Boundary effect in the Euler Limit

Joint work in progress with François Golse and Lionel Paillard

ν is the "human size " viscosity so it is in fact the inverse of the Reynold number

$$\mathcal{R} = \frac{UL}{\nu_{\text{physical}}}$$

I will consider solenoidal solutions of Euler and Navier-Stokes with the pressure as a Lagrange multiplier (called "incompressibles")

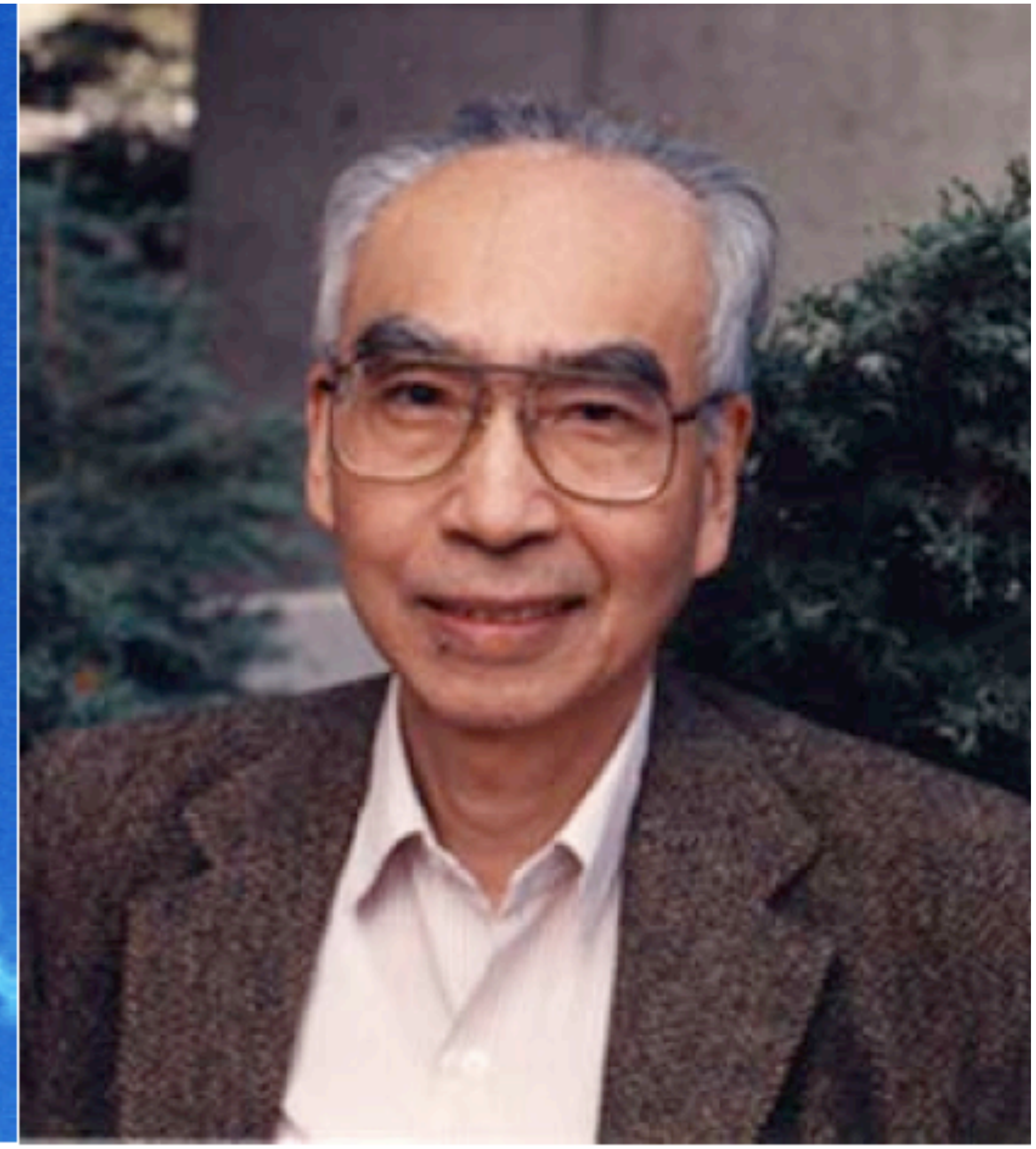
Convergence <u>in presence of boundary effects</u> of solutions of Navier-Stokes or of solutions of the Boltzmann equation to solutions of the Euler equation?

It is a very open problem.

There is no construction of a sequence of solution that (say in a weak sense) do not converge to solution of the Euler equation...But there are good reason to believe that this is the general situation.

- If the air around a wing is a solenoidal with no vorticity solution of the Euler equation there is no force on the wing and plane or birds cannot fly the d'Alembert paradoxe.

