Lecture one: Total positivity and networks

Thomas Lam

U. Michigan, IAS
tfylam@umich.edu

Curia, April 15, 2019

More details in the survey "Totally nonnegative Grassmannian polytopes".

Totally positive matrices

Definition

A real matrix A is *totally positive* (resp. *totally nonnegative*) if every minor is positive (resp. nonnegative).
Let $(\mathrm{GL}_n)_{>0}$ denote the set of $n \times n$ totally positive matrices, and $(\mathrm{GL}_n)_{\geq 0}$ denote the set of non-singular $n \times n$ totally nonnegative matrices.

Totally positive matrices

Definition

A real matrix A is *totally positive* (resp. *totally nonnegative*) if every minor is positive (resp. nonnegative).
Let $(\mathrm{GL}_n)_{>0}$ denote the set of $n \times n$ totally positive matrices, and $(\mathrm{GL}_n)_{\geq 0}$ denote the set of non-singular $n \times n$ totally nonnegative matrices.

Example:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix} \in (\mathrm{GL}_4)_{>0}$$

Two classical results

Theorem (Gantmacher-Krein (1937))

The eigenvalues of a totally positive matrix are all real, positive, and distinct.

Theorem (Loewner-Whitney Theorem (1955,1952))

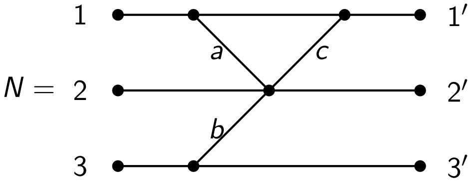
$(\mathrm{GL}_n)_{\geq 0}$ *is the semigroup generated by the elementary Jacobi matrices with positive parameters.*

With $t > 0$,

$$x_2(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad h_2(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y_3(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t & 1 \end{bmatrix}$$

From networks to matrices

Let N be a planar acyclic directed graph with sources $1, 2, \ldots, n$ and sinks $1', 2', \ldots, n'$.



$$M(N) = \begin{bmatrix} 1+ac & a & 0 \\ c & 1 & 0 \\ bc & b & 1 \end{bmatrix}$$

All edges are directed to the right. Unlabeled edges have weight 1.

Theorem (Lindström-Gessel-Viennot)

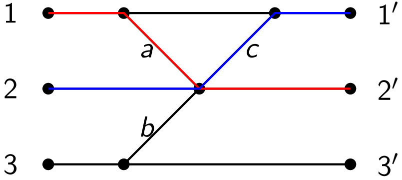
The minor $\det M(N)_{I,J}$ *is equal to the weighted sum of families of non-intersecting paths from sources I to sinks J'.*

Corollary

For any N with positive edge weights, $M(N)$ is TNN.

Idea of proof

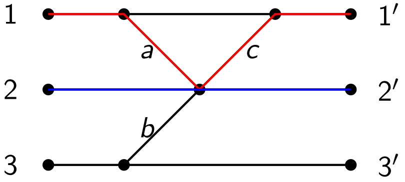
Produce a sign-reversing involution on *intersecting* path families.



Contributes to $m_{12}m_{21}$.

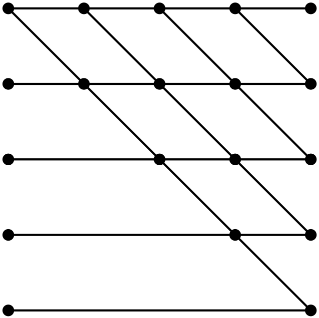
Idea of proof

Produce a sign-reversing involution on *intersecting* path families.



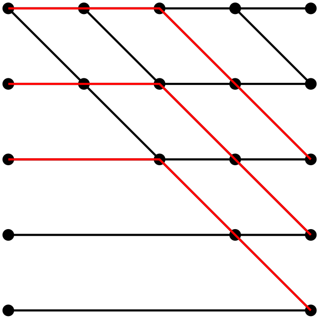
Contributes to $m_{11}m_{22}$.

