to do this might be to assume simple forms for the covariance functions, with a few undetermined parameters, and then use the direct-interaction equations to fix the values of the parameters.

...Kraichnan (1964).

But how to actually do this?

current approach

## 'Hilbert expansion' for two-point closure

(S. L. Woodruff and R. Rubinstein, *Multiple-scale perturbation analysis of slowly evolving turbulence*, JFM (2006))

We treat the problem as finding approximate solutions of

$$\dot{E}(\kappa,t) = P(\kappa,t) - \frac{\partial}{\partial\kappa}\mathcal{F}[E(\kappa,t)] - 2\nu\kappa^2 E(\kappa,t)$$

where, for the Heisenberg model (provisionally elevated to 'truth'),

$$\mathcal{F}[E(\kappa)] = \mathsf{C}\underbrace{\int_0^\kappa d\mu\ \mu^2 E(\mu,t)}_{\text{squared strain at }\kappa}\underbrace{\int_\kappa^\infty dp\ E(p,t)\theta(p,t)}_{\text{turb. viscosity at }\kappa}$$

This model was chosen for 'analytical convenience;' it is a limit of DIA (Kraichnan). We use $\theta(p) = 1/\sqrt{p^3 E(p)}$.

Consider the steady equation

$$P(\kappa) - \frac{\partial}{\partial \kappa}\mathcal{F}[E(\kappa)] - 2\nu\kappa^2 E(\kappa) \quad (*) = 0$$

where production $P(\kappa)$ is characterized by total production $P = \epsilon$ and a forcing scale $L_P$: $P(\kappa) = P(\kappa; \epsilon, L_P)$.

According to Kolmogorov, $E(\kappa) = E(\kappa; \epsilon, L)$ with $L = L_P$, a two-parameter family of steady spectra.

'Inspired' (or perhaps misled) by kinetic theory, we allow $\epsilon$ and $L$ to be slowly varying functions of time, leading to a normal solution $E(\kappa, t) = E(\kappa; \epsilon(t), L(t))$ (time-dependence only through $\epsilon$ and $L$).

$E(\kappa; \epsilon(t), L(t))$ satisfies (*), but the unsteady equation contains error terms $\propto \dot{\epsilon}, \dot{L}$.

Adding a correction to $E(\kappa; \epsilon(t), L(t))$ to cancel these error terms generates a perturbation expansion.

Introduce the time-dependent perturbation

$$P(\kappa,\tau) = P_0(\kappa;\epsilon(\tau),L_P(\tau)) + \delta P_1(\kappa,\tau) \quad (\sharp)$$

with $P_0$ and $P_1$ of order one, $\tau = \delta t$, and $\delta$ a small parameter.

Corresponding to ($\sharp$), let

$$E(\kappa,t) = E_0(\kappa;\epsilon(\tau),L(\tau)) + \delta E_1(\kappa,\tau) + \ldots$$

Substituting in spectral evolution equation, we get

$$P_0 - \partial\mathcal{F}[E_0]/\partial\kappa - D_0 = 0$$

To lowest order, $E_0$ adjusts instantaneously to changes production. At the next order

$$\frac{\partial}{\partial \kappa}\mathcal{L}[E_1(\kappa,\tau)] = -\frac{\partial}{\partial \tau}E_0(\kappa,\epsilon(\tau),L(\tau)) + P_1(\kappa,\tau) - D_1(\kappa,\tau).$$

The linear operator $\mathcal{L} = (\delta\mathcal{F}/\delta E)|_{E_0}$ is the energy transfer linearized about $E_0$: compatibility condition(s) to solve for $E_1$ are of the form

$$\int_0^\infty d\kappa\ \Psi_i(\kappa)\frac{\partial}{\partial \tau}E_0(\kappa,\epsilon(\tau),L(\tau)) = \int_0^\infty d\kappa\ \Psi_i(\kappa)\left[P_1(\kappa,\tau) - D_1(\kappa,\tau)\right]$$

where $\Psi_i$ are solutions of the homogeneous adjoint equation $\mathcal{L}^\dagger\left[\frac{\partial \Psi}{\partial \kappa}\right] = 0$ and $\mathcal{L}^\dagger$ is the adjoint of $\mathcal{L}$.