- The common hypothesis about turbulence is that the high Reynold limit of the energy dissipation is strictly positive...It this in the issue of the boundary effect (production of vorticity at the boundary) that this appears the most clearly in particular with a theorem of Kato (below).

Therefore it seems interesting to exhibit the similarities between the Navier-Stokes and the Boltzmann limit.

An issue different from the question of the existence of large time smooth solutions of the Euler equation.

Dissipative solution is a good tool to underline this difference.

The notion of dissipative solution for the Euler Equation

$$S(w)=\frac{1}{2}(\nabla w+(\nabla w)^t)\,,\quad \partial_t w+P(w\cdot\nabla w)=E(x,t)=E(w)$$

with P denoting the projection:

$$P(w\cdot\nabla w)=w\cdot\nabla w+\nabla q$$

$$\text{in } \Omega -\Delta q=\sum_{ij}\partial_{x_i}w\partial_{x_j}w\,;\ \text{On } \partial\Omega \quad \frac{\partial q}{\partial\vec n}=w\cdot\nabla w\cdot\vec n=-\sum_{i,j}w_iw_j\partial_{x_j}\vec n_i\,.$$

Assume that u is a smooth solution:

$$\partial_t u+\nabla\cdot(u\otimes u)+\nabla p=0\,,\nabla u=0\,,u\cdot\vec n=0\,,$$
$$\partial_t w+w\cdot\nabla w+\nabla q=E$$

one deduces the formula:

$$\frac{1}{2}\partial_t\int_\Omega|u(x,t)-w(x,t)|^2+\int(u(x,t)-w(x,t)S(w)u(x,t)-w(x,t))dx$$
$$=\int(E(x,t),u(x,t)-w(x,t))dx\,.$$

Hence the definition of a dissipative solution as a divergence free tangent to the boundary vector field which for any test function w as introduced above satisfies the relation:

$$|u(x,t)-w(x,t)|^2_{L^2(\Omega)} \le \int_0^t e^{\int_0^t 2|S(w(s))|_{L^\infty(\Omega)}ds}|u(x,0)-w(x,0)|^2_{L^2(\Omega)}$$
$$+2\int_0^t e^{\int_s^t 2|S(w(\tau))|_{L^\infty(\Omega)}d\tau}(E(x,s),u(x,s)-w(x,s))dxds,.$$

Oberve that if u is a dissipative solution and w a smooth solution one has

$$|u(x,t)-w(x,t)|^2_{L^2(\Omega)} \leq \int_0^t e^{\int_0^t 2|S(w(s))|_{L^\infty(\Omega)}ds}|u(x,0)-w(x,0)|^2_{L^2(\Omega)}$$

Hence the stability of dissipative solutions with respect to smooth solutions and in particular the fact that whenever exists a smooth solution $u(x,t)$ any dissipative solution which satisfies $w(.,0) = u(.,0)$ coincides with u for all time.

However it is important to notice that to obtain this property one needs to include in the class of test functions w vector field that may have non 0 tangent to the boundary component.

Navier-Stokes equation with boundary condition

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0 \quad \text{in } \Omega \tag{1}$$

$$u_\nu \cdot \vec{n} = 0\,, \nu(\partial_{\vec{n}} u_\nu + C(x) u_\nu)_\tau + \lambda u_\nu = 0 \ \text{ on } \partial\Omega \tag{2}$$

$$\lambda(\nu, x) \geq 0! \ \ C(x) \in C(\mathbb{R}^n \mapsto \mathbb{R}^n) \tag{3}$$

- $\lambda = \infty \Leftrightarrow$ Dirichlet, $(C = 0, \lambda = 0) \Rightarrow u_\nu \cdot \vec{n} = 0$ and $(\partial_{\vec{n}} u_\nu)_\tau = 0$.

- With $S(u_\nu) = \frac{1}{2}(\nabla u_\nu + \nabla^t u_\nu)$ and $u_\nu \cdot \vec{n} = 0$ other similar conditions:

$$(S(u_\nu) \cdot \vec{n})_\tau = (\partial_{\vec{n}} u_\nu)_\tau - (\nabla^t \vec{n} \cdot u_\nu)_\tau \Rightarrow \nu (S(u_\nu) \cdot \vec{n})_\tau + \lambda u_\nu = 0\,,$$

$$(\nabla \wedge u_\nu) \wedge \vec{n} = (\partial_{\vec{n}} u_\nu)_\tau + (\nabla^t \vec{n} \cdot u_\nu)_\tau \Rightarrow \nu (\nabla \wedge u_\nu) \wedge \vec{n} + \lambda u_\nu = 0\,.$$