Application: Pascal's triangle



$$\longrightarrow \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Application: Pascal's triangle



$$\longrightarrow \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

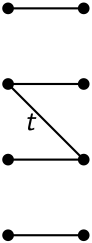
For example,

$$\det \begin{bmatrix} 6 & 4 & 1 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} = 1 \geq 0$$

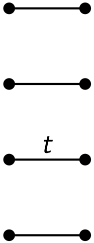
Observation

Concatenating networks corresponds to multiplying matrices:
$M(N * N') = M(N)M(N')$. *(Proof: Cauchy-Binet formula.)*

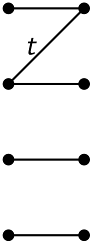
$x_2(t)$ " $=$ "



$h_3(t)$ " $=$ "



$y_1(t)$ " $=$ "



Corollary

Every $g \in (\mathrm{GL}_n)_{\geq 0}$ *can be represented by a planar directed network.*

Stembridge's theorem

Let $\chi : S_n \to \mathbb{C}$ be a function on the symmetric group.

Definition

The *immanant* is the function on $n \times n$ matrices defined by

$$\mathrm{Imm}_\chi(A) = \sum_{w \in S_n} \chi(w) a_{1,w(1)} \cdots a_{n,w(n)}.$$

When $\chi = \chi_\lambda$ is an irreducible character of S_n, we call $\mathrm{Imm}_\lambda = \mathrm{Imm}_{\chi_\lambda}$ the *irreducible immanant*. For the sign and trivial characters, we have

$$\mathrm{Imm}_{(1^n)}(A) = \det(A) \qquad \mathrm{Imm}_{(n)}(A) = \mathrm{perm}(A).$$

Also, $\mathrm{Imm}_{(21)}(A) = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}$.

Theorem (Stembridge)

For totally nonnegative A, we have $\mathrm{Imm}_\lambda(A) \geq 0$.

Haiman's theorem

Let $H_{\mu/\nu} = (h_{\mu_i - \nu_j})$ be a (skew) Jacobi-Trudi matrix, i.e., a submatrix of

$$\begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_0 & h_1 & h_2 & \cdots \\ 0 & h_0 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

satisfying $\det(H_{\mu/\nu}) = s_{\mu/\nu}$, the skew Schur function.

Theorem (Haiman)

$\mathrm{Imm}_\lambda(H_{\mu/\nu})$ *is Schur-positive.*

Earlier, Greene showed that $\mathrm{Imm}_\lambda(H_{\mu/\nu})$ is monomial positive.

Cluster algebras (just a hint)

$$U_{>0} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} a > 0, & b > 0 \\ c > 0, & \Delta = ac - b > 0 \end{matrix} \right\}$$

Cluster algebras (just a hint)

$$U_{>0} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} a > 0, & b > 0 \\ c > 0, & \Delta = ac - b > 0 \end{matrix} \right\}$$

We don't need to check all four inequalities! It is enough to have either

$$a > 0, \qquad b > 0, \qquad \Delta > 0$$

or

$$c > 0, \qquad b > 0, \qquad \Delta > 0$$

Cluster algebras (just a hint)

$$U_{>0} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} a > 0, & \quad b > 0 \\ c > 0, & \quad \Delta = ac - b > 0 \end{matrix} \right\}$$

We don't need to check all four inequalities! It is enough to have either

$$a > 0, \qquad b > 0, \qquad \Delta > 0$$

or

$$c > 0, \qquad b > 0, \qquad \Delta > 0$$

In modern language, $\{a, b, \Delta\}$ and $\{c, b, \Delta\}$ are *clusters*. The variables b, Δ are "frozen", and the relation

$$a = \frac{b + \Delta}{c} \Leftrightarrow c = \frac{b + \Delta}{a}$$

is an *exchange relation*.