One solution is immediately obvious: $\Psi_1 = 1$. It leads to an energy balance condition.

There proves to be exactly one more solution,

$$\Psi_2(\kappa) = \int_0^\kappa d\mu \, \frac{W(\mu)}{\beta(\mu)} \exp\left(-\int_0^\mu dp \; W(p)\right).$$

(please see paper for details).

Explicitly evaluating the compatibility equations, we obtain a two-equation model in the form

$$\epsilon^{-1/3}L^{2/3}I_i^1(\epsilon,L)\frac{\partial\epsilon}{\partial\tau}+(\epsilon L)^{2/3}I_i^2(\epsilon,L)\frac{\partial L}{\partial\tau}=\epsilon I_i^3(P,L_P)\quad i=1,2.$$

where

$$I_i^1(\epsilon,L)=\frac{2}{3}\int_0^\infty dp\ \Psi_i(p)\widehat{\phi}(pL),$$
$$I_i^2(\epsilon,L)=\int_0^\infty dp\ \Psi_i(p)\left(\frac{5}{3}\widehat{\phi}(pL)+(\kappa L)\widehat{\phi}'(pL)\right),$$
$$I_i^3(P,L_P)=\int_0^\infty dp\ \Psi_i(p)\widehat{p}_1(pL).$$

**This is the required two-equation model.** It describes the slow variation of the spectral parameters $\epsilon(\tau)$ and $L(\tau)$ due to slow changes of production through $P(\tau)$ and $L_P(\tau)$.

## demonstration of the Tennekes-Lumley balance

– Perturbations $E_1$ satisfy

$$\dot{E}_0(\kappa,t) = P_1(\kappa,t) - \frac{\partial}{\partial\kappa}\mathcal{L}[E_1(\kappa,t)] - 2\nu\kappa^2 E_1(\kappa,t)$$

where $\mathcal{L}$ is the energy transfer linearized about the steady state.

– If the solution exists, then for scaling,

$$\dot{\epsilon}\epsilon^{-1/3}\kappa^{-5/3} \sim \epsilon^{1/3}\kappa^{-4/3}\kappa^2 E_1$$

therefore

$$E_1 \sim \underbrace{\left[\frac{\dot{\epsilon}}{\epsilon}\frac{1}{\epsilon^{1/3}\kappa^{2/3}}\right]}_{\text{'Knudsen number'}} \epsilon^{2/3}\kappa^{-5/3}$$

thus, $\delta E \sim \kappa^{-7/3}$ (compare Yoshizawa).

– $S = S_0 + S_1$ and $G = G_0 + G_1$ where $S_1, G_1$ come from $E_1$.

– Since $E_0$ is quasi-static, $S_0 = G_0$ : cancellation of leading order divergence $\sim \kappa_d^{2/3}$.

– The scaling $E_1 \sim \kappa^{-7/3}$ implies $S_1, G_1 \sim \kappa_d^0$.

– Therefore, $\dot{\epsilon} = (S_0 - G_0) + (S_1 - G_1) = S_1 - G_1$ is (1) nonzero and (2) finite in limit $\kappa_d \to \infty$.

## remarks

The expansion requires that $\dot{\epsilon}/\epsilon$ and $\dot{L}/L$ be small compared to $\epsilon^{1/3}\kappa^{2/3}$ (ratios are 'Knudsen numbers'). These are the conditions under which the model is valid.

We would like to know whether the number of parameters in the 'normal solution' equals the number of compabitility conditions.