Trace theorem and energy estimate

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u_{\nu}(x,t)|^2dx + \nu\int_{\Omega}|\nabla u_{\nu}|^2dx + \int_{\partial\Omega}\lambda(x)|u_{\nu}(x,t)|^2d\sigma$$

$$= \nu\int_{\partial\Omega}C(u_{\nu})_{\tau}u_{\nu}d\sigma$$

$$\int_{\Omega}|u_{\nu}(x,t)|^2dx \le e^{C\nu t}\int_{\Omega}|u_0(x)|^2dx$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|u_{\nu}(x,t)|^2dx + \nu\int_{\Omega}|\nabla u_{\nu}|^2dx + \int_{\partial\Omega}\lambda(x)|u_{\nu}(x,t)|^2d\sigma$$

$$= \nu\int_{\partial\Omega}C(u_{\nu})_{\tau}u_{\nu}d\sigma \to 0\,.$$

An easy theorem

Given initial data $u_\nu(x,0) = u_0(x)$ with:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu\int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x)|u_\nu(x,t)|^2 d\sigma \to 0$$

If u_ν converges weakly to a function $u(x,t)$ which for $0 < t < T$ is a smooth solution of the Euler equation it converges strongly and at the limit no dissipation of energy!:

$$\lim_{\nu\to 0}\int_0^T (\nu\int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(\nu)|u_\nu(x,t)|^2 d\sigma)dt = 0\,.$$

Convergence of the solution of Navier Stokes to a Dissipative solution.

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu\Delta u_\nu + \nabla p = 0$$
$$\partial_t w - \nu\Delta w + w \cdot \nabla w + \nabla q = E(w) - \nu\Delta w$$
$$\frac{1}{2}\frac{d}{dt}|u(x,t) - w(x,t)|^2_{L^2(\Omega)} + \nu|\nabla(u(x,t) - w(x,t))|^2_{L^2(\Omega)}$$
$$\leq |(S(w) : (u_\nu - w) \otimes (u_\nu - w))| + |(E(w) - \nu\Delta w, u_\nu - w)|$$
$$+(R(\nu) = \nu(\partial_{\vec{n}} u_\nu - \partial_{\vec{n}} w, u_\nu - w)_{L^2(\partial\Omega)})$$

With no boundary convergence (modulo subsequence is always true). No hypothesis on the existence of a solution of Euler...Even with very bad (De Lellis Szekelyhidi) initial data . *With boundary*

For Dirichlet $R(\nu) = \nu(\partial_{\vec{n}} u_\nu, w)_{L^2(\partial\Omega)} + o(\nu)$

Otherwise $R(\nu) = -(\lambda(\nu)u_\nu, u_\nu)_{L^2(\partial\Omega)} + (\lambda(\nu)u_\nu, w)_{L^2(\partial\Omega)} + o(\nu)$

Theorem Sufficient conditions for convergence to a dissipative solution: $\lambda(\nu) \to 0$

Other cases $u_\nu = 0$ on $\partial\Omega$:or $\liminf \lambda(\nu) > 0$

Theorem For “Leray solutions of Navier-Stokes with boundary" one has the following facts:

$$1 \quad \text{For Dirichlet} \lim_{\nu\to 0} \nu \frac{\partial u_\nu}{\partial \vec{n}} = 0 \text{ otherwise } \lim_{\nu\to 0} \lambda(\nu) u_\nu = 0 \tag{4}$$

in $\mathcal{D}'(]0,T[\times\partial\Omega)$ implies that this subsequence converge to a “dissipative " solution.

2 For a subsequence u_ν the local (near boundary) control of the energy dissipation:

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dxdt = 0 \tag{5}$$

implies the relation (4) hence also the convergence to a dissipative solution.

Moreover if the Euler equation with the same initial, has for $0 < t < T$, a smooth solution $u(x,t)$, then this convergence is equivalent to the following facts are equivalent.

$$
\begin{array}{ll}
(i) & u_\nu(t) \to u(t) \text{ in } L^2(\Omega) \text{ uniformly in } t \in [0,T] \\
(ii) & u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t \in [0,T] \\
(iii) & \lim\limits_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dxdt = 0 \\
(iv) & \lim\limits_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dxdt = 0\,.
\end{array}
$$

- Validity of a Prandlt boundary layer anaylisis$\Rightarrow$ above criteria

- In general super open problem... The most realistic case where dissipation of energy may appear!!!!

Extension of these results to the Boltzmann equation:

Issue: Microscopic effects of the boundary generate or do not generate turbulence ! Similar to macroscopic effects!