The Grassmannian

The *Grassmannian* $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.
We represent $V \in \mathrm{Gr}(k, n)$ by a $k \times n$ matrix whose rows are a basis for V.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \end{bmatrix} \qquad \longrightarrow \qquad \mathrm{span}\{(1, 0, 2, 3), (0, 1, -1, 4)\} \subset \mathbb{R}^4$$

The Grassmannian

The *Grassmannian* $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.
We represent $V \in \mathrm{Gr}(k, n)$ by a $k \times n$ matrix whose rows are a basis for V.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \end{bmatrix} \qquad \longrightarrow \qquad \mathrm{span}\{(1,0,2,3),(0,1,-1,4)\} \subset \mathbb{R}^4$$

Two matrices represent the same point in $\mathrm{Gr}(k, n)$ if they are related by left-multiplication by $g \in \mathrm{GL}_k$.

The Grassmannian

The *Grassmannian* $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.
We represent $V \in \mathrm{Gr}(k, n)$ by a $k \times n$ matrix whose rows are a basis for V.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \end{bmatrix} \qquad \begin{matrix} \Delta_{12} = 1 & \Delta_{13} = -1 & \Delta_{14} = 4 \\ \Delta_{23} = -2 & \Delta_{24} = -3 & \Delta_{34} = 11 \end{matrix}$$

Two matrices represent the same point in $\mathrm{Gr}(k, n)$ if they are related by left-multiplication by $g \in \mathrm{GL}_k$.
For $I = \{i_1, i_2, \ldots, i_k\}$, let $\Delta_I(V)$ denote the *Plücker coordinate*: the $k \times k$ minor indexed by columns $i_1, i_2, \ldots, i_k$. The Plücker coordinates are only defined up to a common scalar.

The Grassmannian

The *Grassmannian* $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.
We represent $V \in \mathrm{Gr}(k, n)$ by a $k \times n$ matrix whose rows are a basis for V.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \end{bmatrix} \qquad \begin{matrix} \Delta_{12} = 1 & \Delta_{13} = -1 & \Delta_{14} = 4 \\ \Delta_{23} = -2 & \Delta_{24} = -3 & \Delta_{34} = 11 \end{matrix}$$

Two matrices represent the same point in $\mathrm{Gr}(k, n)$ if they are related by left-multiplication by $g \in \mathrm{GL}_k$.
For $I = \{i_1, i_2, \ldots, i_k\}$, let $\Delta_I(V)$ denote the *Plücker coordinate*: the $k \times k$ minor indexed by columns $i_1, i_2, \ldots, i_k$. The Plücker coordinates are only defined up to a common scalar.
If $\Delta_{1,2,\ldots,k}(V) \neq 0$, then V belongs to the *open Schubert cell*:

$$\begin{bmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{bmatrix} \subset \mathrm{Gr}(3, 6)$$

The dimension of $\mathrm{Gr}(k, n)$ is thus $k(n-k)$.

Totally nonnegative Grassmannian I

Definition (Postnikov)

A point $V \in \mathrm{Gr}(k,n)$ lies in the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ if $\Delta_I(V) \geq 0$ for all I.

The *totally positive Grassmannian* $\mathrm{Gr}_{>0}(k,n)$ is the locus where $\Delta_I(V) > 0$. Example: $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 3 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}_{>0}(2,4)$.

Totally nonnegative Grassmannian I

Definition (Postnikov)

A point $V \in \mathrm{Gr}(k,n)$ lies in the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ if $\Delta_I(V) \geq 0$ for all I.

The *totally positive Grassmannian* $\mathrm{Gr}_{>0}(k,n)$ is the locus where $\Delta_I(V) > 0$. Example: $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 3 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}_{>0}(2,4)$.

The Grassmannian $\mathrm{Gr}(k,n)$ contains $\binom{n}{k}$ *torus-fixed points*

$$e_I = e_{\{i_1,i_2,\ldots,i_k\}} = \mathrm{span}(e_{i_1}, e_{i_2}, \ldots, e_{i_k}).$$

Definition (Lusztig)

Define

$$\mathrm{Gr}_{\geq 0}(k,n) := \overline{(\mathrm{GL}_n)_{>0} \cdot e_{\{1,2,\ldots,k\}}}.$$

The two definitions coincide, but this is not obvious.