Consider the pair of adjoint equations

$$\frac{\partial}{\partial\kappa}\left(\frac{\delta\mathcal{F}}{\delta E}\right)_{E_0}[E_1] = \frac{\partial}{\partial\kappa}\mathcal{L}[E_1] = 0 \qquad \left(\frac{\delta\mathcal{F}}{\delta E}\right)^{\dagger}_{E_0}\left[\frac{\partial\Psi}{\partial\kappa}\right] = \mathcal{L}^{\dagger}_{E_0}\left[\frac{\partial\Psi}{\partial\kappa}\right] = 0$$

The first has only one solution (perturb the inertial range flux). Because $\mathcal{L}$ is (1) *not self-adjoint* and (2) *infinite dimensional*, there can be more than one solution of the second.

row $\neq$ column rank possible in function spaces; difference is the *index* of $\mathcal{L}$

Example: for the *Kovaznay model* in which $\mathcal{F} = \eta(\kappa)E(\kappa)$ with $\eta = \sqrt{\kappa^3 E}$, there is only one compatibility condition (index zero); no two-equation model.

We think that this implies that $L$ and $\epsilon$ cannot vary independently in slow evolution under this model (Steve Woodruff).

**two questions**

What property of a closure permits a two-equation model?
How strongly does this model depend on the closure?

## kinetic theory analogies

| | |
|---|---|
| distribution function | energy spectrum |
| equilibrium Maxwellian | steady state Kolmogorov spectrum |
| Boltzmann equation | DIA-based closure |
| Hilbert expansion | multiple-scale perturbation theory |
| Navier-Stokes equations | 'our' model equations |

## difficulties

kinetic theory: Gaussian equilibrium, moments of all orders exist
present: power law, divergence of moments

kinetic theory: self-adjoint linearized collision operator; number of compatibility equations equals number of conserved moments
present: linearized transfer is not self-adjoint; the number of compatibility equations need not equal the number of degrees of freedom of the spectrum

kinetic theory: local Maxwellian is a 'universal' state
present: an infinite number of self-similar states exists

## some possible generalizations

In self-similar decay, $A$ in $E(\kappa) \sim A\kappa^2$ for $\kappa \approx 0$ is a constant.

This fixes time-dependence of moments ($k \sim A^{2/5} t^{-6/5}$).

More fundamentally, $E_0(\kappa, t) = e(A\kappa^5 t^2)$ is a similarity solution of (any) spectral evolution equation.

**What if $A = A(\tau)$ is slowly varying in time?** (for example, due to a small production term).

**same story?**

$A(\tau)$ introduces error terms; adding a correction term $E_1$ generates a perturbation expansion.

$E_1$ satisfies a linear integral equation determined by the transfer linearized about $E_0$.

Compatibility condition to solve this equation give equation of motion for $A(\tau)$.

**But can we do something more useful?**:

Basic state: a growing self-similar boundary layer described by $U_\infty$ and $\tau_w$.

Derive equations for response of $\tau_w$ to slow changes of $U_\infty$?

**stress transport model?**

It would require a basic seven-parameter state described by $\ell$ and (arbitrary) $\langle u_i u_j \rangle$, say

$$U_{ij} = U_{ij}(\ell, \langle \mathbf{uu} \rangle)$$

Given a tractable anisotropic closure, could we derive equations for slowly varying

$$U_{ij}(\mathbf{x}, t) = U_{ij}(\ell(\mathbf{x}, t), \langle \mathbf{uu} \rangle(\mathbf{x}, t))$$

## limits of models

Homogeneous isotropic turbulence with strong transient effects: time-dependent forcing (Rubinstein, Clark, Livescu, Luo JoT 2004); periodically forced turbulence (Bos, Clark, Rubinstein, PoF 2007)

The energy spectrum may not admit description by finitely many parameters:
–'unbalanced vortex stretching' (Speziale and Bernard)
– consequent failure of Tennekes-Lumley balance
– no alternative to complete spectral evolution equation

# CONCLUSIONS

Finite dimensional modeling can be justified in some cases.

**Open questions:**

–Can we 'control' or even predict the number of model equations? (index of a linear operator)

–How strongly do the models depend on the underlying closure theory?

–How to treat more meaningful problems? (requires an analytically tractable inhomogeneous closure theory)

**Further research is needed!**