$F_\epsilon(x,v,t)$ solution in $\Omega \times \mathbb{R}^n_v$ of the (rescaled in time) Boltzmann equation:

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\mathrm{Knudsen}(\epsilon)} \mathcal{B}(F_\epsilon, F_\epsilon)$$

with Maxwell Boundary Condition for $v \cdot \vec{n} < 0$ in term of $v \cdot \vec{n} > 0$

$$F_\epsilon^-(x,v) = (1-\alpha(\epsilon))F_\epsilon^+(x,v^*) + \alpha(\epsilon) M(v) \sqrt{2\pi} \int_{v\cdot\vec{n}<0} |v\cdot\vec{n}| F_\epsilon^+(x,v)dv$$

$$0 \le \alpha(\epsilon) \le 1\,, v^* = v - 2(v\cdot\vec{n})\vec{n} = \mathcal{R}(v)\,,$$

$$M(v) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|v|^2}{2}} \quad \Lambda(\phi) = \sqrt{2\pi} \int_{\mathbb{R}^n_v} (v\cdot\vec{n})_+ \phi(v) M(v) dv.$$

$$\Lambda(1) = 1(\text{proba!}) \quad F_\epsilon^-(x,v) = (1-\alpha(\epsilon))F_\epsilon^+(x,\mathcal{R}(v)) + \alpha(\epsilon)\Lambda(\frac{F_\epsilon}{M})$$

Analysis Theorem

- Existence of the weak solutions of Navier-Stokes

- Existence of Renormalized solutions of the Boltzmann equation (in the whole space Di Perna Lions, With Maxwell Boundary conditions Mischler 2000.

- Scaling and convergence. Law Mach number configurations are described by “incompressible " Navier-Stokes or Euler equation this leads to the following rescaled Boltzmann equation.

$$\epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}}\mathcal{B}(F_\epsilon, F_\epsilon)$$

$$F_\epsilon^-(x,v) = (1-\alpha(\epsilon))F_\epsilon^+(x,\mathcal{R}(v)) + \alpha(\epsilon)\Lambda(\frac{F_\epsilon}{M})$$

$$F_\epsilon = G_\epsilon M(v) = (1+\epsilon g_\epsilon)M(v)$$

- For $q = 0$, $u_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}^n_v} vF_\epsilon dv$ converges to a Leray solution of Navier-Stokes with the boundary condition:

$$u\cdot\vec{n} = 0 \quad \text{and} \quad \nu((\nabla u + \nabla^t u)\cdot n)_\tau + \lambda u = 0$$
$$\lambda = \lim_{\epsilon\rightarrow 0}\frac{\alpha(\epsilon)}{\epsilon} \quad \text{Dirichlet} \Leftrightarrow \lim_{\epsilon\rightarrow 0}\frac{\alpha(\epsilon)}{\epsilon} = \infty\,.$$

With no boundary: Formal proof B. Golse Levermore (1991), Complete results with Di Perna Lions solution Golse Saint Raymond (2009).

With boundary effect Aoki, Inamuro, Onishi (1979) Stationary solution linearized regime and Hilbert expansion; Masmoudi Saint Raymond for Mischler solutions towards Leray solutions. General formal proof B. Golse Paillard.

Natural hypothesis to ensure the convergence to Euler

$$\lim_{\nu\rightarrow 0}\lambda(\nu) = 0 \Leftrightarrow \lim_{\epsilon\rightarrow 0}\frac{\alpha(\epsilon)}{\epsilon} = 0\,.$$

The Euler limit

Theorem (In a periodic box $\mathbf{T}^3$ Saint-Raymond (2003)) Let F_ϵ be a family of renormalized solutions in $\Omega\mathbb{R}^n_v$ of the Boltzmann equation:

$$\epsilon\partial_t F_\epsilon + v\nabla_x F_\epsilon = \frac{1}{\epsilon^q}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad q > 1$$

with initial data

$$F_\epsilon(x,v,0) = M_{1,\epsilon u^{in},1} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v-\epsilon u^{in}|^2}{2}}\,,\ \ \nabla\cdot u^{in}(x) = 0.$$

Then the family $(\frac{1}{\epsilon}\int vF_\epsilon dv)$ is relatively compact in $w-L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^3))$ and each of its limit points is a dissipative solution of the $3d$ Euler equation.

Theorem $\Omega \times \mathbb{R}^3_v$; F_ϵ a family of renormalized solutions BE+BC:

$$\epsilon\partial_t F_\epsilon + v\nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad q > 0$$

$$F_\epsilon(x,v,0) = M_{1,\epsilon u^{in},1} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v-\epsilon u^{in}|^2}{2}}\,,$$

$$\nabla_x \cdot u^{in}(x) = 0\,,\; u^{in} \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

$$F_\epsilon^-(x,v,t) = (1-\alpha(\epsilon))F_\epsilon^+(x,\mathcal{R}v,t) + \alpha(\epsilon)M(v)\Lambda(\frac{F_\epsilon}{M(v)})$$