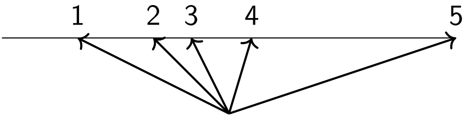
We'll use Postnikov's definition.

Totally nonnegative Grassmannian II

The torus $\mathbb{R}^n_{>0}$ acts on $\mathrm{Gr}_{\geq 0}(k,n)$ by scaling columns. A generic point in $\mathrm{Gr}_{\geq 0}(2,n)$ can be scaled to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$$

The positivity condition for $\mathrm{Gr}_{>0}(2,n)$ is that $a_1 < a_2 < a_3 < \cdots < a_5$.

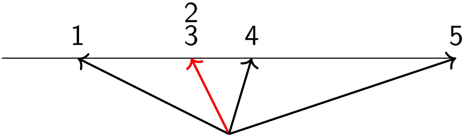


Totally nonnegative Grassmannian II

The torus $\mathbb{R}^n_{>0}$ acts on $\mathrm{Gr}_{\geq 0}(k, n)$ by scaling columns. A generic point in $\mathrm{Gr}_{\geq 0}(2, n)$ can be scaled to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$$

The positivity condition for $\mathrm{Gr}_{>0}(2, n)$ is that $a_1 < a_2 < a_3 < \cdots < a_5$.



When some Plücker coordinates go to 0, the configuration degenerates.

Positroid stratification I

Given $V \in \mathrm{Gr}(k,n)$

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix} \rightsquigarrow \begin{bmatrix} & | & | & & | & | & \\ \dots & v_0 & v_1 & \dots & v_n & v_{n+1} & \dots \\ & | & | & & | & | & \end{bmatrix}$$

with $v_{i+n} = (-1)^{(k-1)} v_i$.

Given $V \in \mathrm{Gr}(k,n)$

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \ldots & v_n \\ | & | & & | \end{bmatrix} \rightsquigarrow \begin{bmatrix} \ldots & | & | & & | & | & \\ & v_0 & v_1 & \ldots & v_n & v_{n+1} & \ldots \\ & | & | & & | & | & \end{bmatrix}$$

with $v_{i+n} = (-1)^{(k-1)} v_i$.
Define $f_V : \mathbb{Z} \to \mathbb{Z}$ by

$$f_V(i) = \min\{j \geq i \mid v_i \in \mathrm{span}(v_{i+1}, v_{i+2}, \ldots, v_j)\}$$

Example:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \to f(1) = 2, \quad f(2) = 5, \quad f(3) = 3, \quad f(4) = 6, \quad f(5) = 9, \ldots$$

Positroid stratification II

Proposition (Postnikov, Knutson–L.–Speyer)

The function $f = f_V : \mathbb{Z} \to \mathbb{Z}$ *is a (k, n)-bounded affine permutation:*

1. $f(i + n) = f(i)$ *for all* $i \in \mathbb{Z}$,
2. $i \leq f(i) \leq i + n$,
3. $f(1) + f(2) + \cdots + f(n) = 1 + 2 + \cdots + n + kn$,
4. $f : \mathbb{Z} \to \mathbb{Z}$ *is a bijection.*

Let $\mathrm{Bound}(k, n)$ be the set of (k, n)-bounded affine permutations or *juggling patterns*.

Definition

Define the *open positroid variety* and *(closed) positroid variety*

$\mathring{\Pi}_f := \{V \in \mathrm{Gr}(k, n) \mid f_V = f\}$ $\quad\Pi_f := \overline{\mathring{\Pi}_f}$

and the totally nonnegative *open positroid cell* and *closed positroid cell*

$\Pi_{f,>0} := \mathring{\Pi}_f \cap \mathrm{Gr}_{\geq 0}(k, n)$ $\quad\Pi_{f,\geq 0} := \Pi_f \cap \mathrm{Gr}_{\geq 0}(k, n)$.