and the local conservation of moments (for simplicity, could be removed)

$$\epsilon\frac{\partial}{dt}\int_{\mathbb{R}^3_v} vF_\epsilon(x,v,t)dv + \nabla_x \int_{\mathbb{R}^3_v} v\otimes vF_\epsilon(x,v,t)dv = 0$$

then under the hypothesis $\lim_{\epsilon\to 0}\frac{\alpha(\epsilon)}{\epsilon} = 0$ the family $(\frac{1}{\epsilon}\int vF_\epsilon dv)$ is relatively compact in $w - L^\infty(\mathbf{R}_+; L^1(\Omega))$. Any limit point a dissipative solution of the Euler equation in $\Omega \times \mathbb{R}_t^+$.

The proof of Laure uses equation (or inequation) for the moment of the conserved quantities plus the dissipation of entropy.

Here we focus on the terms coming from the boundary.

Entropies and entropies estimates

The following functions (and their Legendre transforms) are used below (and where also present in the ϵ independent results):

$$h(z) = (1+z)\log(1+z) - z$$

$$\text{with} \quad h^*(p) = \sup_z (zp - h(z)) = e^p - p - 1$$

$$l(z) = h(z+z_0) - h(z_0) - h'(z_0)z$$

$$\text{with} \quad l^*(p) = \sup_z (zp - l(z)) = (1+z_0)(e^p - p - 1)$$

$$H(F|G) = \int_\Omega \int_{\mathbb{R}^n_v} h(F|G)\,dx\,dv$$

The local entropy dissipation:

$$\partial_t \int_{\mathbb{R}^n_v} (F_\epsilon \log F_\epsilon - F_\epsilon + M + \frac{|v|^2}{2} F_\epsilon) dv$$
$$+ \nabla_x \int_{\mathbb{R}^n_v} v (F_\epsilon \log F_\epsilon - F_\epsilon + M + \frac{|v|^2}{2} F_\epsilon) dv$$
$$- \int_v \mathcal{B}(F_\epsilon, F_\epsilon) \log(F_\epsilon) dv = 0 \,.$$

For renormalized solutions:

$$\frac{1}{\epsilon^2} \frac{d}{dt} H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+3}} \int_\Omega \int_{\mathbb{R}^3_v} DE(F_\epsilon) dv dv_1 d\sigma + \frac{1}{\epsilon^3} \int_{\partial\Omega} \mathrm{DG} \leq 0$$

$$\mathrm{DE}(F)(v, v_1, \sigma) = \frac{1}{4} (F' F'_1 - F F_1) \log(F' F'_1 - F F_1) b(|v - v_1|, \sigma)$$

$$\mathrm{DG}(F) = \int_{\mathbb{R}^3_v} v \cdot \vec{n} H(F_\epsilon | M) d\sigma dv$$

The Darrozes-Guiraud local entropy

$$\sqrt{2\pi}\mathrm{DG} = \int_{\mathbb{R}^3_v} v\cdot\vec{n}H(F_\epsilon|M)d\sigma dv =$$

$$\sqrt{2\pi}\int_{\mathbb{R}^3_v} v\cdot\vec{n}H(M(1+\epsilon g_\epsilon)|M)dv = \sqrt{2\pi}\int_{\mathbb{R}^3_v} v\cdot\vec{n}M(v)h(1+\epsilon g_\epsilon)dv$$

$$= \sqrt{2\pi}\int_{\mathbb{R}^3_v}(v\cdot\vec{n})_+M(v)h(\epsilon g_\epsilon(v))dv - \sqrt{2\pi}\int_{\mathbb{R}^3_v}(v\cdot\vec{n})_+M(v)h(\epsilon g_\epsilon(\mathcal{R}v))d$$

$$= \Lambda(h(\epsilon g_\epsilon)) - \Lambda(h[(1-\alpha(\epsilon))\epsilon g_\epsilon + \alpha(\epsilon)\Lambda(\epsilon g_\epsilon)])$$

$$\geq \alpha(\epsilon)\left[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\right] \geq 0$$

Hence the final entropy estimate:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+3}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$

$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))]d\sigma \leq 0\,.$$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+3}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma \leq 0$$
$$\frac{1}{\epsilon^2}H(F_\epsilon(0)|M) = \frac{1}{2}\int |u_{\text{in}}(x)|^2dx$$

Theorem (simple) If $\frac{1}{\epsilon}\int_{\mathbb{R}^3_v} vF_\epsilon(x,v,t)dv$ converges weakly to function $u(x,t)$ which is for (for $0 < t < T$) a C^1 solution of the Euler equation the entropy dissipation and the Darrozes Guiraud entropy converge to 0 :

$$\lim_{\epsilon\to 0}\int_0^T(\frac{1}{\epsilon^{q+3}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma)dt = 0$$

Returning to $\frac{\alpha(\epsilon)}{\epsilon} \to 0$

$$\frac{1}{\epsilon^2} H(M_{(1,\epsilon u_0,1)}|M_{(1,\epsilon w,1)}) = \frac{1}{2}\int_\Omega |u_{in} - w(x,0)|^2 dx$$

$$\frac{1}{\epsilon^2} H(F_\epsilon|M_{(1,\epsilon w,1)})(t) = \frac{1}{\epsilon^2} H(F_\epsilon|M)(t)$$
$$+ \iint (\frac{w^2}{2} - \frac{v}{\epsilon} w) F_\epsilon(t,x,v) dxdv$$

$$\frac{1}{2\epsilon^2}\frac{d}{dt}\int_\Omega\int F_\epsilon(t,x,v)(\epsilon^2 w^2 - 2\epsilon v\cdot w) dxdv$$
$$= \int_\Omega\int \partial_t w \cdot (w - \frac{1}{\epsilon} v) F_\epsilon(t,x,v) dxdv$$
$$+ \int_\Omega \left(\frac{w^2}{2}\partial_t \int F_\epsilon(t,x,v) dv - \frac{w}{\epsilon} \cdot \int \partial_t F_\epsilon(t,x,v) v dv \right) dx\,.$$

For $\partial_t \int F_\epsilon(t,x,v)dv$ and $\partial_t \int F_\epsilon(t,x,v)vdv$ use the local conservation laws : For the first term:

$$\int_\Omega \frac{1}{2}w^2\partial_t \int F_\epsilon(t,x,v)dx = -\frac{1}{\epsilon}\int_\Omega \frac{1}{2}w^2\nabla_x \cdot \int vF_\epsilon(t,x,v)vdvdx$$

$$= \frac{1}{\epsilon}\int_\Omega\int (v\cdot\nabla_x w)\cdot wF_\epsilon(t,x,v)dvdx$$

$$-\frac{1}{\epsilon}\int_{\partial\Omega} d\sigma\frac{1}{2}w^2\int v\cdot\vec{n}F_\epsilon(t,x,v)dv =$$

$$= \int_\Omega\int \frac{1}{\epsilon}(v\cdot\nabla_x w)\cdot wF_\epsilon(t,x,v)dvdx\,.$$

For the second term:

$$-\int_\Omega \frac{w}{\epsilon} \cdot \int \partial_t F_\epsilon(t,x,v) v dv = \int_\Omega \int_{\mathbb{R}^3} \frac{w}{\epsilon^2} \cdot \int \nabla_x F_\epsilon(t,x,v) v \otimes v dv =$$
$$-\frac{1}{\epsilon^2} \int_\Omega \int (v \cdot \nabla_x) w \cdot v F_\epsilon(t,x,v) dv dx + \int_{\partial\Omega} \frac{1}{\epsilon^2} \int F_\epsilon(t,x,v)(w \cdot v)(\vec{n} \cdot v) dv d$$

Since w is tangent to the boundary one has for $x \in \partial\Omega$:

$$\frac{1}{\epsilon^2} \int F_\epsilon(t,x,v)(w \cdot v)(\vec{n} \cdot v) =$$
$$\frac{\alpha(\epsilon)}{\epsilon^2} \int F_\epsilon(t,x,v)(w \cdot v)(\vec{n} \cdot v)_+ dv =$$
$$\frac{1}{\sqrt{2\pi}} \frac{\alpha(\epsilon)}{\epsilon^2} \Lambda(\epsilon g_\epsilon(x,v,t)(w \cdot v)) .$$

Therefore one obtains:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) +$$
$$\frac{1}{\epsilon^{3+q}}\mathrm{DE}(F_\epsilon) + \frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^3}\int_{\partial\Omega}\Big[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\Big]d\sigma$$
$$\leq \int_\Omega\int(\partial_t w + w\cdot\nabla w)(w - \frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv -$$
$$\int_\Omega\int(w - \frac{v}{\epsilon})\nabla_x w(w - \frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv$$
$$+\frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^2}\int_{\partial\Omega}\Lambda(\epsilon g_\epsilon(x,v,t)(w\cdot v))d\sigma\,.$$

The exotic terms coming from the boundary are

$$\text{Good} \quad \frac{1}{\sqrt{2\pi}} \frac{\alpha(\epsilon)}{\epsilon^3} \int_{\partial\Omega} \left[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))) \right] d\sigma$$

$$\text{Bad} \quad \frac{1}{\sqrt{2\pi}} \frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(x, v, t)(w \cdot v)) d\sigma \, .$$

The bad has to be balanced by the good.