Closure partial order

Theorem (Postnikov, Rietsch, Knutson–L.–Speyer)

We have

$$\Pi_f = \bigsqcup_{g \le f} \mathring{\Pi}_g \qquad \text{and} \qquad \Pi_{f,\ge 0} = \bigsqcup_{g \le f} \Pi_{g,>0}$$

where $\le$ is the partial order on $\mathrm{Bound}(k,n)$ that is dual to affine Bruhat order.

Example: (writing window notation $[f(1), f(2), \ldots, f(5)]$)

$$[4,5,3,6,7] \qquad \ge \qquad [2,5,3,6,9] \qquad \ge \qquad [5,2,3,6,9]$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 3 & 1 \end{bmatrix}$$

Positroid stratification III

Theorem (Postnikov)

For $f \in \mathrm{Bound}(k,n)$, we have that $\Pi_{f,>0}$ is homeomorphic to an open ball $\mathbb{R}^d_{>0}$. In particular, it is nonempty.

Postnikov gave a construction of points in $\Pi_{f,>0}$ via *plabic graphs*. We will explain a version of this using the *dimer model*.

Theorem (Galashin–Karp–L.)

The cells $\{\Pi_{f,>0} \mid f \in \mathrm{Bound}(k,n)\}$ give $\mathrm{Gr}_{\geq 0}(k,n)$ the structure of a regular CW-complex.

A regular CW-complex is a CW-complex X where the attaching maps are homeomorphisms

$$\iota : \overline{B} \longrightarrow X$$

onto its image in X. We will discuss this result and its motivation in Lecture 3.

Postiroids

Definition (Matroid)

Let $X \in \mathrm{Gr}(k, n)$. The *matroid* $\mathcal{M}_X$ is the collection $\mathcal{M}_X \subset \binom{[n]}{k}$

$$\mathcal{M}_X = \{I \mid \Delta_I(X) \neq 0\}.$$

When $X \in \mathrm{Gr}_{\geq 0}(k, n)$, we call $\mathcal{M}_X$ a *positroid*.

Postiroids

Definition (Matroid)

Let $X \in \mathrm{Gr}(k,n)$. The *matroid* $\mathcal{M}_X$ is the collection $\mathcal{M}_X \subset \binom{[n]}{k}$

$$\mathcal{M}_X = \{I \mid \Delta_I(X) \neq 0\}.$$

When $X \in \mathrm{Gr}_{\geq 0}(k,n)$, we call $\mathcal{M}_X$ a *positroid*.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \to \mathcal{M} = \{14, 15, 24, 25, 45\}$$

Postiroids

Definition (Matroid)

Let $X \in \mathrm{Gr}(k,n)$. The *matroid* $\mathcal{M}_X$ is the collection $\mathcal{M}_X \subset \binom{[n]}{k}$

$$\mathcal{M}_X = \{I \mid \Delta_I(X) \neq 0\}.$$

When $X \in \mathrm{Gr}_{\geq 0}(k,n)$, we call $\mathcal{M}_X$ a *positroid*.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \to \mathcal{M} = \{14, 15, 24, 25, 45\}$$

Postnikov's positroid cells are determined by specifying a positroid. Set

$$\Pi_{\mathcal{M},>0} := \{X \in \mathrm{Gr}_{\geq 0}(k,n) \mid \Delta_I(X) > 0 \text{ for all } I \in \mathcal{M}\}.$$

Theorem

There is a bijection $\mathrm{Bound}(k,n) \to (k,n)$*-positroids,* $f \mapsto \mathcal{M}$ *such that* $\Pi_{f,>0} = \Pi_{\mathcal{M},0}$.