Proposition

$$\forall \eta > 0$$

$$\int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v)))d\sigma \leq (\frac{1}{\eta} + \frac{\eta C(w)}{\epsilon}) \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda g_\epsilon) d\sigma$$
$$+ C_2 \eta \int_{\partial\Omega}\int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With $\eta = 2\epsilon$

$$\frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v)))d\sigma$$
$$\leq (1 + 2\epsilon\alpha(\epsilon)C(w))\frac{\alpha(\epsilon)}{2\epsilon^3} \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon\Lambda g_\epsilon)d\sigma$$
$$+ C_2 \frac{\alpha(\epsilon)}{\epsilon} \int_{\partial\Omega}\int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With $\frac{\alpha(\epsilon)}{\epsilon} \longrightarrow 0$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) \leq \int_\Omega \int (\partial_t w + w \cdot \nabla w)(w - \frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv$$
$$- \int_\Omega \int (w - \frac{v}{\epsilon})\nabla_x w(w - \frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv + o(\epsilon)$$

Then (cf. Saint Raymond) for

$$u = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}\int_{\mathbb{R}^3_v} vF_\epsilon(x,v,t)dv$$

$$\begin{aligned}\frac{1}{2}\partial_t \int_\Omega |u(x,t) - w(x,t)|^2 \ &+ \ \int (u(x,t) - w(x,t)S(w)u(x,t) - w(x,t))dx \\ &\leq \ \int (E(x,t), u(x,t) - w(x,t))dx \,.\end{aligned}$$

Proof of the Proposition 2 steps

- Symmetry: $\Lambda(\Lambda(g_\epsilon)(w \cdot v)) = 0$ and Legendre duality between

$$l(\epsilon g_\epsilon - \Lambda(\epsilon g_\epsilon))) = h((\epsilon g_\epsilon - \Lambda(\epsilon g_\epsilon)) + \Lambda(\epsilon g_\epsilon)) - h(\Lambda(\epsilon g_\epsilon))$$
$$-h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon - \Lambda(\epsilon g_\epsilon))$$

$$l^*(p) = (1 + \Lambda(\epsilon g_\epsilon))(e^p - p - 1)$$

$$(\epsilon g_\epsilon(t,x,v)-\Lambda(\epsilon g_\epsilon))(w\cdot v)) = \frac{1}{\eta}(\epsilon g_\epsilon(t,x,v)-\Lambda(\epsilon g_\epsilon))(\eta w\cdot v))$$

$$\le \frac{1}{\eta}\Big(h((\epsilon g_\epsilon - \Lambda(\epsilon g_\epsilon)) + \Lambda(\epsilon g_\epsilon)) - h(\Lambda(\epsilon g_\epsilon)) - h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon - \Lambda(\epsilon g_\epsilon))\Big)$$

$$+(1+\Lambda(\epsilon g_\epsilon))\frac{(e^{\eta|w||v|}-\eta|w||v|-1)}{\eta}$$

$$\Lambda(h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon - \Lambda(\epsilon g_\epsilon))) = 0 \quad \text{Proba!}$$

$$\Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v))) \le \frac{1}{\eta}(\Lambda(h(\epsilon g_\epsilon)) - h((\Lambda(\epsilon g_\epsilon)) + \eta C(w)(1+\Lambda(\epsilon g_\epsilon))$$

- Step 2

$$\int_{\partial\Omega}(1+\Lambda(\epsilon g_\epsilon)d\sigma$$
$$\leq C_1\int_{\partial\Omega}\Lambda(h(\epsilon g_\epsilon)-h(\epsilon\Lambda g_\epsilon)d\sigma + C_2\int_{\partial\Omega}\int_{\mathbb{R}^3}F_\epsilon(v\cdot\vec{n}_x)^2 dv d\sigma$$

Proof With $G_\epsilon = F_\epsilon/M$ and $c = \int(v\cdot\vec{n})_+^2 \wedge 1 M dv$

$$c\int_{\partial\Omega}(1+\Lambda(\epsilon g_\epsilon)d\sigma = \int_{\partial\Omega}\Lambda(G_\epsilon)\int(v\cdot\vec{n})_+^2\wedge 1 dv M(v)d\sigma_x$$
$$= I_1 + I_2$$
$$\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|>\beta}(v\cdot\vec{n})_+^2\wedge 1 M(v)d\sigma_x dv$$
$$+$$
$$\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|\leq\beta}(v\cdot\vec{n})^2\wedge 1 M(v)d\sigma_x dv$$

$h(z) = (z+1)\log(z+1) - z\,,\, h(z) \geq h(|z|)$ and h is increasing on $\mathbb{R}_+$

$$I_1 \leq \frac{1}{h(\beta)} \int_{\partial\Omega} \int_{\mathbb{R}^3_v} \Lambda(G_\epsilon) h\left(|G_\epsilon/\Lambda(G_\epsilon) - 1| \right) (v \cdot \vec{n})^2_+ \wedge 1 M(v) d\sigma_x dv$$

$$\leq \frac{1}{h(\beta)} \int_{\partial\Omega} \int_{\mathbb{R}^3_v} \Lambda(G_\epsilon) h\left(G_\epsilon/\Lambda(G_\epsilon) - 1 \right) (v \cdot \vec{n})_+ M(v) d\sigma_x dv$$

$$\leq \frac{1}{h(\beta)} \int_{\partial\Omega} \int_{\mathbb{R}^3_v} \left(G_\epsilon \log(\frac{G_\epsilon}{\Lambda(G_\epsilon)}) - G_\epsilon + \Lambda(G_\epsilon) \right) (v \cdot \vec{n})_+ M(v) d\sigma_x dv$$

$$= \frac{1}{h(\beta)} \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon\Lambda(g_\epsilon)) d\sigma$$

For I_2 with $\beta < 1$

$$|G_\epsilon/\Lambda(G_\epsilon) - 1| \leq \beta \Rightarrow (\Lambda(G_\epsilon)) \leq \frac{1}{1-\beta} G_\epsilon$$

Hence

$$I_2 = \int_{\partial\Omega}\int_{\mathbb{R}^3_v} \Lambda(G_\epsilon) 1_{|G_\epsilon/\Lambda(G_\epsilon)-1|\leq\beta} (v\cdot\vec{n})^2 \wedge 1 M(v) d\sigma_x dv$$

$$\leq \frac{1}{1-\beta}\int_{\partial\Omega}\int_{\mathbb{R}^3_v} G_\epsilon (v\cdot\vec{n})^2_+ \wedge 1 M(v) d\sigma_x dv$$

$$\leq \frac{1}{1-\beta}\int_{\partial\Omega}\int_{\mathbb{R}^3_v} F_\epsilon (v\cdot\vec{n})^2_+ d\sigma_x dv$$

Use trace theorems introduced by Mischler!!!

Boltzmann version of Kato Theorem

- If $u_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}^n_v} vF_\epsilon dv$ converges weakly to a C^1 solution with :

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+3}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma \le 0\,.$$

One has:

$$\lim \frac{1}{\epsilon^{q+3}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma = 0 \quad \text{and}$$
$$\lim \frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\epsilon^2}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma = 0$$

Conversely

Proposition With $\frac{\alpha(\epsilon)}{\epsilon} \to \lambda < \infty$ the convergence to zero of the Darrozes Guiraud entropy implies the convergence to a dissipative solution.

Proof Just show that in this case the term

$$\frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w \cdot v)))d\sigma$$

goes to zero.

Starting from

$$\forall \eta > 0$$

$$\int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v)))d\sigma \;\leq\; (\frac{1}{\eta} + \frac{\eta C(w)}{\epsilon}) \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda g_\epsilon) d\sigma$$
$$+ C_2 \eta \int_{\partial\Omega} \int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With

$$\frac{\alpha(\epsilon)}{\epsilon^3} \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon\Lambda g_\epsilon) d\sigma = d(\epsilon) \to 0 \quad \eta = \epsilon D(\epsilon)\,,\; \frac{d(\epsilon)}{D(\epsilon)} \to 0$$

$$\frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v)))d\sigma \leq (\frac{d(\epsilon)}{D(\epsilon)} + C(w)\epsilon d(\epsilon) D(\epsilon))$$
$$+ \quad C_2 \frac{\alpha(\epsilon)}{\epsilon} D(\epsilon) \int_{\partial\Omega} \int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma\,.$$

And the conclusion follows.

Remarks

- Formally with g_ϵ converging to $u \cdot v$ and u tangent to boundary one has:

$$\frac{\alpha}{\epsilon^3}\int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda g_\epsilon) d\sigma \simeq \frac{1}{2} \lim \frac{\alpha(\epsilon)}{\epsilon} \int_{\partial\Omega} |u_\epsilon|^2 d\sigma \,, u_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}^n_v} vF_\epsilon(x,v,t)d$$

and therefore the macroscopic equivalent condition is

$$\lambda(\nu)\int_{\partial\Omega} |u_\nu|^2 dx \to 0$$

which is (for $0 < \lambda(\nu) < \infty$) a sufficient condition for the convergence.

- More generally what has been underlined are the striking similarities between convergence (or non convergence) between the Boltzmann and the Navier-Stokes limit.

- One thing that remain to do is to prove that a local (near the boundary) decay of dissipative entropy would be (as it is the case for the dissipation of energy) to ensure the convergence.

- In some sense this program validates the boundary conditions both at the level of Boltzmann and Navier-Stokes through their coherence.