Grassmann necklaces and Oh's Theorem

Let $X \in \mathrm{Gr}(k,n)$. The *Grassmann necklace* is the n-tuple $(I_1, I_2, \ldots, I_n)$ where I_a is the lexicographically minimal non-vanishing Plücker coordinate under the a-cyclically rotated order.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \to \{14, 24, 45, 45, 15\}$$

Grassmann necklaces and Oh's Theorem

Let $X \in \mathrm{Gr}(k, n)$. The *Grassmann necklace* is the n-tuple $(I_1, I_2, \ldots, I_n)$ where I_a is the lexicographically minimal non-vanishing Plücker coordinate under the a-cyclically rotated order.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix} \to \{14, 24, 45, 45, 15\}$$

A *Schubert matroid* is $\mathcal{M}_I := \{J \in \binom{[n]}{k} \mid J \geq I\}$.

Theorem (Oh)

Every positroid $\mathcal{M}$ is the intersection of cyclically rotated Schubert matroids of its Grassmann necklace:

$$\mathcal{M} = \bigcap_{a=1}^{n} \mathcal{M}_I^{(a)}$$

For example, $\mathcal{M}_{24} = \{24, 25, 34, 35, 45\}$
$\mathcal{M}_{24}^{(2)} = \{12, 13, 14, 15, 24, 25, 34, 35, 45\}$.

$\mathrm{Gr}_{\geq 0}(1, n)$ and $\mathrm{Gr}_{\geq 0}(2, n)$

$\mathrm{Gr}_{\geq 0}(1, n)$ is a $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$.
Each positroid cell $\Pi_{f,>0}$ is of the form

$$\{[a_0 : 0 : a_2 : a_3 : 0 : 0 : 0 : a_7]\}$$

where some coordinates are 0, and the rest (a_0, a_2, a_3, a_7) take arbitrary values in $\mathbb{R}_{>0}$.

$\mathrm{Gr}_{\geq 0}(1, n)$ and $\mathrm{Gr}_{\geq 0}(2, n)$

$\mathrm{Gr}_{\geq 0}(1, n)$ is a $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$.
Each positroid cell $\Pi_{f,>0}$ is of the form

$$\{[a_0 : 0 : a_2 : a_3 : 0 : 0 : 0 : a_7]\}$$

where some coordinates are 0, and the rest (a_0, a_2, a_3, a_7) take arbitrary values in $\mathbb{R}_{>0}$.
The topology and combinatorics of $\mathrm{Gr}_{\geq 0}(2, n)$ is more complicated. Each positroid cell $\Pi_{f,>0}$ is given by a collection of conditions of the form

1. the column vector $v_i = 0$
2. the column vectors $v_j, v_{j+1}, \ldots, v_k$ are parallel

$\mathrm{Gr}_{\geq 0}(1, n)$ and $\mathrm{Gr}_{\geq 0}(2, n)$

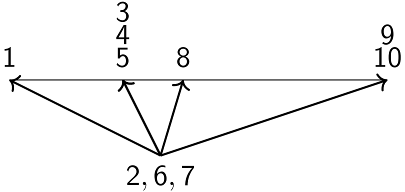
$\mathrm{Gr}_{\geq 0}(1, n)$ is a $(n-1)$-dimensional simplex in $\mathbb{P}^{n-1}$.
Each positroid cell $\Pi_{f,>0}$ is of the form

$$\{[a_0 : 0 : a_2 : a_3 : 0 : 0 : 0 : a_7]\}$$

where some coordinates are 0, and the rest (a_0, a_2, a_3, a_7) take arbitrary values in $\mathbb{R}_{>0}$.
The topology and combinatorics of $\mathrm{Gr}_{\geq 0}(2, n)$ is more complicated. Each positroid cell $\Pi_{f,>0}$ is given by a collection of conditions of the form

1. the column vector $v_i = 0$
2. the column vectors $v_j, v_{j+1}, \ldots, v_k$ are parallel

After rescaling the columns, we obtain the picture:



$$\begin{bmatrix} 1 & 0 & a & \alpha a & \beta a & 0 & 0 & c & 0 & 0 \\ 0 & 0 & b & \alpha b & \beta b & 0 & 0 & d & \lambda & 1 \end{bmatrix}$$

$$a, b, c, d, \lambda, \alpha, \beta, ad - bc > 0$$